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RADIATIVE TRANSFER EQUATION SOLUTION FOR MANY SCATTERING TYPES

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Abstract: The radiative transfer equation is a mathematical equation that describes the changes in the number of photons within a specified volume of a medium over time, taking into account phenomena such as scattering, absorption, and re-emission resulting from photon interactions with the medium. In this study, the radiative transfer equation is considered for a finite slab which anisotropic scattering in a homogeneous medium. The equation solution is done by Legendre polynomials for linear anisotropic, pure quadratic and Rayleigh scattering types. The numerical results are displayed in the tables up to the 13th iteration of the Legendre polynomials. Tables are obtained using different scattering coefficients and single scattering albedo values. The results contain a wide range of data obtained from the method of solving the Legendre polynomial of the radiative transfer equation. Thus, with this study, the effect of different scattering types on the solution of the radiative transfer equation has been demonstrated.

Keywords: Radiative transfer equation, Legendre polynomial method, Anisotropic scattering

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1. Introduction

Radiative transfer theory can be applied to many fields in physics. The radiative transfer equation expresses the states of photons in a medium such as absorption, scattering and propagation. Radiative transfer (radiation emission) is known as a kind of physical energy transfer of electromagnetic radiation (Herman and Browning, 1965). Radiative transfer equation is an equation that depends on time, direction, energy and space variables. The concept of radiative transfer is widely used in many fields of physics such as astrophysics, optics, atmospheric physics, marine sciences and remote sensing systems (Stamnes et al., 2017).

In 1960, Chandrasekhar showed that statistical physics methods can obtain the Boltzmann equation for radiational transfer as photon gas transfer (Chandrasekhar, 1960). The eigenvalue calculations in one-dimensional geometry showed the applicability of the new scattering function to the radiative transfer equation (Anlı and Güngör, 2005). A Legendre polynomial approximation method has been developed to find approximate solutions of higher order Volterra type integro-differential equations by Güçlü (Güçlü İ. 2009). At the same time, he concluded that the method is a good method that can also be used to find the exact solutions of the equations of the mentioned property.

Elghazaly et al. (Elghazaly A, El-Konsol S, Sabbah AS, Hosni M. 2017) have solved the integral form of the radiative transfer equation in a two-layer inhomogeneous plate for linear anisotropic scattering by using the Galerkin iterative method. Biçer and Kaşkaş (Biçer M, Kaşkaş A., 2018) solved the radiative transfer equation for Rayleigh scattering using the infinite medium Green's function. Tapimo et al. (R. Tapimo, H.T. Tagne Kamdem, & D. Yemele, 2018) used the discrete spherical harmonics method to calculate the discrete spherical harmonics method for radiative transfer in the scalar planar inhomogeneous atmosphere and extended the radiative density with a finite series of Legendre polynomials.

This study uses Legendre polynomial methods to discuss the radiative transfer equation solution for linear anisotropic, pure quadratic and Rayleigh scattering types. The scattering phenomena is efficient for the analyzing of the photon behavior in the medium. The discrete eigenvalues from the calculation of the equation are obtained for the 13th iteration of the method to approach the accurate roots of the equation. When the iteration steps are increased, the accuracy is increased to. The closeness of the roots is nearly one digit after comma. The calculated eigenvalues precision is accepted enough for the study.

2. Legendre Polynomial Solution Method

2.1 Linear Anisotropic Scattering Solution

The radiative transfer equation has used with the most convenient with some restrictions that are no thermal emission, plane-parallel atmosphere, phase function expandable in Legendre polynomial series and azimuthal symmetry. After all the considerations of the RTE is written as given in Equation 1;

$$\mu \frac{d}{dx} I(x,\mu) + I(x,\mu) = \frac{\omega}{2} \int_{-1}^{1} f(\mu,\mu') I(x,\mu) d\mu' \tag{1}$$

where $I(x, \mu)$ is the angular intensity, μ is the direction cosine of the propagating radiation, x is the optical variable, ω is the single scattering albedo and $f(\mu,\mu')$ is the scattering phase function. It can be shown as given in Equation 2;

$$f(\mu,\mu') = \sum_{n=0}^{N} (2n+1) f_n P_n(\mu) P_n(\mu') \tag{2}$$

here f_n represent the scattering coefficients belong to scattering type. The linear anisotropic scattering is known as the case of the N=1. So, the function becomes (Equation 3);

$$f(\mu,\mu') = f_0 P_0(\mu) P_0(\mu') + 3 f_1 P_1(\mu) P_1(\mu') \tag{3}$$

Now the scattering function for the linear anisotropic case in Equation 3 is substituted in to the general form of the radiative transfer equation in Equation 1 so Equation 4.

