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Research Article

The Ruled Surfaces Generated by Quasi-Vectors in $\mathbb{E}^4$ Space

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Abstract
In this article, firstly, it is aimed to introduce the ruled surfaces, which is generated by quasi-vectors, by using the relationship between the Frenet frame and the quasi-frame, the quasi-equations, the quasi-curvatures in the spaces $\mathbb{E}^3$ and $\mathbb{E}^4$. Calculating the coefficients of the first fundamental form, Gaussian and mean curvatures of ruled surfaces, which are generated by quasi vectors are obtained in 4-dimensional Euclidean space. In addition to these, the relation between the Gaussian and mean curvatures of the ruled surfaces is given. Then, some geometric properties such as developability, minimality and striction line for those surfaces are investigated. Also, an example of surface curvatures by using the coefficients of fundamental form is obtained and the shapes of the ruled surface sample in projection spaces are plotted.

Keywords and 2020 Mathematics Subject Classification
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1. Introduction

The theory of surfaces and curves, as explored by researchers such as Kuhnel, Gray and Do-Carmo, continues to intrigue scholars in the field of differential geometry. Gaspard Mongea's contributions to the study of these surfaces are notable. Ruled surfaces, generated by the movement of a straight line along a curve, have taken considerable attention due to the work of Otsuki, Shiohama, Ravani, and Ku. Investigations by Aydemir, Kasap, Sarıoğlugil, Tutar, Şentürk, Yüce, Dede, and others have facilitated understanding these surfaces both in Euclidean and Minkowski spaces. In 3-dimensional Euclidean space, many scientists have published work, in particular the canal surface, which is [1–4], and the tubular surface, which is [5, 6], the ruled surface, which is the [4, 7–15]. Various geometric properties of ruled surfaces in Minkowski space, have analysed a lot of studies [16–19].

The construction of a moving frame or the Frenet frame, composed of mutually orthonormal vectors, becomes possible when dealing with differentiable curves in an open interval. The curvatures measures the deviation of the curve from a straight line and these curve elements form what is known as the Frenet apparatus. The Bishop's parallel transport frame provides an alternative to the Frenet frame, particularly well- suited for smooth curves, while the quasi-frame offers an alternative that simplifies calculations and serves as a more generalized version of the parallel transport frame.

The quasi-normal vector introduced by Coquillart is central to the concept of the quasi-frame, which leverages fixed projection vectors and Euclidean angles to create a frame consisting of the unit tangent, unit quasi-normal, and unit quasi-binormal vectors. The quasi-frame proves valuable, especially in cases where second-order derivatives are absent, offering a broader scope than the Frenet frame. In their studies, the authors have utilized the Bishop [6, 20], the Darboux [8, 9, 12, 13, 15], and the q-frame [21–24] for the theory of curves in $\mathbb{E}^3$ and $\mathbb{E}^4$ spaces. Some researchers [9, 16, 25–31] have also examined the theory of curves in 3 and 4-dimensional Euclidean spaces too.

Researchers like Kaymanlı have reseached ruled surfaces generated by quasi vectors $\mathbf{T}$, $\mathbf{N}_q$ and $\mathbf{B}_q$ in Euclidean 3-space, uncovering properties like the Gaussian and mean curvatures [7]. Different frames such as the quasi-frame in [5,7,16] and the Darboux frame in [12,13,15,17] have been used to conduct studies on surfaces in space $\mathbb{E}^3$. Focusing on Euclidean 4-space $\mathbb{E}^4$, reseaches like Alessio, Elsayied, Bayram, Bulca, Öztürk and Mello have investigated the Frenet elements and derivative equations for space curves with unit speed, and have extended extending this study to superconformal ruled surfaces [23,25,31–34]. Furthermore, the differential geometry of ruled surface, which is the [32,35–37], canal surface, which is the [33] and tubular surfaces surface, which is the [38,39], particularly with the aid of Frenet and various frames in the 4-dimensional Euclidean space $\mathbb{E}^4$, has been addressed. Also, Yüce has worked Weingarten map of the hypersurfaces in $\mathbb{E}^4$ [40].

This article aims to contribute on 2-dimensional ruled surfaces by explaining the quasi-frame and the quasi-curvature of a space curve in four-dimensional Euclidean space using the k_x and k_z projection vectors in the xz-plane. It establishes definitions and parametric expressions for surfaces such as ruled surfaces in both 3 and 4-dimensional Euclidean spaces. The ruled surfaces generated by the quasi-frame vectors in Euclidean 3-space and 2-dimensional ruled surfaces in Euclidean 4-space are presented, along with their respective first and second partial derivatives, fundamental form coefficients, and properties like striction lines, Gaussian curvatures, and mean curvatures. To enhance clarity, the calculation of quasi-vectors and quasi-curvatures for a specific space curve in 4-dimensional Euclidean space, including the equations of ruled surfaces are shown with an example. Moreover, an illustration demonstrates surface curvatures using fundamental form coefficients, visually represented in projection spaces.

