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Research Article

The Isometry Groups of $\mathbb{R}^3_{DH}$, $\mathbb{R}^3_{PD}$ and $\mathbb{R}^3_{TI}$

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Abstract
There are two aims of this paper. First one, we want to give a detailed exposition of basic properties of deltoidal hexacontahedron, pentakis dodecahedron and triakis icosahedron which are Catalan solids. Also, we construct the spaces by covering related metrics. The spheres of these spaces are deltoidal hexacontahedron, pentakis dodecahedron and triakis icosahedron. Second one is to find the isometry group of these solids. In fact, the main aim of this paper is the second one. We show that the group of isometries of the spaces covering with deltoidal hexacontahedron, pentakis dodecahedron, and triakis icosahedron metrics is the semi-direct product of the icosahedral group I_h and $T(3)$, where I_h is the (Euclidean) symmetry group of the icosahedron and $T(3)$ is the group of all translations of the 3-dimensional space.

Keywords and 2020 Mathematics Subject Classification
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1. Introduction

To a classical Greek geometer, a polyhedron was solid. Over the past 200 years, it has become more convenient to think of polyhedra as surfaces. It has been said that the only thing all polyhedra have in common is the name. However, there is some common ground to be found. Their most obvious property is that they are made of polygons. This fundamental property constituted a definition of polyhedron for many centuries. We shall make a distinction the constituent parts of polygons and those of polyhedra. Thereby a polygon has sides and corners, whereas a polyhedron has faces, edges and vertices. Each edge of a polyhedron is formed from the sides of two faces [1]. Examples of polyhedra in architecture, art, ornament, nature and cartography. The ancient pyramids in Egypt which were built four thousand years ago. Much modern abstract sculpture has a polyhedral form. This is as simple as a cube with one corner embedded in the ground. Mostly, platonic solids are used [1].

In three-dimensional space, a Platonic solid is a regular convex polyhedron. It consists of congruent, regular polygonal faces that meet at each vertex. Five solids meet these criteria, and each is named after the number of its faces. Geometers have studied the mathematical beauty and symmetry of the Platonic solids for thousands of years. They are named after the ancient Greek philosopher Plato, who theorised in his dialogue Timaeus that the classical elements are made up of these regular solids. Examples of semi regular polyhedra are Archimedean solids. In geometry, an Archimedean solid is a highly symmetric, semi-regular convex polyhedron consisting of two or more types of regular polygons meeting at identical vertices. They differ from Platonic solids, which consist of only one kind of polygons meeting at identical vertices, and from Johnson solids, whose regular polygon faces do not meet in identical vertices. In mathematics, a Catalan solid or Archimedean dual is a dual polyhedron to an Archimedean solid. Catalan solids are named after the Belgian mathematician Eugène Catalan, who first described them in 1865. Catalan solids are all convex. They are surface-transitive, but not vertex-transitive. The reason for this is that the dual Archimedean solids are vertex-transitive and not face-transitive. Unlike the Platonic solids and the Archimedean solids, the faces of the Catalan solids are not regular polygons. However, the vertex figures of the Catalan solids are regular.

The fundamental problem of geometric investigations for given space S with metric d is to describe the group G of isometries. It is known that for the Euclidean space, $G = E(3)$ is the semi-direct product of its two subgroups $O(3)$ (the orthogonal group) and $T(3)$ (the translation group) consist of all translations of 3-dimensional space. During this work we use the following descriptions, quoted from Martin [2]:

i. A *transformation* is one to one equivalence from the set of points in space onto itself. If $d(X,Y) = d(\alpha(X), \alpha(Y))$ for every point X and Y, then transformation α is named an *isometry.*

ii. For all points X, if $i(X) = X$, then i is called *identity.*

iii. If α fixes which set of points then isometry α is called a symmetry.

iv. For plane Δ, If $\sigma_\Delta(X) = X$ for point X on Δ and if $\sigma_\Delta(X) = Y$ for point X on Δ and Δ is perpendicular bisector of line segment XY, then σ_Δ, which is mapping on the points in $\mathbb{R}^3$, is called reflection.

v. $\sigma_\Delta\sigma_\Gamma$ is defined as *a rotation* about axis *l,* if Γ and Δ are two intersecting planes at line *l.*

vi. $\sigma_\Pi\sigma_\Delta\sigma_\Gamma$ is defined as *a rotary reflection* about the common point to Γ, Δ and Π if Γ, Δ ,which each one perpendicular to Π, are intersecting planes.

vii. If $\sigma_N(X) = Y$ for every X points and N is midpoint of X and Y, then σ_N inversion about N is called *a transformation*. At the same time σ_N is defined *a point reflection.*

Some mathematicians have studied isometry groups of some planes and spaces covering with different metrics [3–19]. In this work, we show that isometry groups of Deltoidal Hexacontahedron, Pentakis Dodecahedron and Triakis Icosahedron spaces are the semi direct product of icosahedral group I_h and translation group $T(3)$.

2. Preliminaries

A deltoidal hexacontahedron (sometimes called a trapezoidal hexacontahedron, strombic hexacontahedron, or tetragonal hexacontahedron) is a Catalan solid that is the dual polyhedron of the rhombicosidodecahedron, an Archimedean solid. The 60 faces are deltoids or kites. A pentakis dodecahedron or kisdodecahedron is a dodecahedron with a pentagonal pyramid covering each face, that is, it is the cloetop of the dodecahedron. The usual Catalan pentakis dodecahedron, a convex hexacontahedron with sixty isosceles triangular faces. It is a Catalan solid, dual to the truncated icosahedron, an Archimedean solid. The Pentakis dodecahedron is also a model for some icosahedron symmetric viruses, such as the adeno-associated virus. These have 60 symmetry-related capsid proteins, which together give the 60 symmetric faces of a Pentakis dodecahedron. The pentakis dodecahedron in a model of Buckminsterfullerene, with each surface segment representing a carbon atom. Similarly, a truncated icosahedron is a model of Buckminsterfullerene where each vertex represents a carbon atom. The Triakis icosahedron (or Kisikosahedron) is an Archimedean dual body or a Catalan body. Its dual is the truncated dodecahedron. It can be considered as an icosahedron with triangular pyramids on each face, that is, it is the cloetop of the icosahedron. This interpretation is expressed in the name Triakis. The 60 faces are isosceles triangular, (see [20–22]).