$$\mu \frac{d}{dx} I(x,\mu) + I(x,\mu) = \frac{\omega}{2} \int_{-1}^{1} \left(f_0 P_0(\mu) P_0(\mu') + 3 f_1 P_1(\mu) P_1(\mu')\right) I(x,\mu) d\mu' \tag{4}$$

The angular flux of the radiative transfer equation can be displayed through series expansion of the Legendre polynomials (Equation 5);

$$I(x,\mu) = \sum_{n=0}^{N} \frac{2n+1}{2} \phi_n(x) P_n(\mu) \tag{5}$$

here $\phi_n(x)$ is called as flux moments that are demonstrated as for n=0 and 1 (Equations 6 and 7);

$$\phi_0(x) = \int_{-1}^{1} P_0(\mu') I(x,\mu) d\mu' \tag{6}$$

$$\phi_1(x) = \int_{-1}^{1} P_1(\mu') I(x,\mu) d\mu' \tag{7}$$

After determining the angular flux in Equation 5, scattering function in Equation 3 and the flux moments in Equations 6 and 7, the radiative transfer equation in Equation 1 is composed again like;

$$\mu \frac{d}{dx} \left[\sum_{n=0}^{N} \frac{2n+1}{2} \phi_n(x) P_n(\mu) \right] + \sum_{n=0}^{N} \frac{2n+1}{2} \phi_n(x) P_n(\mu) = \frac{\omega}{2} [f_0 \phi_0(x) + 3\mu f_1 \phi_1(x)] \tag{8}$$

By solving Equation 8 the recurrence relation of the Legendre polynomials is used and then the series is expanded for n=0,1,2,3,…N. The resultant equation is multiplied by $\int_{-1}^{1} P_m(\mu) d\mu$. The orthogonality relation of the Legendre polynomials is applied to obtain the discrete differential sets for m=0,1,2,3,…,. (Equations 9a-9d)

$$m = 0\ ; \quad \frac{d}{dx} \phi_1(x) + \phi_0(x) = \omega f_0 \phi_0(x) \tag{9a}$$

$$m = 1\ ; \quad \frac{1d}{3dx} \phi_0(x) + \frac{2d}{3dx} \phi_2(x) + \phi_1(x) = \omega f_1 \phi_1(x) \tag{9b}$$

$$m = 2\ ; \quad \frac{2d}{5dx} \phi_1(x) + \frac{3d}{5dx} \phi_3(x) + \phi_2(x) = 0 \tag{9c}$$

$$m = 3 \quad \frac{3d}{7dx} \phi_2(x) + \phi_3(x) = 0 \tag{9d}$$

The eigen-spectrum of the differential sets in Equations 9.a-d are obtained by employing the well-known ansatz;

$$\phi_n(x) = A_n(\nu) e^{x/\nu} \tag{10}$$

The Equation 10 is used to determine the eigenvalues of the radiative transfer equation by Legendre polynomial solution (Equations 11a-11c).

$$A_1(\nu) = -\nu A_0(\nu)(1 - \omega f_0) \tag{11a}$$

$$A_2(\nu) = (-3\nu A_1(\nu)(1 - \omega f_1) - A_0(\nu))/2 \tag{11b}$$

$$A_3(\nu) = (-5\nu A_2(\nu) - 2A_1(\nu))/3 \tag{11c}$$

with $A_0(\nu)$ = 1, the eigenvalues of ν can be found by solving the equation $A_{n+1}(\nu)$ = 0, (Taşdelen, 2017). These eigenvalues are the roots needed to solve the equation. As the iteration steps increase in the method, convergence in the roots is increased. If N symbolizes the iteration step, N+1/2 eigenvalues are found for P_N iteration. For example, there are seven positive eigenvalues for the 13th iteration of the polynomial expressed as P_{13}.

