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Hybrid Numbers with Fibonacci and Lucas Hybrid Number Coefficients

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Abstract

In this paper, we introduce hybrid numbers with Fibonacci and Lucas hybrid number coefficients. We give the Binet formulas, generating functions, exponential generating functions for these numbers. Then we define an associate matrix for these numbers. In addition, using this matrix, we present two different versions of Cassini identity of these numbers.

1. Introduction

Recently, in [1], Özdemir defined the set of hybrid numbers which contains complex, dual and hyperbolic numbers as

$$\mathbb{K} = \left\{ a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a,b,c,d \in \mathbb{R}, \mathbf{i}^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i} \right\}.$$

This number system is a generalization of complex $\left(\mathbf{i}^2 = -1\right)$, hyperbolic $\left(\mathbf{h}^2 = 1\right)$ and dual number $\left(\varepsilon^2 = 0\right)$ systems. Here, $\mathbf{i}$ is complex unit, ε is dual unit and $\mathbf{h}$ is hyperbolic unit. We call these units as hybrid units. In the last few years, researchers from many different fields have taken this number system and used it in various fields of applied sciences. For some applications of hybrid numbers we refer the reader to [2, 3]. There is no doubt that this number system will be studied by other applied science researchers in the near future.
The conjugate of a hybrid number $K = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is defined by

$$\overline{K} = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}.$$

From the definition of hybrid numbers, the multiplication table of the hybrid units is given by the following table:

$\bullet$	1	$\mathbf{i}$	ε	$\mathbf{h}$
1	1	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	$\mathbf{h}+\mathbf{1}$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

Table 1: The Multiplication Table for Hybrid Units

This table shows that the multiplication of hybrid numbers is not commutative. Using the above datas, Özdemir [1] investigated various algebraic and geometric properties of hybrid numbers. For instance, he defined a ring isomorphism between the hybrid number ring $\mathbb{K}$ and the ring of real 2×2 matrices $\mathbb{M}_{2\times 2}$. This map is $\varphi : \mathbb{K} \to \mathbb{M}_{2\times 2}$ where

$$\varphi\left(a + b\mathbf{i} + c\varepsilon + d\mathbf{h}\right) = \begin{bmatrix} a+c & b-c+d \\ c-b+d & a-c \end{bmatrix}. \tag{1.1}$$

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We refer the reader to [1] for more details and properties about hybrid numbers.
The well-known Fibonacci and Lucas sequences are defined as (for $n \geq 0$)

$$F_{n+2} = F_{n+1} + F_n$$

and

$$L_{n+2} = L_{n+1} + L_n$$

where $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$. Note that for $n \geq 1$, $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ and $L_{n-1}L_{n+1} - L_n^2 = 5(-1)^{n+1}$.
In [4], the authors introduced the Fibonacci hybrid numbers and derived some combinatorial properties of these numbers. For $n \geq 0$, they defined the nth Fibonacci hybrid and nth Lucas hybrid numbers as

$$FH_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\varepsilon + F_{n+3}\mathbf{h}$$

and

$$LH_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\varepsilon + L_{n+3}\mathbf{h}$$

where $FH_0 = \mathbf{i} + \varepsilon + 2\mathbf{h}$, $FH_1 = 1 + \mathbf{i} + 2\varepsilon + 3\mathbf{h}$, $LH_0 = 2 + \mathbf{i} + 3\varepsilon + 4\mathbf{h}$ and $LH_1 = 1 + 3\mathbf{i} + 4\varepsilon + 7\mathbf{h}$. They also gave the Binet formulas of these hybrid numbers as