We asserted the deltoidal hexacontahedron, pentakis dodecahedron and triakis icosahedron metrics in [23,24]. Let $\mathbb{R}^3_{DH}$, $\mathbb{R}^3_{PD}$ and $\mathbb{R}^3_{TI}$ which are called deltoidal dodecahedron, pentakis dodecahedron and triakis icosahedron space point out 3-dimensional analytical space furnishing deltoidal hexacontahedron metric, pentakis dodecahedron metric and triakis icosahedron metric, respectively. $\mathbb{R}^3_{DH}$, $\mathbb{R}^3_{PD}$, $\mathbb{R}^3_{TI}$ are almost the same the Euclidean 3-dimensional space $\mathbb{R}^3$. All of them are Minkowski geometry. The points, lines and planes in Minkowski geometry are the same Euclidean geometry's points, lines and planes, but the distance function is different. The taxicab (Manhattan) and the maximum (Chebyshev) norms are defined as $\|X\|_1 = |x| + |y| + |z|$ and $\|X\|_\infty = \max\{|x|, |y|, |z|\}$, respectively and they are special cases of l_p-norm; $\|X\|_p = (|x|^p + |y|^p + |z|^p)^{1/p}$, where $X = (x,y,z) \in \mathbb{R}^3$. Among l_p-metrics only crystalline metrics, i.e., metrics having polygonal unit balls are l_1- and $l_\infty-$ metrics [25].

The deltoidal hexacontahedron, pentakis dodecahedron and triakis icosahedron metrics and some properties of them are given shortly from [21] and [22]. First, we give some notions that will be used in the descriptions of distance functions we define. For $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$, M denotes $\|P_1 - P_2\|_\infty$ and S denotes $\|P_1 - P_2\|_1$. Moreover, $X - Y - Z - X$ and $Z - Y - X - Z$ orientations are called positive (+) direction and negative (-) direction, respectively. M^+ and M^- expresses the next term in the respective direction according to M. For example, if $M = |x_1 - x_2|$, then $M^+ = |y_1 - y_2|$ and $M^- = |z_1 - z_2|$. The metrics for which the unit spheres are the deltoidal hexacontahedron, the pentakis dodecahedron and the triakis icosahedron are defined as following:

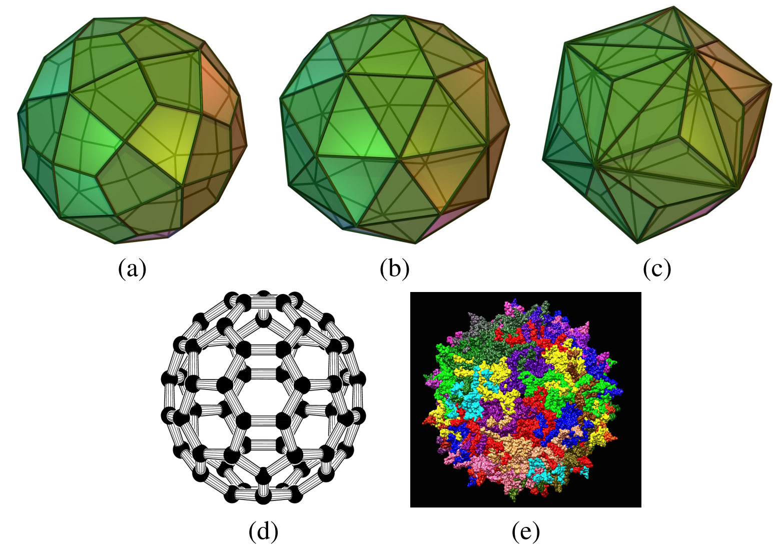


Fig. 1. a) Deltoidal hexecontahedron, b) Pentakis dodecahedron, c) Triakis icosahedron, d) A model of buckminsterfullerene, e) Adeno-associated virüs

Definition 1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. The distance functions for deltoidal hexacontahedron, pentakis dodecahedron and triakis icosahedron distance, respectively between P_1 and P_2 are defined by*

$$\begin{cases} d_{DH}(P_1,P_2) = \max\left\{(3\varphi-4)M+(\varphi-1)M^-, M+(2\varphi-3)\left[M^+ + M^-\right], (4-2\varphi)M+(\varphi-1)M^+ + (2-\varphi)M^-\right\}, \\ d_{PD}(P_1,P_2) = \max\left\{M+\frac{\varphi-1}{3}M^+, \frac{\varphi+1}{3}M+\frac{2(\varphi-1)}{3}M^+ + \frac{1}{3}M^-, \frac{2(\varphi+1)}{3}M+\frac{\varphi-1}{3}M^+ + \frac{2}{3}M^-\right\}, \\ d_{TI}(P_1,P_2) = \max\left\{M+\frac{3\varphi-4}{5}M^-, \frac{2\varphi+4}{5}M+\frac{2\varphi-1}{5}M^+ + \frac{6-2\varphi}{5}M^-, \frac{4\varphi-2}{5}M+\frac{3\varphi-4}{5}M^+ + \frac{2\varphi-1}{5}M^-\right\}, \end{cases}$$

where $\varphi = \frac{\sqrt{5}+1}{2}$ the golden ratio.

Geometrically, there are three possible ways for the shortest paths between the points the P_1 and P_2 in $\mathbb{R}^3_{DH}$ as shown in Figure 2 (a)-(b). These paths are:

i. The DH-path from P_1 and P_2 is union of three line segments, one segment is parallel to one of the coordinate axes and the other two make $\arctan(2)$ radian angle with one of the other coordinate axes.

ii. The DH-path from P_1 and P_2 is union of two line segments, one segment is parallel to one of the coordinate axes and the other makes $\arctan\left(\frac{10-3\sqrt{5}}{10}\right)$ radian angle with one of the other coordinate axes.

iii. The DH-path from P_1 and P_2 is union of three line segments, one segment is parallel to one of the coordinate axes and the other two make $\arctan\left(\frac{3\sqrt{5}-5}{8}\right)$ and $\arctan\left(\frac{3}{4}\right)$ radians angle with one of the other coordinate axes.