2.2. Pure-Quadratic Anisotropic Scattering Solution

The scattering function is taken into account for the case of N=2 which is called as pure quadratic anisotropic scattering type. In this situation, the scattering coefficients are assumed as f_1=0 and f_2≠0. Described the scattering function in Equation 2 takes the form of pure quadratic anisotropic scattering (Equation 12);

$$f(\mu,\mu') = f_0 P_0(\mu) P_0(\mu') + 5 f_2 P_2(\mu) P_2(\mu') \tag{12}$$

The Equation 12 is used in Equation 1 (Equation 13);

$$\mu \frac{d}{dx} I(x,\mu) + I(x,\mu) = \frac{\omega}{2} \int_{-1}^{1} \left(f_0 P_0(\mu) P_0(\mu') + 5 f_2 P_2(\mu) P_2(\mu')\right) I(x,\mu) d\mu' \tag{13}$$

The angular moments are applied (Equation 14).

$$\mu\frac{d}{dx}I(x,\mu)+I(x,\mu) = \frac{\omega}{2}[f_0P_0(\mu)\phi_0(x) + 5f_2P_2(\mu)\phi_2(x)] \quad (14)$$

The mentioned angular flux Legendre polynomial expansion form in Equation 7 is substituted into Equation 15.

$$\mu\frac{d}{dx}\left[\sum_{n=0}^{N}\frac{2n+1}{2}\phi_n(x)P_n(\mu)\right] + \sum_{n=0}^{N}\frac{2n+1}{2}\phi_n(x)P_n(\mu) = \frac{\omega}{2}\left[f_0\phi_0(x) + \frac{5(3\mu^2-1)}{2}f_2\phi_2(x)\right] \quad (15)$$

The solution of the Equation 15 is done by using the recurrence relation of Legendre polynomials and then multiplying by $\int_{-1}^{1}P_n(\mu)d\mu$. After some algebra the differentials sets are found as given in Equation 16a-16c;

$$\frac{1d}{3dx}\phi_0(x)+\frac{2d}{3dx}\phi_2(x)+\phi_1(x)=0 \quad (16a)$$

$$\frac{2d}{5dx}\phi_1(x)+\frac{3d}{5dx}\phi_3(x)+\phi_2(x)=\omega f_2\phi_2(x) \quad (16b)$$

$$\frac{3d}{7dx}\phi_2(x)+\phi_3(x)=0 \quad (16c)$$

The differential equations are solved by applying to the Equation 10. The eigenfunctions obtained as given in Equations 17a-17c;

$$A_1(\nu)=-\nu A_0(\nu)(1-\omega f_0) \quad (17a)$$

$$A_2(\nu)=(-3\nu A_1(\nu)-A_0(\nu))/2 \quad (17b)$$

$$A_3(\nu)=(-5\nu A_2(\nu)(1-\omega f_0))-2A_1(\nu))/3 \quad (17c)$$

2.3. Rayleigh Scattering Solution

Rayleigh scattering occurs when the incident wavelength is greater than the size of the scattering particles. Lord Rayleigh explained why the sky is blue and the sunset is red by using this scattering function. This scattering function can be used in astrophysics, oceanography and tissue optics. Chandrasekhar formulated the scattering function for the polarized case using the Stokes parameters (Chandrasekhar, 1950). The scattering function can be written for the unpolarized Rayleigh case in the following form with ($f_0=1\ ve\ f_2=1/2$) (Sallah and Selim, 2008) (Equation 18).