$$FH_n = \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta}$$

and

$$LH_n = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n,$$

respectively, where $\underline{\alpha} = 1 + \alpha\mathbf{i} + \alpha^2\varepsilon + \alpha^3\mathbf{h}$, $\underline{\beta} = 1 + \beta\mathbf{i} + \beta^2\varepsilon + \beta^3\mathbf{h}$, $\alpha = \left(1+\sqrt{5}\right)/2$ and $\beta = \left(1-\sqrt{5}\right)/2$.
Hybrid number sequences have been studied by many researchers. For instance, in [5], Cerda-Morales studied generalized hybrid Fibonacci numbers and their properties. In [6], using an associate matrix, Irmak gave various identities about Fibonacci and Lucas quaternions by matrix methods. In [7], Kızılateş investigated the q-Fibonacci and the q-Lucas hybrid numbers and gave some algebraic properties of these numbers. In [8], the same author introduced the Horadam hybrid polynomials called Horadam hybrinomials. Liana et al. studied Pell hybrinomials in [9]. In [10–15], Liana and Wloch introduced several hybrid number sequences and polynomials and gave various properties of them. In [16], Şentürk et al. studied Horadam hybrid numbers and obtained various properties. In [17], the author examined the ratios of Fibonacci hybrid and Lucas hybrid numbers. Karaca and Yılmaz [18] gave some fundamental definitions and theorems about harmonic complex numbers and harmonic hybrid Fibonacci numbers in detail. Moreover, they examined some algebraic properties such as Binet-like-formula, partial sums related to these sequences.
In this paper, motivated by the above papers, we introduce hybrid numbers with Fibonacci and Lucas hybrid number coefficients. We give the Binet formulas, generating functions, exponential generating functions for these numbers. Then we define an associate matrix for these numbers. Finally, using this matrix, we present two different versions of Cassini identity of these numbers.

2. Main Results

In this section, we define hybrid numbers with Fibonacci and Lucas hybrid number coefficients. Then we give Binet formulas, generating functions, exponential generating functions, and some summation formulas for these numbers.

Definition 2.1. *For $n \geq 0$, the nth term of hybrid number with Fibonacci hybrid number coefficients is given by*

$$\mathbb{F}_n = FH_n + FH_{n+1}\mathbf{i} + FH_{n+2}\varepsilon + FH_{n+3}\mathbf{h}. \tag{2.1}$$

Definition 2.2. *For $n \geq 0$, the nth term of hybrid numbers with Lucas hybrid number coefficients is given by*

$$\mathbb{L}_n = LH_n + LH_{n+1}\mathbf{i} + LH_{n+2}\varepsilon + LH_{n+3}\mathbf{h}. \tag{2.2}$$

Remark 2.3. *If we expand the definitions of $\mathbb{F}_n$ and $\mathbb{L}_n$, we get*

$$\mathbb{F}_n = F_n - F_{n+2} + 2F_{n+3} + F_{n+6} + 2F_{n+1}\mathbf{i} + 2F_{n+2}\varepsilon + 2F_{n+3}\mathbf{h}$$

and

$$\mathbb{L}_n = L_n - L_{n+2} + 2L_{n+3} + L_{n+6} + 2L_{n+1}\mathbf{i} + 2L_{n+2}\varepsilon + 2L_{n+3}\mathbf{h},$$

respectively.

For $n \geq 0$, it is clear that

$$\mathbb{F}_{n+2} = \mathbb{F}_{n+1} + \mathbb{F}_n \tag{2.3}$$

and

$$\mathbb{L}_{n+2} = \mathbb{L}_{n+1} + \mathbb{L}_n. \tag{2.4}$$

Theorem 2.4. *For $n \geq 0$, Binet formulas of hybrid numbers with Fibonacci and Lucas hybrid number coefficients are given by*

$$\mathbb{F}_n = \frac{(\underline{\alpha})^2 \alpha^n - \left(\underline{\beta}\right)^2 \beta^n}{\alpha - \beta} \tag{2.5}$$

and

$$\mathbb{L}_n = (\underline{\alpha})^2 \alpha^n + \left(\underline{\beta}\right)^2 \beta^n, \tag{2.6}$$

respectively, where $\underline{\alpha} = 1 + \alpha \mathbf{i} + \alpha^2 \varepsilon + \alpha^3 \mathbf{h}$, $\underline{\beta} = 1 + \beta \mathbf{i} + \beta^2 \varepsilon + \beta^3 \mathbf{h}$, $\alpha = \left(1 + \sqrt{5}\right)/2$ *and* $\beta = \left(1 - \sqrt{5}\right)/2$.