Thus deltoidal hexacontahedron distance between P_1 and P_2 is for part i the sum of Euclidean lengths of mentioned three line segments, for part ii $\frac{3\sqrt{5}-5}{2}$ times the sum of Euclidean lengths of mentioned two line segments, for part iii $\left(3-\sqrt{5}\right)$ times the sum of Euclidean lengths of mentioned three line segments.

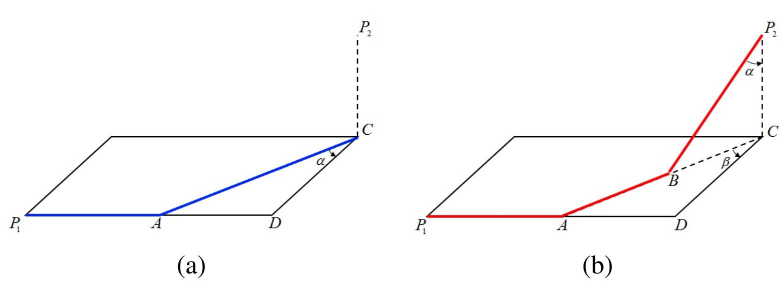


Fig. 2. The shortest paths between the points the P_1 and P_2 in $\mathbb{R}^3_{DH}$

By a similar discussion pentakis dodecahedron and triakis icosahedron ways from P_1 and P_2 are would easily be considered. To the definition of d_{PD}- distance there are three possible ways for the shortest paths between the points the P_1 and P_2 as shown in Figure 2 (a)-(b). These paths are:

i. The PD-path from P_1 and P_2 is union of two line segments, one segment is parallel to one of the coordinate axes and the other makes $\arctan\left(\frac{4\sqrt{5}+5}{6}\right)$ radian angle with one of the other coordinate axes.

ii. The PD-path from P_1 and P_2 is union of two line segments, one segment is parallel to one of the coordinate axes and the other makes $\arctan\left(\frac{538\sqrt{5}-117}{76}\right)$ and $\arctan\left(\frac{3033+299\sqrt{5}}{19}\right)$ radians angle with one of the other coordinate axes.

iii. The PD-path from P_1 and P_2 is union of three line segments, one segment is parallel to one of the coordinate axes and the other two make $\arctan\left(\frac{2730+424\sqrt{5}}{31}\right)$ and $\arctan\left(\frac{4645\sqrt{5}+867}{31}\right)$ radians angle with one of the other coordinate axes.

Thus pentakis dodecahedron distance between P_1 and P_2 is for part i the sum of Euclidean lengths of mentioned two line segments, for part ii $\frac{9-\sqrt{5}}{6}$ times the sum of Euclidean lengths of mentioned three line segments, for part iii $\frac{6-\sqrt{5}}{3}$ times the sum of Euclidean lengths of mentioned three line segments.

To the definition of d_{TI}- distance there are three possible ways for the shortest paths between the points the P_1 and P_2 as shown in Figure 2 (a)-(b). These paths are:

i. The TI-path from P_1 and P_2 is union of two line segments, one segment is parallel to one of the coordinate axes and the other makes $\arctan\left(15-3\sqrt{5}\right)$ radian angle with one of the other coordinate axes.

ii. The TI-path from P_1 and P_2 is union of two line segments, one segment is parallel to one of the coordinate axes and the other makes $\arctan\left(\frac{3}{4}\right)$ and $\arctan\left(2\sqrt{5}-3\right)$ radians angle with one of the other coordinate axes.

iii. The TI-path from P_1 and P_2 is union of three line segments, one segment is parallel to one of the coordinate axes and the other two make $\arctan\left(\frac{91\sqrt{5}-12}{44}\right)$ and $\arctan\left(\frac{1281\sqrt{5}-690}{44}\right)$ radians angle with one of the other coordinate axes.

Thus triakis icosahedron distance between P_1 and P_2 is for part i the sum of Euclidean lengths of mentioned two line segments, for part ii $\frac{2\sqrt{5}}{5}$ times the sum of Euclidean lengths of mentioned three line segments, for part iii $\frac{15+\sqrt{5}}{10}$ times the sum of Euclidean lengths of mentioned three line segments.

Corollary 2. *Let $M_0 = \|X - X_0\|_\infty$ for $X = (x,y,z)$ and $X_0 = (x_0,y_0,z_0)$. The equation of the deltoidal hexacontahedron with center (x_0,y_0,z_0) and radius r,*

$$\max\left\{(3\varphi-4)M_0+(\varphi-1)M_0^-, M_0+(2\varphi-3)\left[M_0^++M_0^-\right], (4-2\varphi)M_0+(\varphi-1)M_0^++(2-\varphi)M_0^-\right\}=r$$

that is a polyhedra which has 60-faces with vertices. The coordinates of the vertices are translations to (x_0,y_0,z_0) such that all possible $+/-$ sign changes of each axis component of $(0,0,r)$, $(r,0,0)$, $(0,r,0)$, $\left(0,\frac{3\sqrt{5}+1}{22}r,\frac{5\sqrt{5}+9}{22}r\right)$, $\left(\frac{5\sqrt{5}+9}{22}r,0,\frac{3\sqrt{5}+1}{22}r\right)$, $\left(\frac{\sqrt{5}+3}{6}r,\frac{\sqrt{5}+1}{6}r,0\right)$, $\left(0,\frac{\sqrt{5}+3}{6}r,\frac{\sqrt{5}+1}{6}r\right)$, $\left(\frac{\sqrt{5}-1}{4}r,\frac{1}{2}r,\frac{\sqrt{5}+1}{4}\right)$, $\left(\frac{\sqrt{5}+1}{4}r,\frac{\sqrt{5}-1}{4}r,\frac{1}{2}r\right)$, $\left(\frac{1}{2}r,\frac{\sqrt{5}+1}{4}r,\frac{\sqrt{5}-1}{4}r\right)$ and $\left(\frac{\sqrt{5}+4}{11}r,\frac{\sqrt{5}+4}{11}r,\frac{\sqrt{5}+4}{11}r\right)$.