$$f(\mu,\mu')=P_0(\mu)P_0(\mu')+\frac{1}{2}P_2(\mu)P_2(\mu') \quad (18)$$

The Rayleigh scattering function is applied to radiative transfer equation in Equation 1 and the same procedure is used to solve the eigenvalues of the Rayleigh scattered radiative transfer equation some of them are found as given in Equations 19a-19c;

$$A_1(\nu)=-\nu A_0(\nu)(1-\omega) \quad (19a)$$

$$A_2(\nu)=(-3\nu A_1(\nu)-A_0(\nu))/2 \quad (19b)$$

$$A_3(\nu)=(-\nu A_2(\nu)(\frac{\omega}{2}-5)-2A_1(\nu))/3 \quad (19c)$$

3. Results and Discussion

The radiative transfer equation is solved by the P_N method with linear anisotropic scattering, pure quadratic anisotropic scattering and Rayleigh scatterings. The solutions are done to obtain the discrete eigenvalues of the radiative transfer equation solutions. The anisotropic scattering function comprises the scattering coefficient belonging to the type of anisotropy. The scattering coefficients are not arbitrary numbers. They are chosen from the numbers supplying the condition that the scattering function must be -1 to 1. The numerical results of the radiative transfer equation solution are demonstrated for many possibilities of the scattering coefficients and also single scattering albedo numbers. The discrete eigenvalues of the linear anisotropic scattering solution is tabulated in Table 1 for f_1=-0.3 to 0.3 in 0.1 with the 13th iteration of the P_N method for the single scattering albedo numbers from 0.1 to 0.9.

The series of the scattering function is expanded up to the second term called as the pure quadratic anisotropic scattering. So, it contains two scattering coefficients that have to be determined by verifying the condition of the scattering function. It occurs with many possibilities and complexity. That's why the first term is assumed zero. The conceivable numbers are selected for the second scattering coefficient. Table 2 shows the discrete eigenvalues of the solution of pure quadratic anisotropic scattering corresponding to the scattering coefficients f_2=-0.20, -0.14, -0.12, -0.08, 0.2, 0.4 with the 13th iteration of the P_N method for the single scattering albedo numbers from 0.1 to 0.9.

Physically, Rayleigh scattering is considered when the size of scattering particles of the stochastic media are much smaller than the incident radiation wavelength. On the other hand, if particles sizes are comparable to, or larger than, wavelength, the scattering is customarily referred to as Mie scattering. Frequently, these phenomena are related to the radiative transfer in astrophysical setting, Hussein and Selim 2012 (Hussein A, Selim MM., 2012). Rayleigh scattering occurs for particles that are smaller than the radiation's wavelength, Cherry et al. (Cherry SR, Sorenson JA, Phelps ME, 2012). The obtained discrete eigenvalues from the solution of the Rayleigh scattering are presented in Table 3. The numerical results are found for many single-scattering albedo numbers. The iteration of the method up to 13th are demonstrated in the Table 3.

Table 1. Discrete eigenvalues of the P_{13} iteration for linear anisotropic scattering