2. Preliminaries

Let $\alpha(s)$ be a space curve with a non-vanishing second derivative. The Frenet frame is defined as follows,

$$\mathbf{T} = \frac{\alpha'}{\|\alpha'\|},\ \mathbf{B} = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|},\ \mathbf{N} = \mathbf{B} \wedge \mathbf{T}.$$

The curvature κ and the torsion τ are given by

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}.$$

The well-known Frenet formulas are given by

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where $v = \|\alpha'(s)\|$.

As an alternative to the Frenet frame, we use a new adapted frame along a space curve, the quasi-frame. Given a space curve $\alpha(t)$, the quasi-frame consists of three orthonormal vectors, these are the unit tangent vector $\mathbf{T}$, the quasi-normal N_q and the quasi-binormal vector B_q. The quasi-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q, \mathbf{k}\}$ is given by

$$\mathbf{T} = \frac{\alpha'}{\|\alpha'\|},\ \mathbf{N}_q = \frac{\mathbf{T} \wedge \mathbf{k}}{\|\mathbf{T} \wedge \mathbf{k}\|},\ \mathbf{B}_q = \mathbf{T} \wedge \mathbf{N}_q,$$

where $\mathbf{k}$ is the projection vector [21]. For simplicity, we have chosen the projection vector $\mathbf{k} = (0,0,1)$ in this paper. However, the quasi-frame is singular in all cases where $\mathbf{T}$ and $\mathbf{k}$ are parallel. In that case the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (0,1,0)$ or $\mathbf{k} = (1,0,0)$. The quasi-frame and the Frenet frame along a space curve are shown in Fig. 1.

Let $\alpha(s)$ be a curve that is parameterized by arc length s. The variation equations of the directional quasi-frame [5] is given by

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_q' \\ \mathbf{B}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix},$$

where the quasi-curvatures are

$$k_1 = <\mathbf{T}', \mathbf{N}_q> \quad k_2 = <\mathbf{T}', \mathbf{B}_q> \quad k_3 = <\mathbf{N}_q', \mathbf{B}_q>.$$

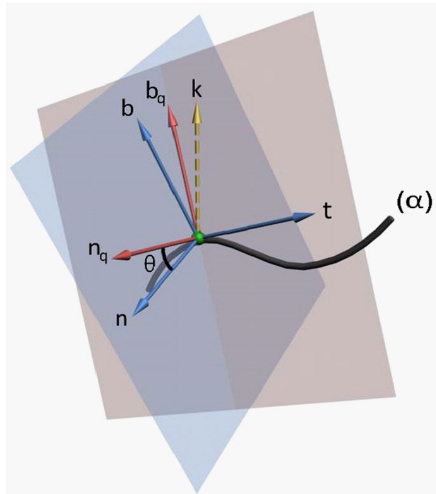

Fig. 1. The quasi frame and Frenet frame

Let $\alpha(t) = \alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be any space curve in Euclidean 4-space. Let $\mathbf{X} = (x_1, x_2, x_3, x_4)$, $\mathbf{Y} = (y_1, y_2, y_3, y_4)$ and $\mathbf{Z} = (z_1, z_2, z_3, z_4)$ be three vectors in $\mathbb{E}^4$, with the standard inner product as

$$< \mathbf{X}, \mathbf{Y} >= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

The norm of the vector $\mathbf{X}$ in $\mathbb{E}^4$ is given by $||\mathbf{X}|| = \sqrt{< \mathbf{X}, \mathbf{X} >}$. The curve is said to be parameterized by arc length s if $< \alpha', \alpha' >= 1$. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}_4$ be orthonormal basis vectors in $\mathbb{E}^4$. The vector product of the vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ is given by the determinant as follows

$$\mathbf{X} \wedge \mathbf{Y} \wedge \mathbf{Z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

or in vector form

$$\begin{aligned} \mathbf{X} \wedge \mathbf{Y} \wedge \mathbf{Z} \quad = \quad & (x_2y_3z_4 - x_2y_4z_3 - x_3y_2z_4 + x_3y_4z_2 + x_4y_2z_3 - x_4y_3z_2, \\ & -x_1y_3z_4 + x_1y_4z_3 + x_3y_1z_4 - x_3y_4z_1 - x_4y_1z_3 + x_4y_3z_1, \\ & x_1y_2z_4 - x_1y_4z_2 - x_2y_1z_4 + x_2y_4z_1 + x_4y_1z_2 - x_4y_2z_1, \\ & -x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 - x_2y_3z_1 - x_3y_1z_2 + x_3y_2z_1), \end{aligned}$$

where

$$\begin{cases} \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_4, & \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 = \mathbf{e}_1, \\ \mathbf{e}_3 \wedge \mathbf{e}_4 \wedge \mathbf{e}_1 = \mathbf{e}_{2,} & \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_4, \end{cases}$$

[23, 25].