The equation of pentakis dodecahedron with center (x_0,y_0,z_0) and radius r,

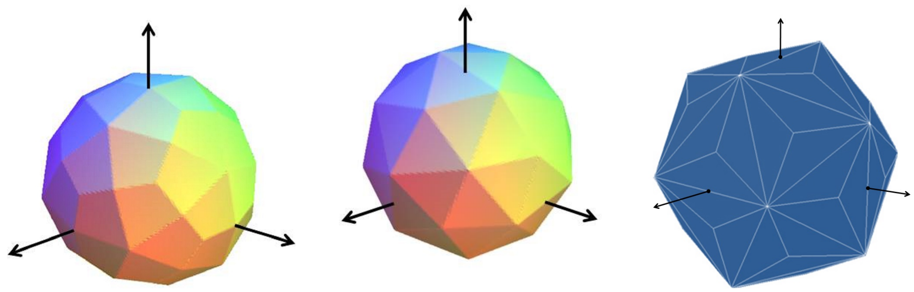
$$\max\left\{M_0+\frac{\varphi-1}{3}M_0^+, \frac{\varphi+1}{3}M_0+\frac{2(\varphi-1)}{3}M_0^++\frac{1}{3}M_0^-, \frac{2(\varphi+1)}{3}M_0+\frac{\varphi-1}{3}M_0^++\frac{2}{3}M_0^-\right\}=r$$

that is a polyhedra which has 60-faces with vertices. The coordinates of the vertices are translations to (x_0,y_0,z_0) such that all possible $+/-$ sign changes of each axis component of $(0,\frac{3-\sqrt{5}}{2}r,r)$, $(r,0,\frac{3-\sqrt{5}}{2}r)$, $(\frac{3-\sqrt{5}}{2}r,r,0)$, $(\frac{6\sqrt{5}-3}{19}r,0,\frac{3\sqrt{5}+27}{38}r)$, $(\frac{3\sqrt{5}+27}{38}r,\frac{6\sqrt{5}-3}{19}r,0)$, $(0,\frac{3\sqrt{5}+27}{38}r,\frac{6\sqrt{5}-3}{19}r)$, $(\frac{\sqrt{5}-1}{2}r,\frac{\sqrt{5}-1}{2}r,\frac{\sqrt{5}-1}{2}r)$ (see Figure 3 (b)).

The equation of the triakis icosahedron with center (x_0,y_0,z_0) and radius r,

$$\max\left\{M_0+\frac{3\varphi-4}{5}M_0^-, \frac{2\varphi+4}{5}M_0+\frac{2\varphi-1}{5}M_0^++\frac{6-2\varphi}{5}M_0^-, \frac{4\varphi-2}{5}M_0+\frac{3\varphi-4}{5}M_0^++\frac{2\varphi-1}{5}M_0^-\right\}=r$$

that is a polyhedra which has 60-faces with vertices. The coordinates of the vertices are translations to (x_0,y_0,z_0) such that all permutations of the three axis components and all possible $+/-$ sign changes of each axis component of $(0,r,\frac{\sqrt{5}-1}{2}r)$, $(0,\frac{4\sqrt{5}-5}{11}r,\frac{7\sqrt{5}+5}{22}r)$, $(\frac{7\sqrt{5}+5}{22}r,0,\frac{4\sqrt{5}-5}{11}r)$, $(\frac{4\sqrt{5}-5}{11}r,\frac{7\sqrt{5}+5}{22}r,0)$, $(\frac{15-\sqrt{5}}{22}r,\frac{15-\sqrt{5}}{22}r,\frac{15-\sqrt{5}}{22}r)$ (see Figure 3 (c)).



(a) (b) (c)

Fig. 3. a) The spheres of spaces of Deltoidal hexecontahedron, b) Pentakis dodecahedron, c) Triakis icosahedron

Lemma 3. *Let $M_d = \|P\|_\infty$ for $P = (p,q,r)$. Let l be the line through the points $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$ in the analytical 3-dimensional space and d_E denotes the Euclidean metric. If l has direction vector (p,q,r), then*

$$\begin{cases} d_{DH}(P_1,P_2) &= \mu_{DH}(P_1P_2)d_E(P_1,P_2), \\ d_{PD}(P_1,P_2) &= \mu_{PD}(P_1P_2)d_{PD}(P_1,P_2), \\ d_{TI}(P_1,P_2) &= \mu_{TI}(P_1P_2)d_{TI}(P_1,P_2), \end{cases}$$

where

$$\mu_{DH}(P_1P_2) = \frac{\max\left\{(3\varphi-4)M_d + (\varphi-1)M_d^-, M_d + (2\varphi-3)\left[M_d^+ + M_d^-\right], (4-2\varphi)M_d + (\varphi-1)M_d^+ + (2-\varphi)M_d^-\right\}}{\sqrt{p^2+q^2+r^2}},$$

$$\mu_{PD}(P_1P_2) = \frac{\max\left\{M_d + \frac{\varphi-1}{3}M_d^+, \frac{\varphi+1}{3}M_d + \frac{2(\varphi-1)}{3}M_d^+ + \frac{1}{3}M_d^-, \frac{2(\varphi+1)}{3}M_d + \frac{\varphi-1}{3}M_d^+ + \frac{2}{3}M_d^-\right\}}{\sqrt{p^2+q^2+r^2}}$$

and

$$\mu_{TI}(P_1P_2) = \frac{\max\left\{M_d + \frac{3\varphi-4}{5}M_d^-, \frac{2\varphi+4}{5}M_d + \frac{2\varphi-1}{5}M_d^+ + \frac{6-2\varphi}{5}M_d^-, \frac{4\varphi-2}{5}M_d + \frac{3\varphi-4}{5}M_d^+ + \frac{2\varphi-1}{5}M_d^-\right\}}{\sqrt{p^2+q^2+r^2}}.$$