Proof. Using the Binet formula of hybrid Fibonacci numbers, we have

$$\begin{aligned}
\mathbb{F}_n &= \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta} + \frac{\underline{\alpha}\alpha^{n+1} - \underline{\beta}\beta^{n+1}}{\alpha - \beta}\mathbf{i} + \frac{\underline{\alpha}\alpha^{n+2} - \underline{\beta}\beta^{n+2}}{\alpha - \beta}\varepsilon + \frac{\underline{\alpha}\alpha^{n+3} - \underline{\beta}\beta^{n+3}}{\alpha - \beta}\mathbf{h} \\
&= \frac{\underline{\alpha}\alpha^n \left(1 + \alpha \mathbf{i} + \alpha^2 \varepsilon + \alpha^3 \mathbf{h}\right) - \underline{\beta}\beta^n \left(1 + \beta \mathbf{i} + \beta^2 \varepsilon + \beta^3 \mathbf{h}\right)}{\alpha - \beta} \\
&= \frac{(\underline{\alpha})^2 \alpha^n - \left(\underline{\beta}\right)^2 \beta^n}{\alpha - \beta}.
\end{aligned}$$

By a similar calculation, we obtain

$$\mathbb{L}_n = (\underline{\alpha})^2 \alpha^n + \left(\underline{\beta}\right)^2 \beta^n$$

□

Theorem 2.5. *The generating functions of hybrid numbers with Fibonacci and Lucas hybrid number coefficients are*

$$F(x) = \sum_{n \geq 0} \mathbb{F}_n x^n = \frac{11 + 7x + 2\mathbf{i} + 2(1 + x)\varepsilon + (4 + 2x)\mathbf{h}}{1 - x - x^2}$$

and

$$L(x) = \sum_{n \geq 0} \mathbb{L}_n x^n = \frac{25 + 15x + 2(1 + 2x)\mathbf{i} + 2(3 + x)\varepsilon + 2(4 + 3x)\mathbf{h}}{1 - x - x^2},$$

recpectively.

Proof. By taking the generating function of both sides of equation (2.1), we directly have

$$\begin{aligned}
\sum_{n \geq 0} \mathbb{F}_n x^n &= \sum_{n \geq 0} FH_n x^n + \left(\sum_{n \geq 0} FH_{n+1} x^n\right)\mathbf{i} + \left(\sum_{n \geq 0} FH_{n+2} x^n\right)\varepsilon + \left(\sum_{n \geq 0} FH_{n+3} x^n\right)\mathbf{h} \\
&= \frac{11 + 7x + 2\mathbf{i} + 2(1 + x)\varepsilon + (4 + 2x)\mathbf{h}}{1 - x - x^2}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\sum_{n \geq 0} \mathbb{L}_n x^n &= \sum_{n \geq 0} LH_n x^n + \left(\sum_{n \geq 0} LH_{n+1} x^n\right)\mathbf{i} + \left(\sum_{n \geq 0} LH_{n+2} x^n\right)\varepsilon + \left(\sum_{n \geq 0} LH_{n+3} x^n\right)\mathbf{h} \\
&= \frac{25 + 15x + 2(1 + 2x)\mathbf{i} + 2(3 + x)\varepsilon + 2(4 + 3x)\mathbf{h}}{1 - x - x^2}.
\end{aligned}$$

□

Theorem 2.6. *For $m, n \in \mathbb{Z}$, generating functions of $\mathbb{F}_{n+m}$ and $\mathbb{L}_{n+m}$ are*

$$\sum_{n \geq 0} \mathbb{F}_{n+m} x^n = \frac{FH_m + FH_{m-1}x}{1 - x - x^2} + \left(\frac{FH_{m+1} + FH_m x}{1 - x - x^2}\right)\mathbf{i} + \left(\frac{FH_{m+2} + FH_{m+1}x}{1 - x - x^2}\right)\varepsilon + \left(\frac{FH_{m+3} + FH_{m+2}x}{1 - x - x^2}\right)\mathbf{h}$$

and

$$\sum_{n \geq 0} \mathbb{L}_{n+m} x^n = \frac{LH_m + LH_{m-1}x}{1 - x - x^2} + \left(\frac{LH_{m+1} + LH_m x}{1 - x - x^2}\right)\mathbf{i} + \left(\frac{LH_{m+2} + LH_{m+1}x}{1 - x - x^2}\right)\varepsilon + \left(\frac{LH_{m+3} + LH_{m+2}x}{1 - x - x^2}\right)\mathbf{h},$$

respectively.