$w\backslash f_2$	-0.30	-0.20	-0.10	0.10	0.20	0.4
0.1	0.10922148	0.10922494	0.10922839	0.10923530	0.10923875	0.10922148
	0.32219243	0.32227921	0.32236568	0.32253769	0.32262323	0.32219243
	0.51918027	0.51952389	0.51986492	0.52053919	0.52087243	0.51918027
	0.69087793	0.69160299	0.69232224	0.69374250	0.69444314	0.69087793
	0.82966620	0.83069108	0.83171640	0.83376409	0.83478432	0.82966620
	0.92969277	0.93069121	0.93171344	0.93382459	0.93491039	0.92969277
	0.98670182	0.98725320	0.98784506	0.98916558	0.98990283	0.98670182
0.2	0.11041966	0.11042538	0.11043110	0.11044252	0.11044822	0.11041966
	0.32548166	0.32562806	0.32577344	0.32606112	0.32620344	0.32548166
	0.52371805	0.52431936	0.52491114	0.52606633	0.52662988	0.52371805
	0.69552988	0.69686517	0.69817373	0.70070702	0.70193025	0.69552988
	0.83342189	0.83542848	0.83741796	0.84131332	0.84320383	0.83342189
	0.93205500	0.93415408	0.93632627	0.94081361	0.94307785	0.93205500
	0.98769054	0.98897814	0.99046838	0.99421394	0.99656147	0.98769054
0.3	0.11164962	0.11165650	0.11166338	0.11167710	0.11168395	0.11164962
	0.32897040	0.32914907	0.32932593	0.32967430	0.32984584	0.32897040
	0.52886564	0.52962213	0.53036016	0.53178205	0.53246656	0.52886564
	0.70138335	0.70314689	0.70484696	0.70805365	0.70956018	0.70138335
	0.83885381	0.84169675	0.84445092	0.84961387	0.85199328	0.83885381
	0.93609830	0.93939343	0.94273627	0.94917588	0.95207554	0.93609830
	0.98975263	0.99224194	0.99535240	1.00405916	1.00996814	0.98975263
0.4	0.11291130	0.11291836	0.11292541	0.11293948	0.11294650	0.11291130
	0.33264081	0.33282569	0.33300823	0.33336642	0.33354213	0.33264081
	0.53457905	0.53537754	0.53615019	0.53762103	0.53832076	0.53457905
	0.70846781	0.71039809	0.71222317	0.71556824	0.71709547	0.70846781
	0.84629826	0.84960969	0.85269048	0.85810452	0.86043851	0.84629826
	0.94265997	0.94694311	0.95092772	0.95737025	0.95975642	0.94265997
	0.99426781	0.99916528	1.00561351	1.02420108	1.03661995	0.99426781
0.5	0.11420447	0.11421088	0.11421729	0.11423007	0.11423644	0.11420447
	0.33646549	0.33663373	0.33679955	0.33712414	0.33728298	0.33646549
	0.54074783	0.54147764	0.54217942	0.54350354	0.54412812	0.54074783
	0.71658598	0.71836771	0.72002196	0.72298041	0.72430106	0.71658598
	0.85564288	0.85875076	0.86150142	0.86603504	0.86788586	0.85564288
	0.95208742	0.95608654	0.95925044	0.96350908	0.96491653	0.95208742
	1.00559135	1.01571858	1.02861238	1.06285561	1.08426029	1.00559135
0.6	0.11552862	0.11553378	0.11553892	0.11554919	0.11555431	0.11552862
	0.34040673	0.34054095	0.34067316	0.34093177	0.34105822	0.34040673
	0.54718592	0.54775892	0.54830840	0.54934148	0.54982726	0.54718592
	0.72521422	0.72657054	0.72781700	0.73002022	0.73099495	0.72521422
	0.86573030	0.86792898	0.86981382	0.87283199	0.87404498	0.86573030
	0.96168007	0.96380309	0.96536066	0.96742550	0.96813127	0.96168007
	1.03618749	1.05500487	1.07695949	1.13072705	1.16309948	1.03618749
0.7	0.11688301	0.11688655	0.11689008	0.11689713	0.11690064	0.11688301
	0.34441690	0.34450741	0.34459666	0.34477148	0.34485709	0.34441690
	0.55364491	0.55401637	0.55437341	0.55504711	0.55536513	0.55364491
	0.73357939	0.73439339	0.73514427	0.73648133	0.73707809	0.73357939
	0.87464629	0.87576819	0.87673949	0.87832916	0.87898525	0.87464629
	0.96775833	0.96848696	0.96906062	0.96990133	0.97021734	0.96775833
	1.10928863	1.13815411	1.17050645	1.24766452	1.29390650	1.10928863
0.8	0.11826658	0.11826845	0.11827033	0.11827407	0.11827594	0.11826658
	0.34844029	0.34848697	0.34853312	0.34862389	0.34866852	0.34844029
	0.55985551	0.56003533	0.56020956	0.56054224	0.56070114	0.55985551
	0.74096975	0.74132138	0.74165184	0.74225611	0.74253295	0.74096975
	0.88128288	0.88168247	0.88204327	0.88266853	0.88294097	0.88128288
	0.97090984	0.97110548	0.97127502	0.97155402	0.97167012	0.97090984
	1.26976876	1.31072484	1.35638228	1.46564646	1.53196937	1.26976876
0.9	0.11967792	0.11967847	0.11967902	0.11968012	0.11968067	0.11967792
	0.35241654	0.35242970	0.35244277	0.35246865	0.35248146	0.35241654
	0.56558411	0.56563071	0.56567648	0.56576562	0.56580904	0.56558411
	0.74702582	0.74710576	0.74718308	0.74733038	0.74740058	0.74702582
	0.88581158	0.88588812	0.88596092	0.88609637	0.88615946	0.88581158
	0.97261221	0.97264393	0.97267370	0.97272807	0.97275295	0.97261221
	1.69054019	1.75321107	1.82353786	1.99446652	2.10042214	1.69054019