The above lemma says that d_{DH}, d_{PD} and d_{TI} distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 4. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then*

$$\begin{cases} d_E(P_1,X) &= d_E(P_2,X) \Leftrightarrow d_{DH}(P_1,X) = d_{DH}(P_2,X), \\ d_E(P_1,X) &= d_E(P_2,X) \Leftrightarrow d_{PD}(P_1,X) = d_{PD}(P_2,X), \\ d_E(P_1,X) &= d_E(P_2,X) \Leftrightarrow d_{TI}(P_1,X) = d_{TI}(P_2,X). \end{cases}$$

Corollary 5. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then*

$$\frac{d_{DH}(P_1,X)}{d_{DH}(P_2,X)} = \frac{d_{PD}(P_1,X)}{d_{PD}(P_2,X)} = \frac{d_{TI}(P_1,X)}{d_{TI}(P_2,X)} = \frac{d_E(P_1,X)}{d_E(P_2,X)}.$$

So the ratios of the d_E, d_{DH}, d_{PD} and d_{TI} distances along a line are the same.

3. Isometries of $\mathbb{R}^3_{DH}$, $\mathbb{R}^3_{PD}$ and $\mathbb{R}^3_{TI}$

The symmetry group of an object is the group of all transformations under which the object is invariant, where composition is the group operation. For a space with a metric, it is a subgroup of the isometry group of the space in question. The icosahedral group known as a regular icosahedron has 60 rotational symmetries (or orientation-preserving symmetries) and a symmetry order of 120 including transformations combining a reflection and a rotation. A regular dodecahedron has the same set of symmetries since it is the dual of the icosahedron. The list of these transformations can be represented as follows;

identity

12 × rotation by 72°, order 5,

12 $\times$ rotation by 144°, order 5,
20 $\times$ rotation by 120°, order 3,
15 $\times$ rotation by 180°, order 2,
inversion
12 $\times$ rotoreflection by 108°, order 10,
12 $\times$ rotoreflection by 36°, order 10,
20 $\times$ rotoreflection by 60°, order 6,
15 $\times$ reflection, order 2.

Deltoidal hexacontahedron, pentakis dodecahedron and triakis icosahedron have I_h icosahedral symmetry. Because of the deltoidal hexacontahedron, pentakis dodecahedron and triakis icosahedron are the study of Euclidean points, lines planes and angles in $\mathbb{R}^3$, an isometry of $\mathbb{R}^3_{DH}$, $\mathbb{R}^3_{PD}$ and $\mathbb{R}^3_{TI}$ are isometry of real with respect to the d_{DH}, d_{PD} and d_{TI} metrics. So, we show that which Euclidean isometries are the isometries for $\mathbb{R}^3_{DH}$, $\mathbb{R}^3_{PD}$ and $\mathbb{R}^3_{TI}$.

There are three main methods of geometric investigations: synthetic, metric and group approach. The group approach takes isometry groups of a geometry and convex sets plays an essential role in specifying the isometry group of geometries. These properties are invariant under the group of motions and geometry studies these properties. There are many studies on isometry groups of a space (See [3–19]).

It was mentioned in the introduction that in Minkowski geometry the linear structure is the same as in Euclidean geometry, but the distance is not the same in all directions. Instead of the usual sphere in Euclidean space, the unit sphere is a certain symmetric closed convex set. In [26] the author gives the following example:

Theorem 6. *If the unit ball C of $(V, \|\|)$ does not intersect a two-plane in an ellipse, then the group $I(3)$of isometries of $(V, \|\|)$ is isomorphic to the semi-direct product of the translation group $T(3)$ of $\mathbb{R}^3$ with a finite subgroup of the group of linear transformations with determinant ± 1.*

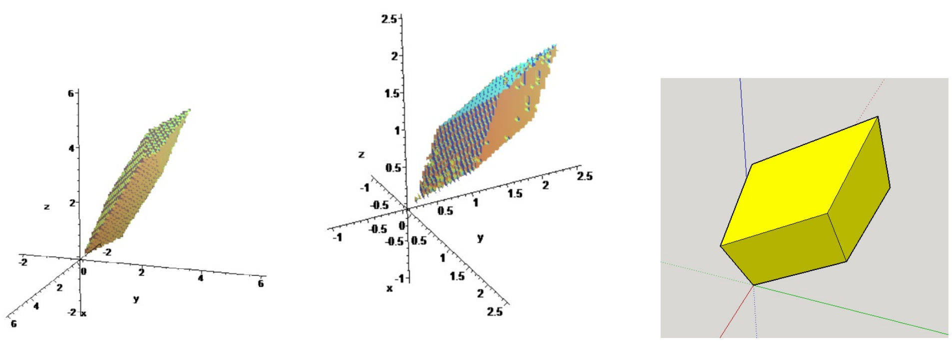
After this theorem, a single question remains. That question is: what is the relevant subgroup? We now show that all isometries of $\mathbb{R}^3_{DH}$ are in $T(3).G(DH)$ and also all isometries $\mathbb{R}^3_{PD}$ are in $T(3).G(PD)$. In the rest of the paper, we take $\Delta = DH$, $\Delta = PD$ or $\Delta = TI$. That is, $\Delta \in \{DH, PD, TI\}$.

Definition 7. *Let A, B be two points in $\mathbb{R}^3_\Delta$. The minimum distance set of A, B*

$$\{X \mid d_\Delta(A,X) + d_\Delta(B,X) = d_\Delta(A,B)\}$$

and denoted by $[AB]$ (see Figure 4).

In general, $[AB]$ stand for a parallelepiped with diagonal AB as in Figure 4 (a)-(b)-(c).