Proof. By the virtue of generating function of Fibonacci hybrid sequence given in [4], we have

$$\begin{aligned}\sum_{n\geq 0}\mathbb{F}_{n+m}x^n &= \sum_{n\geq 0}FH_{n+m}x^n+\left(\sum_{n\geq 0}FH_{n+m+1}x^n\right)\mathbf{i}+\left(\sum_{n\geq 0}FH_{n+m+2}x^n\right)\varepsilon+\left(\sum_{n\geq 0}FH_{n+m+3}x^n\right)\mathbf{h}\\ &= \frac{FH_m+FH_{m-1}x}{1-x-x^2}+\left(\frac{FH_{m+1}+FH_m x}{1-x-x^2}\right)\mathbf{i}+\left(\frac{FH_{m+2}+FH_{m+1}x}{1-x-x^2}\right)\varepsilon+\left(\frac{FH_{m+3}+FH_{m+2}x}{1-x-x^2}\right)\mathbf{h}.\end{aligned}$$

□

Theorem 2.7. *Exponential generating functions of $\mathbb{F}_n$ and $\mathbb{L}_n$ are given by*

$$\sum_{n\geq 0}\mathbb{F}_n\frac{x^n}{n!}=\frac{(\underline{\alpha})^2e^{\alpha x}-\left(\underline{\beta}\right)^2e^{\beta x}}{\alpha-\beta}$$

and

$$\sum_{n\geq 0}\mathbb{L}_n\frac{x^n}{n!}=(\underline{\alpha})^2e^{\alpha x}+\left(\underline{\beta}\right)^2e^{\beta x},$$

respectively.

Proof. Using equation (2.5) and equation (2.6), we get

$$\begin{aligned}\sum_{n\geq 0}\mathbb{F}_n\frac{x^n}{n!} &= \sum_{n\geq 0}\left(\frac{(\underline{\alpha})^2\alpha^n-\left(\underline{\beta}\right)^2\beta^n}{\alpha-\beta}\right)\frac{x^n}{n!}\\ &= \frac{(\underline{\alpha})^2}{\alpha-\beta}\sum_{n\geq 0}\frac{(\alpha x)^n}{n!}-\frac{\left(\underline{\beta}\right)^2}{\alpha-\beta}\sum_{n\geq 0}\frac{(\beta x)^n}{n!}\\ &= \frac{(\underline{\alpha})^2}{\alpha-\beta}e^{\alpha x}-\frac{\left(\underline{\beta}\right)^2}{\alpha-\beta}e^{\beta x}\\ &= \frac{(\underline{\alpha})^2e^{\alpha x}-\left(\underline{\beta}\right)^2e^{\beta x}}{\alpha-\beta}\end{aligned}$$

and

$$\begin{aligned}\sum_{n\geq 0}\mathbb{L}_n\frac{x^n}{n!} &= \sum_{n\geq 0}\left((\underline{\alpha})^2\alpha^n+\left(\underline{\beta}\right)^2\beta^n\right)\frac{x^n}{n!}\\ &= (\underline{\alpha})^2\sum_{n\geq 0}\frac{(\alpha x)^n}{n!}+\left(\underline{\beta}\right)^2\sum_{n\geq 0}\frac{(\beta x)^n}{n!}\\ &= (\underline{\alpha})^2e^{\alpha x}+\left(\underline{\beta}\right)^2e^{\beta x}\end{aligned}$$

as desired. □

Now we give some summation formulas containing $\mathbb{F}_n$ and $\mathbb{L}_n$.

Proposition 2.8. *The following formulas containing $\mathbb{F}_n$ and $\mathbb{L}_n$ are hold:*

(i) $\sum_{k=0}^{n}\mathbb{F}_k=\mathbb{F}_{n+2}-(18+2\mathbf{i}+4\,\varepsilon+6\mathbf{h})$,
(ii) $\sum_{k=0}^{n}\mathbb{L}_k=\mathbb{L}_{n+2}-(40+6\mathbf{i}+8\,\varepsilon+14\mathbf{h})$,
(iii) $\sum_{k=0}^{n}\binom{n}{k}\mathbb{F}_k=\mathbb{F}_{2n}$,
(iv) $\sum_{k=0}^{n}\binom{n}{k}\mathbb{L}_k=\mathbb{L}_{2n}$.

Proof. We give only the proofs of (i) and (iii). The others can be done in a similar way.