Table 2. Discrete eigenvalues of the P_{13} iteration for pure quadratic anisotropic scattering

w\f$_2$	-0.20	-0.14	-0.12	-0.08	0.2	0.4
0.1	0.10895392	0.10903710	0.10906487	0.10912045	0.10951161	0.10979322
	0.32201417	0.32214493	0.32218862	0.32227616	0.32289470	0.32334281
	0.52011074	0.52013845	0.52014769	0.52016622	0.52029672	0.52039087
	0.69290065	0.69294127	0.69295478	0.69298174	0.69316867	0.69330025
	0.83156661	0.83192279	0.83204079	0.83227570	0.83387855	0.83497840
	0.93048130	0.93116489	0.93139274	0.93184832	0.93501871	0.93723967
	0.98673580	0.98723536	0.98740633	0.98775515	0.99047397	0.99274801
0.2	0.10986620	0.11003659	0.11009354	0.11020766	0.11101506	0.11160108
	0.32495100	0.32523853	0.32533485	0.32552820	0.32690883	0.32792482
	0.52518702	0.52527825	0.52530879	0.52537007	0.52580662	0.52612672
	0.69935152	0.69938262	0.69939296	0.69941358	0.69955607	0.69965587
	0.83760278	0.83814919	0.83832893	0.83868482	0.84104236	0.84258561
	0.93432490	0.93561159	0.93603769	0.93688475	0.94252263	0.94609498
	0.98814878	0.98925826	0.98964727	0.99045547	0.99738803	1.00393672
0.3	0.11079166	0.11105338	0.11114097	0.11131669	0.11256672	0.11348156
	0.32790610	0.32837809	0.32853664	0.32885560	0.33115919	0.33288348
	0.53038197	0.53058838	0.53065776	0.53079736	0.53180748	0.53256582
	0.70646993	0.70647376	0.70647503	0.70647757	0.70649526	0.70650779
	0.84544020	0.84595426	0.84612225	0.84645325	0.84859109	0.84993887
	0.94085982	0.94249134	0.94302123	0.94405804	0.95031347	0.95364596
	0.99166000	0.99367573	0.99439450	0.99590571	1.00953810	1.02291264
0.4	0.11173007	0.11208738	0.11220712	0.11244759	0.11416793	0.11543761
	0.33086075	0.33154597	0.33177685	0.33224233	0.33564520	0.33823878
	0.53557531	0.53596290	0.53609381	0.53635824	0.53831154	0.53982442
	0.71389426	0.71390952	0.71391462	0.71392486	0.71399764	0.71405082
	0.85461337	0.85489670	0.85498915	0.85517113	0.85634232	0.85707930
	0.95046649	0.95178904	0.95220399	0.95299571	0.95726743	0.95927194
	1.00080113	1.00431802	1.00556721	1.00818332	1.03111306	1.05288552
0.5	0.11268119	0.11313843	0.11329188	0.11360036	0.11581988	0.11747201
	0.33379479	0.33472229	0.33503577	0.33566928	0.34035956	0.34400147