Let $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ be vectors in $\mathbb{E}^4$. Then,

i. if these vectors linearly independent, then the vector $\mathbf{U} \wedge \mathbf{V} \wedge \mathbf{W} \in \mathbb{E}^4$ is orthogonal to the vectors $\mathbf{U}, \mathbf{V}, \mathbf{W}$ and, if any two vectors replace, the sign changes.

ii. if the vectors are not linearly independent, the cross product must be the zero vector.

iii. in four dimension space, $\mathbf{U} \wedge \mathbf{V}$ has not been defined. Since the matrix of type 3×4 has on determined [25].

For the curve with unit speed in Euclidean 4-space $\mathbb{E}^4$ such that $\alpha : I \to \mathbb{E}^4$ and $\alpha''(s) \neq 0$, the Frenet vectors are given by, [25],

$$\begin{cases} \mathbf{T}(s) = \alpha'(s), & \mathbf{N}_2(s) = \mathbf{N}_3(s) \times \mathbf{T}(s) \times \mathbf{N}_1(s), \\ \mathbf{N}_1(s) = \dfrac{\alpha''(s)}{\|\alpha''(s)\|}, & \mathbf{N}_3(s) = \dfrac{\alpha'(s) \times \alpha''(s) \times \alpha'''(s)}{\|\alpha'(s) \times \alpha''(s) \times \alpha'''(s)\|}. \end{cases}$$

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. Let us denote $\mathbf{T}(s) = \alpha'(s)$ and call as a unit tangent vector of α at s. We denote the first Serret-Frenet curvature of α by $\kappa(s) = \|\alpha''(s)\|$. If $\kappa(s) \neq 0$, then the unit principal normal vector $\mathbf{N}_1(s)$ of the curve α at s is given by $N_1'(s) + \kappa(s)\mathbf{T}(s) = \tau(s)N_2(s)$; where τ is the second Serret-Frenet curvature of α. If $\tau(s) \neq 0$, then the unit second principal normal vector $N_2(s)$ of the curve α at s is given by $N_2'(s) + \tau(s)N_1(s) = \eta(s)N_3(s)$, where η is the third Serret-Frenet curvature of α. Then we have the Serret-Frenet formulae [29]:

$$\begin{cases} \mathbf{T}'(s) &= \kappa(s)\mathbf{N}_1(s), \\ \mathbf{N}_1'(s) &= -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{N}_2(s), \\ \mathbf{N}_2'(s) &= -\tau(s)\mathbf{N}_1(s) + \eta(s)\mathbf{N}_3(s), \\ \mathbf{N}_3'(s) &= -\eta(s)\mathbf{N}_2(s). \end{cases} \tag{1}$$

Here Frenet curvatures $\kappa = k_1$, $\tau = k_2$ and $\eta = k_3$ are the first, second and third curvature functions of the curve α, respectively, [31].

In this part, we investigate the quasi-frame as an adapted frame along a space curve in $\mathbb{E}^4$. Let $\alpha = \alpha(s)$ be a space curve, the quasi-frame in $\mathbb{E}^4$ consists of four orthonormal vectors $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q, \mathbf{C}_q\}$, where $\mathbf{T}$ is the unit tangent vector field, $\mathbf{N}_q$ is the quasi-normal vector field, $\mathbf{B}_q$ and $\mathbf{C}_q$ are the first and second qausi-binormal vector fields respectively. The frame is given by

$$\begin{cases} \mathbf{T} = \dfrac{\alpha'(s)}{\|\alpha'(s)\|}, & \mathbf{N}_q = \dfrac{\mathbf{T} \wedge \mathbf{k}_x \wedge \mathbf{k}_y}{\left\|\mathbf{T} \wedge \mathbf{k}_x \wedge \mathbf{k}_y\right\|}, \\ \mathbf{B}_q = \mathbf{C}_q \wedge \mathbf{T} \wedge \mathbf{N}_q, & \mathbf{C}_q = \dfrac{\alpha'(s) \wedge \mathbf{N}_q\ (\mathbf{s}) \wedge \alpha'''(s)}{\left\|\alpha'(s) \wedge \mathbf{N}_q\ (\mathbf{s}) \wedge \alpha'''(s)\right\|}, \end{cases} \tag{2}$$

where $\mathbf{k}_x$ and $\mathbf{k}_y$ are the projection vectors. For simplicity, we choose $\mathbf{k}_x = (1,0,0,0)$ and $\mathbf{k}_y = (0,1,0,,0)$ in our calculations. It is also singular whenever $\mathbf{T}$ lies in the plane spanned by $\mathbf{k}_x$ and $\mathbf{k}_y$. In those cases we may change our projection vectors [23, 24].