(i) From the equation (2.3), we can write the following equations:

$$\begin{aligned}\mathbb{F}_0 &= \mathbb{F}_2-\mathbb{F}_1,\\ \mathbb{F}_1 &= \mathbb{F}_3-\mathbb{F}_2,\\ \mathbb{F}_2 &= \mathbb{F}_4-\mathbb{F}_3,\\ &\vdots\\ \mathbb{F}_{n-1} &= \mathbb{F}_{n+1}-\mathbb{F}_n,\\ \mathbb{F}_n &= \mathbb{F}_{n+2}-\mathbb{F}_{n+1}.\end{aligned}$$

If we add the above equations side by side, then we obtain

$$\begin{aligned}\sum_{k=0}^{n}\mathbb{F}_k &= \mathbb{F}_{n+2}-\mathbb{F}_1\\ &= \mathbb{F}_{n+2}-(18+2\mathbf{i}+4\,\varepsilon+6\mathbf{h}).\end{aligned}$$

(iii) With the help of the equation (2.5) and binomial theorem, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\mathbb{F}_k &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{(\underline{\alpha})^2\alpha^k-\left(\underline{\beta}\right)^2\beta^k}{\alpha-\beta}\right)\\ &= \frac{(\underline{\alpha})^2}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}\alpha^k-\frac{\left(\underline{\beta}\right)^2}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}\beta^k\\ &= \frac{(\underline{\alpha})^2}{\alpha-\beta}(1+\alpha)^n-\frac{\left(\underline{\beta}\right)^2}{\alpha-\beta}(1+\beta)^n\\ &= \frac{(\underline{\alpha})^2\alpha^{2n}-\left(\underline{\beta}\right)^2\beta^{2n}}{\alpha-\beta}\qquad(\text{since } 1+\alpha=\alpha^2 \text{ and } 1+\beta=\beta^2)\\ &= \mathbb{F}_{2n}.\end{aligned}$$

□

3. A Matrix Approach For Hybrid Numbers with Fibonacci and Lucas Hybrid Number Coefficients

Firstly, let us consider the following matrix:

$$Q=\begin{pmatrix}1&1\\1&0\end{pmatrix}.$$

This Q-matrix was studied by Charles H. King [19] in 1960 for his Master's thesis. It is well-known that

$$Q^n=\begin{pmatrix}F_{n+1}&F_n\\F_n&F_{n-1}\end{pmatrix}.$$

In 1963, Hoggatt and Ruggles [20] introduced the following R-matrix:

$$R=\begin{pmatrix}1&2\\2&-1\end{pmatrix}.$$

It is easily seen that

$$RQ^n=\begin{pmatrix}1&2\\2&-1\end{pmatrix}\begin{pmatrix}F_{n+1}&F_n\\F_n&F_{n-1}\end{pmatrix}=\begin{pmatrix}L_{n+1}&L_n\\L_n&L_{n-1}\end{pmatrix}.$$

Now, motivated by [6], we define an associate matrix as

$$A=\begin{pmatrix}18+2\mathbf{i}+4\varepsilon+6\mathbf{h}&11+2\mathbf{i}+2\,\varepsilon+4\mathbf{h}\\11+2\mathbf{i}+2\varepsilon+4\mathbf{h}&7+2\varepsilon+2\mathbf{h}\end{pmatrix}.$$

Then we can easily see that

$$Q^nA=\begin{pmatrix}\mathbb{F}_{n+1}&\mathbb{F}_n\\\mathbb{F}_n&\mathbb{F}_{n-1}\end{pmatrix}\tag{3.1}$$

and

$$RQ^nA=\begin{pmatrix}\mathbb{L}_{n+1}&\mathbb{L}_n\\\mathbb{L}_n&\mathbb{L}_{n-1}\end{pmatrix}.\tag{3.2}$$

Theorem 3.1 (First Type of Cassini Identity)**.** *For $n\geq 1$, we have*

$$\mathbb{F}_{n-1}\mathbb{F}_{n+1}-\mathbb{F}_n^2=(-1)^n(1-34\mathbf{i}+12\varepsilon-6\mathbf{h})$$

and

$$\mathbb{L}_{n-1}\mathbb{L}_{n+1}-\mathbb{L}_n^2=5(-1)^{n+1}(1-34\mathbf{i}+12\varepsilon-6\mathbf{h}),$$

respectively.