(a) (b) (c)

Fig. 4. a) The minimum distance sets of spaces of Deltoidal hexecontahedron, b) Pentakis dodecahedron, c) Triakis icosahedron

Proposition 8. *Let $\phi : \mathbb{R}^3_\Delta \to \mathbb{R}^3_\Delta$ be an isometry and $[AB]$ be the parallelepiped. Then*

$$\phi([AB]) = [\phi(A)\phi(B)].$$

Proof. Let $Y \in \phi([AB])$ Then

$$\begin{aligned} Y \in \phi([AB]) &\Leftrightarrow \exists X \in [AB] \ni Y = \phi(X) \\ &\Leftrightarrow d_\Delta(A,X) + d_\Delta(B,X) = d_\Delta(A,B) \\ &\Leftrightarrow d_\Delta(\phi(A),\phi(X)) + d_\Delta(\phi(B),\phi(X)) = d_\Delta(\phi(A),\phi(B)) \\ &\Leftrightarrow Y = \phi(X) \in [\phi(A)\phi(B)]. \end{aligned}$$

$\blacksquare$

Corollary 9. *Let $\phi : \mathbb{R}^3_\Delta \to \mathbb{R}^3_\Delta$ be an isometry and $[AB]$ be the parallelepiped. Then ϕ maps vertices to vertices and preserves the lengths of the edges of $[AB]$.*

Proposition 10. *Let $\phi : \mathbb{R}^3_\Delta \to \mathbb{R}^3_\Delta$ be an isometry such that $\phi(O) = O$. Then $\phi \in G(\Delta)$.*

Proof. Because of $\Delta \in \{DH, PD, TI\}$, there are three possibility for Δ. Let $\Delta = DH$ and let $A_6 = (0, \frac{3\sqrt{5}+1}{22}, \frac{5\sqrt{5}+9}{22})$, $A_{26} = (0, \frac{\sqrt{5}+3}{6}, \frac{\sqrt{5}+1}{6})$, $A_{30} = (\frac{\sqrt{5}-1}{4}, \frac{1}{2}, \frac{\sqrt{5}+1}{4})$ and $D = (\frac{13\sqrt{5}-25}{44}, \frac{\sqrt{5}+6}{6}, \frac{73\sqrt{5}+61}{132})$ be four points in $\mathbb{R}^3_{DH}$. Consider $[OD]$ that is the parallelepiped with diagonal $[OD]$ (see Figure 5 (a)). The points A_6, A_{26}, A_{30} are on the minimum distance set $[OD]$ and unit sphere with center at origin. However three points are the corner points of deltoidal hexacontahedron by Corollary 9. Because of ϕ preserves the lengths of the edges, $\phi(A_6) = A_i$, $\phi(A_{26}) = A_j$, $\phi(A_{30}) = A_k$ such that $i \in \{6,7,8,9,10,11,12,13,14,15,16,17,54,55,56,57,58,59,60,61\}$, $j \in \{18,19,20,21,22,23,24,25,26,27,28,29\}$, $k \in \{0,1,2,3,4,5,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53\}$. Because of deltoidal hexacontahedron have 60 kites, there are 60 possibility to points which they can map and also there are two possibility to points which they can map on the face of deltoidal hexacontahedron. Thus the possibility number are 120. Some of these cases can be seen as below:

i. If $\phi(A_6) = A_6$, $\phi(A_{26}) = A_{26}$ and $\phi(A_{30}) = A_{34}$ then $\phi = \sigma_\Delta$ is the reflection about plane $\Delta : x = 0$

ii. If $\phi(A_6) = A_8$, $\phi(A_{26}) = A_{28}$ and $\phi(A_{30}) = A_{32}$ then $\phi = \sigma_\Delta$ is the reflection about plane $\Delta : y = 0$.

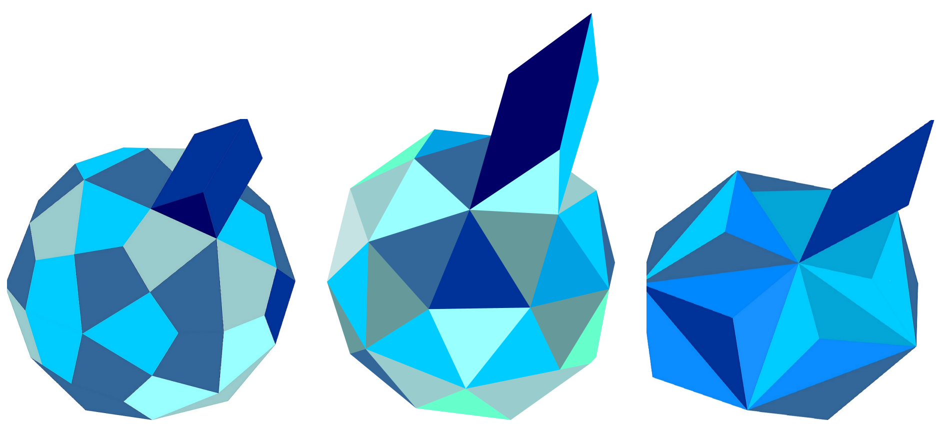
iii. If $\phi(A_6) = A_8$, $\phi(A_{26}) = A_{28}$ and $\phi(A_{30}) = A_{32}$ then $\phi = \sigma_\Delta$ is the reflection about plane $\Delta : y = 0$.

iv. If $\phi(A_6) = A_9$, $\phi A_{26}) = A_{29}$ and $\phi(A_{30}) = A_{33}$ then $\phi = r_\pi$ is the rotation with rotation axis $\parallel (1,0,0)$.

v. If $\phi(A_6) = A_6$, $\phi(A_{26}) = A_{26}$ and $\phi(A_{30}) = A_{30}$ then ϕ is the identity.

vi. If $\phi(A_6) = A_9$, $\phi(A_{26}) = A_{29}$ and $\phi(A_{30}) = A_{37}$ then f is the inversion.

The other cases can be similarly.