Let $\alpha(s)$ be a curve that is parameterized by arc length s [24]. Differentiating (2) with respect to s, the variation equations of the quasi-frame are given by the following form

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_q' \\ \mathbf{B}_q' \\ \mathbf{C}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & 0 \\ -k_1 & 0 & k_3 & 0 \\ -k_2 & -k_3 & 0 & k_4 \\ 0 & 0 & -k_4 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_q \\ \mathbf{C}_q \end{bmatrix}.$$

The q-curvatures are also

$$\begin{cases} k_1 = \dfrac{\langle \mathbf{T}', \mathbf{N}_q \rangle}{\|\alpha'\|}, & k_2 = \dfrac{\langle \mathbf{T}', \mathbf{B}_q \rangle}{\|\alpha'\|}, \\ k_3 = \dfrac{\langle \mathbf{N}_q', \mathbf{B}_q \rangle}{\|\alpha'\|} \quad \text{and} & k_4 = \dfrac{\langle \mathbf{B}_q', \mathbf{C}_q \rangle}{\|\alpha'\|}. \end{cases} \tag{3}$$

Let M, be a regular surface given with the parameterization $\varphi(s,v)$ in $\mathbb{E}^4$ such that where $\varphi : U \subset E^2 \to \mathbb{E}^4$. The tangent space of M at an arbitrary point is spanned by the vectors φ_s and φ_v. The coefficients of the first fundamental form of M are defined as

$$E = \langle \varphi_s, \varphi_s \rangle, F = \langle \varphi_s, \varphi_v \rangle, G = \langle \varphi_v, \varphi_v \rangle \tag{4}$$

and

$$W^2 = EG - F^2, \tag{5}$$

where $\langle , \rangle$ is the Euclidean inner product [32, 34].

If $\alpha(s)$ is a curve and $X(s)$ is a generator vector, then the ruled surface $\varphi(s,u)$ has the following parameter representation:

$$M : \varphi(s,u) = \alpha(s) + uX(s), \tag{6}$$

that is, the ruled surface is a surface generated by the motion of a straight line X along α. The striction point on the ruled surface is the foot of the common perpendicular line successive rulings on the main ruling. The set of the striction points of the ruled surface generates its striction curve [37]. It is given as

$$\beta(s) = \alpha(s) - \frac{\langle \alpha_s, X_s \rangle}{\langle X_s, X_s \rangle} X(s). \tag{7}$$

Theorem 1. *(see [32, 34]) Let M be a ruled surface given with parametrization (6) in $\mathbb{E}^4$. Then the Gaussian curvature of M at point p is*

$$K = -\frac{1}{W}\left(\langle \varphi_{su}, \varphi_{su}\rangle - \frac{1}{E}\langle \varphi_{su}, \varphi_{s}\rangle^2\right). \tag{8}$$

Theorem 2. *(see [32, 34]) Let M be a ruled surface given with parametrization (6) in $\mathbb{E}^4$. Then the mean curvature of M at point p is*

$$4\|H\| = \frac{1}{W^2}\left(\langle \varphi_{ss}, \varphi_{ss}\rangle - \frac{1}{E}\langle \varphi_{ss}, \varphi_{s}\rangle^2 + \frac{1}{G}\langle \varphi_{su}, \varphi_{s}\rangle[2\langle \varphi_{ss}, \varphi_{u}\rangle + \langle \varphi_{su}, \varphi_{s}\rangle] - \frac{2}{EG}\langle \varphi_{ss}, \varphi_{s}\rangle\langle \varphi_{su}, \varphi_{s}\rangle\langle \varphi_{s}, \varphi_{u}\rangle\right). \tag{9}$$

Theorem 3. *(see [10, 30]) The ruled surface is developable if and only if $K = 0$.*

Theorem 4. *(see [10, 30]) The ruled surface is minimal if and only if $H = 0$.*

3. The ruled surfaces generated by quasi vectors in $\mathbb{E}^4$

If $\alpha(s)$ is a curve and $X(s)$ is a generator vector, then the ruled surface $\varphi(s,u)$ has the following parameter representation:

$$M : \varphi(s,u) = \alpha(s) + uX(s),$$

that is, the ruled surface is a surface generated by the motion of a straight line X along α.