Proof. By using matrices (3.1) and (3.2), we have

$$\begin{aligned}
\mathbb{F}_{n-1}\mathbb{F}_{n+1}-\mathbb{F}_n^2 &= \begin{vmatrix} \mathbb{F}_{n+1} & \mathbb{F}_n \\ \mathbb{F}_n & \mathbb{F}_{n-1} \end{vmatrix} \\
&= |Q^n A| \\
&= (-1)^n \left[(7+2\varepsilon+2\mathbf{h})(18+2\mathbf{i}+4\varepsilon+6\mathbf{h})-(11+2\mathbf{i}+2\varepsilon+4\mathbf{h})^2\right] \\
&= (-1)^n (1-34\mathbf{i}+12\varepsilon-6\mathbf{h})
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{L}_{n-1}\mathbb{L}_{n+1}-\mathbb{L}_n^2 &= \begin{vmatrix} \mathbb{L}_{n+1} & \mathbb{L}_n \\ \mathbb{L}_n & \mathbb{L}_{n-1} \end{vmatrix} \\
&= |RQ^n A| \\
&= 5(-1)^{n+1} \left[(7+2\varepsilon+2\mathbf{h})(18+2\mathbf{i}+4\varepsilon+6\mathbf{h})-(11+2\mathbf{i}+2\varepsilon+4\mathbf{h})^2\right] \\
&= 5(-1)^{n+1} (1-34\mathbf{i}+12\varepsilon-6\mathbf{h}).
\end{aligned}$$

□

Theorem 3.2 (Second Type of Cassini Identity)**.** *For $n \geq 1$, we have*

$$\mathbb{F}_{n+1}\mathbb{F}_{n-1}-\mathbb{F}_n^2 = (-1)^n (1-26\mathbf{i}+28\varepsilon-14\mathbf{h})$$

and

$$\mathbb{L}_{n+1}\mathbb{L}_{n-1}-\mathbb{L}_n^2 = 5(-1)^{n+1} (1-26\mathbf{i}+28\varepsilon-14\mathbf{h}).$$

Proof. Again, by using matrices (3.1) and (3.2), we obtain

$$\begin{aligned}
\mathbb{F}_{n+1}\mathbb{F}_{n-1}-\mathbb{F}_n^2 &= (-1)^n \left[(18+2\mathbf{i}+4\varepsilon+6\mathbf{h})(7+2\varepsilon+2\mathbf{h})-(11+2\mathbf{i}+2\,\varepsilon+4\mathbf{h})^2\right] \\
&= (-1)^n (1-26\mathbf{i}+28\varepsilon-14\,\mathbf{h})
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{L}_{n+1}\mathbb{L}_{n-1}-\mathbb{L}_n^2 &= 5(-1)^{n+1} \left[(18+2\mathbf{i}+4\varepsilon+6\mathbf{h})(7+2\varepsilon+2\mathbf{h})-(11+2\mathbf{i}+2\varepsilon+4\mathbf{h})^2\right] \\
&= 5(-1)^{n+1} (1-26\mathbf{i}+28\varepsilon-14\mathbf{h}),
\end{aligned}$$

respectively. □

Now, let us define the conjugate matrix of A as

$$\overline{A} = \begin{pmatrix} 18-2\mathbf{i}-4\varepsilon-6\mathbf{h} & 11-2\mathbf{i}-2\,\varepsilon-4\mathbf{h} \\ 11-2\mathbf{i}-2\varepsilon-4\mathbf{h} & 7-2\varepsilon-2\mathbf{h} \end{pmatrix}. \tag{3.3}$$

Thus, using the matrix $\overline{A}$, we can give two types of Cassini identity for the conjugate hybrid numbers with Fibonacci and Lucas hybrid number coefficient respectively. Note that

$$\begin{aligned}
\overline{A}Q^n &= \begin{pmatrix} 18-2\mathbf{i}-4\varepsilon-6\mathbf{h} & 11-2\mathbf{i}-2\,\varepsilon-4\mathbf{h} \\ 11-2\mathbf{i}-2\varepsilon-4\mathbf{h} & 7-2\varepsilon-2\mathbf{h} \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} \overline{\mathbb{F}}_{n+1} & \overline{\mathbb{F}}_n \\ \overline{\mathbb{F}}_n & \overline{\mathbb{F}}_{n-1} \end{pmatrix}
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\overline{A}RQ^n &= \begin{pmatrix} 18-2\mathbf{i}-4\varepsilon-6\mathbf{h} & 11-2\mathbf{i}-2\,\varepsilon-4\mathbf{h} \\ 11-2\mathbf{i}-2\varepsilon-4\mathbf{h} & 7-2\varepsilon-2\mathbf{h} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} \overline{\mathbb{L}}_{n+1} & \overline{\mathbb{L}}_n \\ \overline{\mathbb{L}}_n & \overline{\mathbb{L}}_{n-1} \end{pmatrix}.
\end{aligned} \tag{3.5}$$