	0.54063303	0.54127709	0.54149592	0.54193998	0.54530379	0.54801081
	0.72111740	0.72124622	0.72128972	0.72137757	0.72202612	0.72252810
	0.86377027	0.86381569	0.86383066	0.86386035	0.86405903	0.86419176
	0.96033036	0.96077981	0.96091991	0.96118665	0.96265017	0.96338561
	1.02534909	1.03060372	1.03243540	1.03622353	1.06802366	1.09782395
0.6	0.11364472	0.11420633	0.11439508	0.11477492	0.11752363	0.11958729
	0.33668724	0.33788546	0.33829171	0.33911470	0.34528653	0.35016755
	0.54542417	0.54640134	0.54673558	0.54741727	0.55273118	0.55719661
	0.72763575	0.72801650	0.72814666	0.72841203	0.73047791	0.73221463
	0.87135469	0.87137800	0.87138586	0.87140168	0.87151701	0.87160464
	0.96644071	0.96646801	0.96647691	0.96649442	0.96660713	0.96667832
	1.08063246	1.08657617	1.08864746	1.09293407	1.12950579	1.16537440
0.7	0.11462029	0.11529078	0.11551644	0.11597106	0.11928000	0.12178556
	0.33951717	0.34101295	0.34152170	0.34255478	0.35040039	0.35671175
	0.54983867	0.55121816	0.55169332	0.55266760	0.56049482	0.56736981
	0.73312078	0.73388027	0.73414348	0.73468582	0.73918112	0.74335675
	0.87676704	0.87699755	0.87707713	0.87724065	0.87857479	0.87978760
	0.96943100	0.96945724	0.96946621	0.96948447	0.96962642	0.96974573
	1.18621471	1.19181411	1.19378196	1.19788197	1.23442256	1.27312845
0.8	0.11560748	0.11639142	0.11665564	0.11718851	0.12108955	0.12406845
	0.34226450	0.34408221	0.34470238	0.34596460	0.35566459	0.36358118
	0.55380035	0.55563702	0.55627403	0.55758709	0.56845053	0.57837485
	0.73748263	0.73870282	0.73913142	0.74002392	0.74791525	0.75602992
	0.88033414	0.88088192	0.88107549	0.88148059	0.88519735	0.88933282
	0.97092928	0.97106182	0.97110871	0.97120694	0.97211711	0.97315890
	1.39020742	1.39487561	1.39652857	1.39999355	1.43220727	1.46920665
0.9	0.11660579	0.11750778	0.11781223	0.11842688	0.12295251	0.12643684
	0.34491088	0.34707162	0.34781100	0.34931913	0.36103194	0.37069121
	0.55727131	0.55960236	0.56041615	0.56210195	0.57642182	0.58986849
	0.74080919	0.74252347	0.74313315	0.74441517	0.75645303	0.76991088
	0.88261170	0.88349502	0.88381341	0.88449047	0.89142971	0.90081789
	0.97172777	0.97197493	0.97206474	0.97225708	0.97435412	0.97771060
	1.89084608	1.89412099	1.89528671	1.89774090	1.92130043	1.95026061