Let $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q, \mathbf{C}_q\}$ be a quasi-frame in $\mathbb{E}^4$. In the expression $\varphi(s,u) = \alpha(s) + uX(s)$, if $X(s) = \mathbf{T}$ or $X(s) = \mathbf{N}_q$, the ruled surface becomes

$$M_1 \to \phi(s,u) = \alpha(s) + u\mathbf{T}(s),$$

or

$$M_2 \to \phi(s,u) = \alpha(s) + u\mathbf{N}_q(s).$$

The ruled surface generated by unit first quasi-binormal vector $X(s) = \mathbf{B}_q$ is

$$M_3 \to \varphi(s,u) = \alpha(s) + u\mathbf{B}_q(s).$$

The ruled surface generated by unit second quasi-binormal vector $X(s) = \mathbf{C}_q$ is

$$M_4 \to \varphi(s,u) = \alpha(s) + u\mathbf{C}_q(s).$$

The components E, F and G of the first fundamental form of the ruled surfaces M_1, M_2, M_3 and M_4 generated by the quasi-vectors $\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q$ and $\mathbf{C}_q$ are obtained from (4) and (5) in the form of

$$M_1 : E = 1 + u^2(k_1^2 + k_2^2),\ \ F = 1,\ \ G = 1,\ \ W = u^2(k_1^2 + k_2^2), \tag{10}$$

$$M_2 : E = 1 - 2uk_1 + u^2(k_1^2 + k_3^2),\ \ F = 0,\ \ G = 1,\ \ W = 1 - 2uk_1 + u^2(k_1^2 + k_3^2), \tag{11}$$

$$M_3 : E = 1 - 2uk_2 + u^2(k_2^2 + k_3^2 + k_4^2),\ \ F = 0,\ \ G = 1,\ \ W = 1 - 2uk_2 + u^2(k_2^2 + k_3^2 + k_4^2) \tag{12}$$

and

$$M_4 : E = 1 + u^2k_4^2,\ \ F = 0,\ \ G = 1,\ \ W = 1 + u^2k_4^2, \tag{13}$$

respectively.

Theorem 5. *The striction curves of four ruled surfaces generated by quasi-vectors along the curve $\alpha(s)$ are given by the following matrix*

$$\begin{bmatrix} \beta_{\mathbf{T}}(s)-\alpha(s) \\ \beta_{\mathbf{N}_q}(s)-\alpha(s) \\ \beta_{\mathbf{B}_q}(s)-\alpha(s) \\ \beta_{\mathbf{C}_q}(s)-\alpha(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{k_1}{k_1^2+k_3^2} & 0 & 0 \\ 0 & 0 & \frac{k_2}{k_2^2+k_3^2+k_4^2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}_q(s) \\ \mathbf{B}_q(s) \\ \mathbf{C}_q(s) \end{bmatrix}.$$

Proof. Let the striction curve of the ruled surface M_1 be $\beta_{\mathbf{T}}(s)$, with respect to the equation (7)

$$\beta_{\mathbf{T}}(s) = \alpha(s) - \frac{\langle \mathbf{T}(s), \frac{\partial}{\partial s}\mathbf{T}(s)\rangle}{\langle \frac{\partial}{\partial s}\mathbf{T}(s), \frac{\partial}{\partial s}\mathbf{T}(s)\rangle}\mathbf{T}(s)$$

the striction curve of the ruled surface M_1 is its directix curve $\alpha(s)$, that is

$$\beta_{\mathbf{T}}(s) = \alpha(s).$$

Similarly, the striction curves of the ruled surface M_1, M_2, M_3 and M_4 are given the following this equations:

$$\begin{cases} \beta_{\mathbf{N}_q}(s) &= \alpha(s) + \dfrac{k_1}{k_1^2+k_3^2}\mathbf{N}_q(s), \\ \beta_{\mathbf{B}_q}(s) &= \alpha(s) + \dfrac{k_2}{k_2^2+k_3^2+k_4^2}\mathbf{B}_q(s), \\ \beta_{\mathbf{C}_q}(s) &= \alpha(s). \end{cases}$$

The proof is completed when these equations are written in the matrix form. ■

The following theorems and corollaries can be found easily using equations (8), (10), (11), (12) and (13) with partial derivatives of the ruled surfaces M_1, M_2, M_3 and M_4.