Theorem 3.3. *For $n \geq 1$, we have*

(i) $\overline{\mathbb{F}}_{n-1}\overline{\mathbb{F}}_{n+1}-\left(\overline{\mathbb{F}}_n\right)^2 = (-1)^n (1+26\mathbf{i}-28\varepsilon+14\mathbf{h})$,
(ii) $\overline{\mathbb{F}}_{n+1}\overline{\mathbb{F}}_{n-1}-\left(\overline{\mathbb{F}}_n\right)^2 = (-1)^n (1+34\mathbf{i}-12\varepsilon+6\mathbf{h})$,
(iii) $\overline{\mathbb{L}}_{n-1}\overline{\mathbb{L}}_{n+1}-\left(\overline{\mathbb{L}}_n\right)^2 = 5(-1)^{n+1} (1+26\mathbf{i}-28\varepsilon+14\mathbf{h})$,
(iv) $\overline{\mathbb{L}}_{n+1}\overline{\mathbb{L}}_{n-1}-\left(\overline{\mathbb{L}}_n\right)^2 = 5(-1)^{n+1} (1+34\mathbf{i}-12\varepsilon+6\mathbf{h})$.

Proof. We give only the proofs of (i) and (iii). The others can be done in a similar way.

(i) By using (3.4), we have

$$\begin{aligned}\overline{\mathbb{F}}_{n-1}\overline{\mathbb{F}}_{n+1}-\left(\overline{\mathbb{F}}_{n}\right)^{2} &= \begin{vmatrix}\overline{\mathbb{F}}_{n+1} & \overline{\mathbb{F}}_{n}\\ \overline{\mathbb{F}}_{n} & \overline{\mathbb{F}}_{n-1}\end{vmatrix}\\ &= \left|\overline{A}Q^{n}\right|\\ &= (-1)^{n}\left[(7-2\varepsilon-2\mathbf{h})(18-2\mathbf{i}-4\varepsilon-6\mathbf{h})-(11-2\mathbf{i}-2\varepsilon-4\mathbf{h})^{2}\right]\\ &= (-1)^{n}(1+26\mathbf{i}-28\varepsilon+14\mathbf{h}).\end{aligned}$$

(iii) With the help of the (3.5), we obtain

$$\begin{aligned}\overline{\mathbb{L}}_{n-1}\overline{\mathbb{L}}_{n+1}-\left(\overline{\mathbb{L}}_{n}\right)^{2} &= \begin{vmatrix}\overline{\mathbb{L}}_{n+1} & \overline{\mathbb{L}}_{n}\\ \overline{\mathbb{L}}_{n} & \overline{\mathbb{L}}_{n-1}\end{vmatrix}\\ &= \left|\overline{A}RQ^{n}\right|\\ &= 5(-1)^{n+1}\left[(7-2\varepsilon-2\mathbf{h})(18-2\mathbf{i}-4\varepsilon-6\mathbf{h})-(11-2\mathbf{i}-2\varepsilon-4\mathbf{h})^{2}\right]\\ &= 5(-1)^{n+1}(1+26\mathbf{i}-28\varepsilon+14\mathbf{h}).\end{aligned}$$

□

Remark 3.4. *This paper is revised version of the preprint* [21].

4. Conclusion

In this paper, we have introduced hybrid numbers with Fibonacci and Lucas hybrid number coefficients. We have given the Binet formulas, generating functions, exponential generating functions, some summation formulas for these numbers. Then we have defined an associate matrix for these numbers. Using this matrix, we have given two different versions of Cassini identitiy of these numbers. For the interest of the readers of our paper, the results given here have the potential to motivate further researchers of the subject of the higher order hybrid numbers with Fibonacci and Lucas hybrid number coefficients.

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