Table 3. Discrete eigenvalues of the P_{13} iteration for Rayleigh scattering

w	P_1	P_3	P_5	P_7	P_9	P_{11}	P_{13}
0.1	0.60858061	0.35222723	0.24488545	0.18718942	0.15136247	0.12699919	0.10937150
		0.88057190	0.67400993	0.53419630	0.43959504	0.37245332	0.32267262
			0.94430192	0.80710092	0.68740631	0.59356701	0.52024994
				0.96807914	0.87340830	0.77693402	0.69310227
					0.97940041	0.91082296	0.83331449
						0.98564509	0.93389202
							0.98944456
0.2	0.64549722	0.36571853	0.25158251	0.19113571	0.15395071	0.12882341	0.11072497
		0.90411068	0.68838211	0.54367652	0.44625702	0.37735381	0.32641023
			0.95942482	0.81897020	0.69627130	0.60039482	0.52564918
				0.97870361	0.88309100	0.78479704	0.69950555
					0.98739798	0.91880792	0.84022709
						0.99197218	0.94058014
							0.99463679
0.3	0.69006555	0.38062956	0.25874573	0.19528384	0.15664289	0.13070776	0.11211622
		0.93344599	0.70444820	0.55400685	0.45338796	0.38253161	0.33032203
			0.97953926	0.83228659	0.70601210	0.60780084	0.53143999
				0.99396365	0.89402772	0.79345871	0.70648897
					0.99982107	0.92791733	0.84786261
						1.00258591	0.94829321
							1.00402138
0.4	0.74535599	0.39714836	0.26641011	0.19964359	0.15944256	0.13265376	0.11354599
		0.97132057	0.72219072	0.56517951	0.46097677	0.38797673	0.33440022
			1.00754661	0.84673828	0.71654195	0.61574158	0.53759277
				1.01708559	0.90562266	0.80273881	0.71397142
					1.02027563	0.93730424	0.85594381
						1.02148094	0.95597296
							1.02196695
0.5	0.81649658	0.41546162	0.27460801	0.20422368	0.16235283	0.13466276	0.11501493
		1.02246601	0.74132376	0.57712215	0.46898805	0.39366829	0.33863169
			1.04860294	0.86151877	0.72764092	0.62411645	0.54404879
				1.05383918	0.91668860	0.81227652	0.72178754
					1.05509988	0.94558001	0.86398985
						1.05542533	0.96219727
							1.05551141
0.6	0.91287092	0.43572393	0.28336587	0.20903107	0.16537617	0.13673582	0.11652358
		1.09577468	0.76117560	0.58967609	0.47735621	0.39957261	0.34299714
			1.11237875	0.87545486	0.73894975	0.63276225	0.55071777
				1.11457086	0.92606913	0.82158858	0.72969547
					1.11489313	0.95192134	0.87147487
						1.11494159	0.96656875
							1.11494890
0.7	1.05409255	0.45800247	0.29269948	0.21407019	0.16851418	0.13887365	0.11807228
		1.21019870	0.78072693	0.60258799	0.48598262	0.40564231	0.34747085
			1.21886551	0.88757282	0.75002833	0.64146483	0.55748160
				1.21948640	0.93340591	0.83022719	0.73742020
					1.21953274	0.95651192	0.87805633
						1.21953620	0.96957269
							1.21953645
0.8	1.29099444	0.48219365	0.30260795	0.21934189	0.17176740	0.14107655	0.11966122
		1.41545544	0.79891941	0.61552907	0.49473821	0.41181688	0.35202086
			1.41868761	0.89756316	0.76047000	0.64999068	0.56420569
				1.41878191	0.93902530	0.83792883	0.74471844
					1.41878467	0.95988991	0.88366025
						1.41878475	0.97173843
							1.41878475
0.9	1.82574185	0.50792847	0.31306648	0.22484233	0.17513496	0.14334429	0.12129035
		1.91074772	0.81506202	0.62814655	0.50347343	0.41802545	0.35660993
			1.91130975	0.90566601	0.77000594	0.65812931	0.57075608
				1.91131339	0.94341956	0.84464332	0.75143000
					1.91131342	0.96249787	0.88839506
						1.91131342	0.97340607
							1.91131342

4. Conclusion

The radiative transfer equation is used to understand the behaviour of light within a medium. In this study we assumed homogeneous medium surrounded with a slab thickness x. The behaviour of the photons is examined with three different scattering types. The numerical results are obtained for each condition separately. By making calculations, The Wolfram Mathematica program is run to find the discrete eigenvalues of the radiative transfer equation by Legendre polynomial method. The P_N method is iterated up to 13^{th} to observe the convergence. The results for all scattering types meet the expectations related with the eigenvalues of the radiative transfer equation. The presented study displays a wide range of numerical data about the radiative transfer equation solution with Legendre polynomial method. The single scattering albedo numbers and the scattering coefficients are taken into account a detailed spectrum. Being a large scale of calculated data causes a great source to compare their own method solutions.

Author Contributions

The percentage of the author(s) contributions is presented below. All authors reviewed and approved the final version of the manuscript.

	D.A.	H.K.
C	50	50
D	100	
S		100
DCP	50	50
DAI	50	50
L	50	50
W	50	50
CR	50	50
SR	50	50

C=Concept, D= design, S= supervision, DCP= data collection and/or processing, DAI= data analysis and/or interpretation, L= literature search, W= writing, CR= critical review, SR= submission and revision.

Conflict of Interest

The authors declared that there is no conflict of interest.

Ethical Consideration

Ethics committee approval was not required for this study because of there was no study on animals or humans.

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