Theorem 6. *The Gaussian curvatures of surfaces M_1, M_2, M_3 and M_4 are*

$$\begin{cases} K_{M_1} &= -\dfrac{1}{u^2(1+u^2(k_1^2+k_2^2))}, \\ K_{M_2} &= -\dfrac{k_3^2}{(1-2uk_1+u^2(k_1^2+k_3^2))^2}, \\ K_{M_3} &= -\dfrac{k_3^2+k_4^2}{(1-2uk_2+u^2(k_2^2+k_3^2+k_4^2))^2}, \\ K_{M_4} &= -\dfrac{k_4^2}{(1+u^2k_4^2)^2}, \end{cases}$$

respectively.

Corollary 7. *The ruled surface M_1 is non developable.*

Corollary 8. *The ruled surface M_2 is developable if and only if $k_3 = 0$.*

Corollary 9. *The ruled surface M_3 is developable if and only if $k_3 = k_4 = 0$.*

Corollary 10. *The ruled surface M_4 is developable if and only if $k_4 = 0$.*

The following theorems and corollaries can be found easily using equations (9), (10), (11), (12) and (13) with partial derivatives of the ruled surfaces M_1, M_2, M_3 and M_4.

Theorem 11. *The mean curvatures of the ruled surfaces M_1, M_2, M_3 and M_4 are*

$$\begin{cases} H_{M_1} &= \dfrac{k_1^2+u^2k_2^2k_3^2+k_2^2+u^2k_1^2k_3^2+u^2k_2^2k_4^2}{4u^4(k_1^2+k_2^2)^2}, \\ H_{M_2} &= \dfrac{k_2^2-2uk_1k_2^2+u^2(k_1^2k_2^2+k_2^2k_3^2+k_3^2k_4^2)}{4(1-2uk_1+u^2(k_1^2+k_3^2))^2}, \\ H_{M_3} &= \dfrac{k_1^2(1-2uk_2+u^2(k_2^2+k_3^2))}{4(1-2uk_2+u^2(k_2^2+k_3^2+k_4^2))^2}, \\ H_{M_4} &= \dfrac{k_1^2+2uk_1k_3k_4+k_2^2+u^2k_4^2(k_2^2+k_3^2)}{(1+u^2k_4^2)^2}, \end{cases}$$

respectively.

Corollary 12. *The ruled surface M_2 is minimal if and only if $k_2 = u = 0$.*

Corollary 13. *The ruled surface M_3 is minimal if and only if $k_1 = 0$.*

Corollary 14. *A relation between K_{M_1} and H_{M_1} is as follows*

$$\frac{H_{M_1}}{K_{M_1}} = -\frac{(k_1^2+u^2k_2^2k_3^2+k_2^2+u^2k_1^2k_3^2+u^2k_2^2k_4^2)(1+u^2(k_1^2+k_2^2))}{4u^2(k_1^2+k_2^2)^2}.$$

Corollary 15. *A relation between K_{M_2} and H_{M_2} is as follows*

$$\frac{H_{M_2}}{K_{M_2}} = -\frac{k_2^2-2uk_1k_2^2+u^2(k_1^2k_2^2+k_2^2k_3^2+k_3^2k_4^2)}{4k_3^2}.$$

Corollary 16. *A relation between K_{M_3} and H_{M_3} is as follows*

$$\frac{H_{M_3}}{K_{M_3}} = -\frac{k_1^2(u^2k_3^2+1-2uk_2+u^2k_2^2)}{4(k_3^2+k_4^2)}.$$

Corollary 17. *A relation between K_{M_4} and H_{M_4} is as follows*

$$\frac{H_{M_4}}{K_{M_4}} = \frac{u^2k_2^2k_4^2+k_1^2+2uk_1k_3k_4+u^2k_3^2k_4^2+k_2^2}{4k_4^2}.$$

Example 18. *Let, in $\mathbb{E}^4$, $\alpha(s)$ be the curve parameterized by*

$$\alpha(s) = \left(-s\cos s+\sin s, s\sin s+\cos s, -s\cos 2s+\frac{1}{2}\sin 2s, s\sin 2s+\frac{1}{2}\cos 2s\right).$$

The Frenet vectors are calculated by

$$\begin{cases} \mathbf{T} &= \dfrac{1}{\sqrt{5}}(\sin s, \cos s, 2\sin 2s, 2\cos 2s), \\ \mathbf{N}_1 &= \dfrac{1}{\sqrt{17}}(\cos s, -\sin s, 4\cos 2s, -4\sin 2s), \\ \mathbf{N}_3 &= \dfrac{1}{\sqrt{17}}(-4\cos s, 4\sin s, \cos 2s, -\sin 2s), \\ \mathbf{N}_2 &= \dfrac{1}{\sqrt{5}}(-2\sin s, -2\cos s, \sin 2s, \cos 2s). \end{cases}$$