(a) (b) (c)

Fig. 5. a) The intersection of the sphere and minimum distance set of spaces of Deltoidal hexecontahedron, b) Pentakis dodecahedron, c) Triakis icosahedron

Let $\Delta = PD$ and $A_0 = (0, \frac{3-\sqrt{5}}{2}, 1)$, $A_{20} = (0, \frac{3\sqrt{5}+27}{38}, \frac{6\sqrt{5}-3}{19})$, $A_{24} = (\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2})$ and $D = (\frac{\sqrt{5}-1}{2}, \frac{3\sqrt{5}+65}{38}, \frac{31\sqrt{5}+13}{38})$ be four points in $\mathbb{R}^3_{PD}$. Consider $[OD]$ that is the parallelepiped with diagonal OD (see Figure 5 (b)).

The points A_0, A_{20}, A_{24} are on the minimum distance set $[OD]$ and unit sphere with center at origin. However three points are the corner points of pentakis dodecahedron by Corollary 9. Because of ϕ preserves the lengths of the edges, $\phi(A_0) = A_i$, $\phi(A_{20}) = A_j$, $\phi(A_{24}) = A_k$ such that $i,k \in \{0,1,2,3,4,5,6,7,8,9,10,11,24,25,26,27,28,29,30,31\}$, $j \in \{12\ ,13,14,14,15,16,7,18,19,20,21,22,23\}$. Because of pentakis dodecahedron 60 isosceles faces, there are 60 possibility to points which they can map and also there are two possibility to points which they can map on the face of pentakis dodecahedron. Thus, the possibility number is 120. Some of these cases can be seen as below:

i. If $\phi(A_0) = A_0$ $\phi(A_{20}) = A_{20}$ $\phi(A_{24}) = A_{24}$, then ϕ is the identity.

ii. If $\phi(A_0) = A_3$ $\phi(A_{20}) = A_{23}$ $\phi(A_{24}) = A_{31}$, then ϕ is the inversion.

iii. If $\phi(A_0) = A_{31}$ $\phi(A_{20}) = A_{15}$ $\phi(A_{24}) = A_3$, then $\phi = \sigma_\Delta$ is the reflection about the plane $\Delta : x + \varphi y + (1-\varphi)z = 0$.

iv. If $\phi(A_0) = A_8$ $\phi(A_{20}) = A_{20}$ $\phi(A_{24}) = A_{24}$, then $\phi = \sigma_\Delta$ is the reflection about the plane $\Delta : -x + \varphi y + (1-\varphi)z = 0$.

v. If $\phi(A_0) = A_6$ $\phi(A_{20}) = A_{18}$ $\phi(A_{24}) = A_{28}$, then $\phi = r_{\frac{8\pi}{5}}$ is the rotation axis $\parallel (0, \frac{\sqrt{50+10\sqrt{5}}}{10}, -\frac{\sqrt{50-10\sqrt{5}}}{10})$.

vi. If $\phi(A_0) = A_2 \phi(A_{20}) = A_{14}$ $\phi(A_{24}) = A_0$, then $\phi = r_{\frac{2\pi}{5}}$ is the rotation axis $\parallel (\frac{\sqrt{50-10\sqrt{5}}}{10}, 0, \frac{\sqrt{50+10\sqrt{5}}}{10})$.

The other cases can be similarly.

Let $\Delta = TI$ and $A_0 = (\frac{\sqrt{5}-1}{2}, 0, 1)$, $A_8 = (0, \frac{4\sqrt{5}-5}{11}, \frac{7\sqrt{5}+5}{22})$, $A_{12} = (0, 1, \frac{\sqrt{5}-1}{2})$ and $D = (\frac{\sqrt{5}-1}{2}, \frac{4\sqrt{5}+6}{11}, \frac{16+18\sqrt{5}}{22})$ be four points in $\mathbb{R}^3_{TI}$. Consider $[OD]$ that is the parallelepiped with diagonal OD (see Figure 5 (c)).

The points A_0, A_8, A_{12} are on the minimum distance set $[OD]$ and unit sphere with center at origin. However three points are the corner points of triakis icosahedron by Corollary 9. Because of ϕ preserves the lengths of the edges, $\phi(A_0) = A_i$, $\phi(A_8) = A_j$, $\phi(A_{12}) = A_k$ such that $j, k \in \{0,1,2,3,4,5,...,26,27,28,29,30,31\}$, $i \in \{0,1,2,3,4,5,6,7,8,9,10,11\}$. Because of triakis icosahedron 60 isosceles faces, there are 60 possibility to points which they can map and also there are two possibility to points which they can map on the face of triakis icosahedron. Thus, the possibility number is 120. Some of these cases can be seen as below:

i. If $\phi(A_0) = A_0$, $\phi(A_8) = A_8$, $\phi(A_{12}) = A_{12}$ then ϕ is the identity.

ii. If $\phi(A_0) = A_1$, $\phi(A_8) = A_{11}$, $\phi(A_{12}) = A_{15}$ then ϕ is the inversion.

iii. If $\phi(A_0) = A_2$ $\phi(A_8) = A_8$ $\phi(A_{12}) = A_{12}$ then $\phi = \sigma_\Delta$ is the reflection about the plane $\Delta : x = 0$.

iv. If $\phi(A_0) = A_0$, $\phi(A_8) = A_{10}$ $\phi(A_{12}) = A_{14}$ then $\phi = \sigma_\Delta$ is the reflection about the plane $\Delta : y = 0$.

v. If $\phi(A_0) = A_{4,}$ $\phi(A_8) = A_0$ $\phi(A_{12}) = A_{16}$ then $\phi = r_{\frac{2\Pi}{3}}$ is the rotation axis $\parallel (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

vi. If $\phi(A_0) = A_{9,}$ $\phi(A_8) = A_6$ $\phi(A_{12}) = A_{22}$ then $\phi = r_{\frac{2\pi}{5}}$ is the rotation axis $\parallel (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

The other cases can be similarly. ■

Theorem 11. *Let $\phi : \mathbb{R}^3_\Delta \to \mathbb{R}^3_\Delta$ be an isometry. Then there exists a unique $T_A \in T(3)$ and $g \in G(\Delta)$ where $\phi = T_A \circ g$.*