The Frenet curvatures from the equations (1) are

$$\kappa(s) = \frac{17}{5\sqrt{17}}, \qquad \tau(s) = -\frac{6}{5\sqrt{17}} \quad \text{and} \quad \eta(s) = \frac{10}{\sqrt{85}}.$$

Furthermore, for $\mathbf{k}_x = (1,0,0,0)$ *and* $\mathbf{k}_y = (0,1,0,0)$, *the quasi-frame vectors are obtained as*

$$\begin{cases} \mathbf{T}_q &= \dfrac{1}{\sqrt{5}}(\sin s, \cos s, 2\sin 2s, 2\cos 2s), \\ \mathbf{N}_q &= (0,0,\cos 2s, -\sin 2s), \\ \mathbf{C}_q &= \dfrac{(-3s\cos s + 2\sin s, 3s\sin s + 2\cos s, -\sin 2s, -\cos 2s)}{\sqrt{5+9s^2}}, \\ \mathbf{B}_q &= \dfrac{(-6s\sin s + 5\cos s, -6s\cos s + 5\sin s, 3s\sin 2s, 3s\cos 2s)}{\sqrt{25+45s^2}}. \end{cases}$$

The quasi-curvatures are found as

$$k_1(s) = \frac{4}{\sqrt{5}}, \ \ k_2(s) = -\frac{1}{s\sqrt{5+9s^2}}, \ \ k_3(s) = -\frac{6s}{\sqrt{25+45s^2}} \ \ \text{and} \ \ k_4(s) = -\frac{18s^2-5}{(5+9s^2)\sqrt{5}}$$

from the equation (3).

If we use the equation given by

$$\varphi(s,u) = \alpha(s) + u\mathbf{B}_q(s)$$

for the ruled surface generated by the first quasi-binormal vector field B_q, *the ruled surface* M_4 *in* 4*-dimensional Euclidean space is given by the parametrization*

$$\begin{aligned} \varphi(s,u) \ = \ & \Bigg(-s\cos s + \sin s - \frac{u(6s\sin s + 5\cos s)}{\sqrt{25+45s^2}}, s\sin s + \cos s + \frac{u(-6s\cos s + 5\sin s)}{\sqrt{25+45s^2}}, \\ & -s\cos 2s + \frac{1}{2}\sin 2s + \frac{3su\sin 2s}{\sqrt{25+45s^2}}, s\sin 2s + \frac{1}{2}\cos 2s + \frac{3su\cos 2s}{\sqrt{25+45s^2}}\Bigg). \end{aligned}$$

Hence, the equation of the striction curve of the ruled surface M_4 *is*

$$\begin{aligned} \beta_{\mathbf{B}_q}(s) \ = \ & \Bigg(-s\cos s + \sin s + \frac{5(5+9s^2)(6s\sin s + 5\cos s)}{(648s^4+45s^2+50)\sqrt{5}}, s\sin s + \cos s - \frac{5(5+9s^2)(-6s\cos s + 5\sin s)}{(648s^4+45s^2+50)\sqrt{5}}, \\ & -s\cos 2s + \frac{1}{2}\sin 2s - \frac{15s(5+9s^2)\sin 2s}{(648s^4+45s^2+50)\sqrt{5}}, s\sin 2s + \frac{1}{2}\cos 2s - \frac{15s(5+9s^2)\cos 2s}{(648s^4+45s^2+50)\sqrt{5}}\Bigg). \end{aligned}$$

The parametrization of the ruled surface in xyz projection space is

$$\begin{aligned} \varphi(s,u) \ = \ & \Bigg(-s\cos s + \sin s - \frac{u(6s\sin s + 5\cos s)}{\sqrt{25+45s^2}}, s\sin s + \cos s + \frac{u(-6s\cos s + 5\sin s)}{\sqrt{25+45s^2}}, \\ & -s\cos 2s + \frac{1}{2}\sin 2s + \frac{3su\sin 2s}{\sqrt{25+45s^2}}\Bigg) \end{aligned}$$

and the striction curve is

$$\begin{aligned} \beta_{\mathbf{B}_q}(s) \ = \ & \Bigg(-s\cos s + \sin s + \frac{5(5+9s^2)(6s\sin s + 5\cos s)}{(648s^4+45s^2+50)\sqrt{5}}, s\sin s + \cos s - \frac{5(5+9s^2)(-6s\cos s + 5\sin s)}{(648s^4+45s^2+50)\sqrt{5}}, \\ & -s\cos 2s + \frac{1}{2}\sin 2s - \frac{15s(5+9s^2)\sin 2s}{(648s^4+45s^2+50)\sqrt{5}}\Bigg). \end{aligned}$$

The graph of the ruled surface in xyz projection space and the striction curve on it is given in Fig. 2. (a).