Proof. Let $\phi(O) = A$ such that $A = (a_1, a_2, a_3)$. ϕ is definition $\phi = T_{-A} \circ \phi$. We know $\phi(O) = O$ and ϕ is an isometry and by Proposition 10, $\phi \in G(\Delta)$. The proof of uniqueness. ■

4. Conclusions

In this paper, the spaces that their sphere are Deltoidal Hexacontahedron, Pentakis Dodecahedron and Triakis icosahedron are introduced and some properties of metrics which are used setting up these space are given. Also, the isometry groups of these spaces are given. So each of the groups of isometries of the spaces covering with deltoidal hexacontahedron, pentakis dodecahedron, and triakis icosahedron metrics is the semi-direct product of the icosahedral group I_h and $T(3)$, where I_h is the (Euclidean) symmetry group of the icosahedron and $T(3)$ is the group of all translations of the 3-dimensional space. In the future works, handled solids in this paper are Catalan solids, the new metric space by considering different solids from these solids can be constructed and investigate their some properties which are related to metrics.

References

[1] Cromwell, P. R. (1997). Polyhedra, Cambridge University Press.

[2] Martin, G. E. (1997). Transformation geometry. Springer-Verlag New York Inc.

[3] Kaya, R., Gelişgen, Ö., Ekmekçi, S., & Bayar, A. (2006). *Group of isometries of CC-plane*. Missouri Journal of Mathematical Sciences, 18(3), 221–233.

[4] Gelişgen, Ö., & Kaya, R. (2009). *The Taxicab space group*, Acta Mathematica Hungarica, 122(1–2), 187–200.

[5] Kaya, R., Gelisgen, Ö., Ekmekçi, S., & Bayar, A. (2009). *On The Group of isometries of the plane with generalized absolute value metric*. Rocky Mountain Journal of Mathematics, 39(2), 591–603.

[6] Çolak, Z., & Gelişgen, Ö. (2015). *New metrics for deltoidal hexacontahedron and pentakis dodecahedron*, Sakarya University Journal of Science, 19(3), 353–360.

[7] Ermiş, T., & Kaya, R. (2015). *On the isometries the of 3- dimensional maximum space*, Konuralp Journal of Mathematics, 3(1), 103–114.

[8] Gelişgen, Ö., & Kaya, R. (2015). *The isometry group of Chinese Checker space*. International Electronic Journal Geometry, 8(2), 82–96.

[9] Gelişgen Ö., & Can, Z. (2016). *On the family of metrics for some platonic and Archimedean polyhedra*. Konuralp Journal of Mathematics, 4(2), 25–33.

[10] Gelişgen Ö., & Çolak, Z. (2016). *A family of metrics for some polyhedra*. Automation Computers Applied Mathematics Scientific Journal, 25 (1), 35–48.

[11] Ekmekçi, S., Akça, Z., Altıntaş, A., & Bayar, A. (2016). *Some coplanar points in tetrahedron*. Journal of Pure and Applied Mathematics: Advances and Applications, 16(2), 109–114.

[12] Gelişgen, Ö., Ermiş, T., & Gunaltılı, I. (2017). *A note about the metrics induced by truncated dodecahedron and truncated icosahedron*. International Journal of Geometry, 6(2), 5–16.

[13] Ermiş, T., Savcı, Ü. Z., & Gelişgen, Ö. (2019). *A note about truncated rhombicuboctahedron and truncated rhombicicosidodecahedron space*, Scientific Studies and Research Series Mathematics and Informatics, 29(1), 73–88.

[14] Gelisgen Ö., & Yavuz, S. (2019). *A note about isometry groups of chamfered dodecahedron and chamfered icosahedron spaces*. International Journal of Geometry, 8(2), 33–45.

[15] Gelisgen Ö., & Yavuz, S. (2019). *Isometry groups of chamfered cube and chamfered octahedron spaces*. Mathematical Sciences and Applications e-Notes, 7(2), 174–182.

[16] Savcı, Ü. Z. (2019). *Truncated truncated dodecahedron and truncated truncated icosahedron spaces*. Cumhuriyet Science Journal (CSJ), 40(2), 457–470.

[17] Ermiş, T. (2020). *Geometric analysis of the some rectified Archimedean solids spaces via their isometry groups*. Mathematical Sciences and Applications E-Notes, 8(2), 96–109.

[18] Gelisgen Ö., & Ermiş, T. (2020). *The metrics for rhombicuboctahedron and rhombicosidodecahedron*. Palestine Journal of Mathematics, 9(1), 15–25.

[19] Akça, Z., & Nazlı, S. (2022). *On the shortest distance in the plane* $R^2_{\pi/3}$. New Trends in Mathematical Sciences, 10(4), 128–132.

[20] Wikipedia, https://en.wikipedia.org/wiki/Deltoidal_hexecontahedron. [Accessed 28 July 2023].

[21] Wikipedia, https://en.wikipedia.org/wiki/Pentakis_dodecahedron. [Accessed 28 July 2023].

[22] Wikipedia, https://en.wikipedia.org/wiki/Triakis_icosahedron. [Accessed 28 July 2023].

[23] Can, Z., Gelişgen, Ö., & Kaya, R. (2015). *On the metrics induced by icosidodecahedron and rhombic triacontahedron*. Scientific and Professional Journal of the Croatian Society for Geometry and Graphics (KoG), 19, 17–23.

[24] Can, Z., Çolak, Z., & Gelişgen, Ö. (2015). *A note On the metrics induced By triakis icosahedron and disdyakis triacontahedron*. Eurasian Academy of Sciences Eurasian Life Sciences Journal / Avrasya Fen Bilimleri Dergisi, 1, 1–11.

[25] Deza, M. M., & Deza, E. (2009). Encyclopedia of distances. German: Springer, Berlin.

[26] Horvath, A. G. (2017). *Isometries of Minkowski geometries*. Linear Algebra and its Applications, 512, 172–190.