The parametrization of the ruled surface in xyt projection space is

$$\begin{aligned} \varphi(s,u) \ = \ & \Bigg(-s\cos s + \sin s - \frac{u(6s\sin s + 5\cos s)}{\sqrt{25+45s^2}}, s\sin s + \cos s + \frac{u(-6s\cos s + 5\sin s)}{\sqrt{25+45s^2}}, \\ & s\sin 2s + \frac{1}{2}\cos 2s + \frac{3su\cos 2s}{\sqrt{25+45s^2}}\Bigg) \end{aligned}$$

and the striction curve is

$$\beta_{\mathbf{B}_q}(s) = \left(-s\cos s+\sin s+\frac{5(5+9s^2)(6s\sin s+5\cos s)}{(648s^4+45s^2+50)\sqrt{5}}, s\sin s+\cos s-\frac{5(5+9s^2)(-6s\cos s+5\sin s)}{(648s^4+45s^2+50)\sqrt{5}},\right.$$
$$\left. s\sin 2s+\frac{1}{2}\cos 2s-\frac{15s(5+9s^2)\cos 2s}{(648s^4+45s^2+50)\sqrt{5}}\right).$$

The graph of the ruled surface in xyt projection space and the striction curve on it is given in Fig. 2. (b).

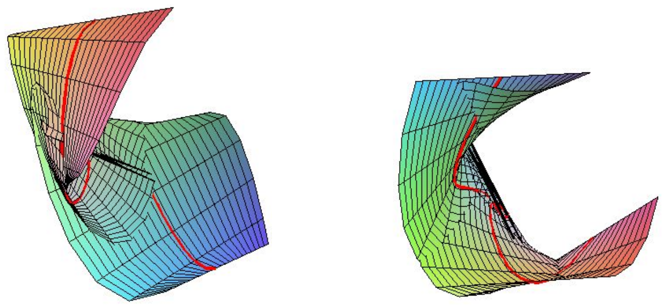

Fig. 2. (a) the surface in xyz space (b) the surface in xyt space and striction curves

The parametrization of the ruled surface in xzt projection space is

$$\varphi(s,u) = \left(-s\cos s+\sin s-\frac{u(6s\sin s+5\cos s)}{\sqrt{25+45s^2}}, -s\cos 2s+\frac{1}{2}\sin 2s+\frac{3su\sin 2s}{\sqrt{25+45s^2}},\right.$$
$$\left. s\sin 2s+\frac{1}{2}\cos 2s+\frac{3su\cos 2s}{\sqrt{25+45s^2}}\right)$$

and the striction curve is

$$\beta_{\mathbf{B}_q}(s) = \left(-s\cos s+\sin s+\frac{5(5+9s^2)(6s\sin s+5\cos s)}{(648s^4+45s^2+50)\sqrt{5}}, -s\cos 2s+\frac{1}{2}\sin 2s-\frac{15s(5+9s^2)\sin 2s}{(648s^4+45s^2+50)\sqrt{5}},\right.$$
$$\left. s\sin 2s+\frac{1}{2}\cos 2s-\frac{15s(5+9s^2)\cos 2s}{(648s^4+45s^2+50)\sqrt{5}}\right).$$

The graph of the ruled surface in xzt projection space and the striction curve on it is given in Fig. 3. (a).

The parametrization of the ruled surface in yzt projection space is

$$\varphi(s,u) = \left(s\sin s+\cos s+\frac{u(-6s\cos s+5\sin s)}{\sqrt{25+45s^2}}, -s\cos 2s+\frac{1}{2}\sin 2s+\frac{3su\sin 2s}{\sqrt{25+45s^2}},\right.$$
$$\left. s\sin 2s+\frac{1}{2}\cos 2s+\frac{3su\cos 2s}{\sqrt{25+45s^2}}\right)$$

and the striction curve is

$$\beta_{\mathbf{B}_q}(s) = \left(s\sin s+\cos s-\frac{5(5+9s^2)(-6s\cos s+5\sin s)}{(648s^4+45s^2+50)\sqrt{5}}, -s\cos 2s+\frac{1}{2}\sin 2s-\frac{15s(5+9s^2)\sin 2s}{(648s^4+45s^2+50)\sqrt{5}},\right.$$
$$\left. s\sin 2s+\frac{1}{2}\cos 2s-\frac{15s(5+9s^2)\cos 2s}{(648s^4+45s^2+50)\sqrt{5}}\right).$$

The graph of the ruled surface in yzt projection space and the striction curve on it is given in Fig. 3. (b).

Additionally, the striction curve of the ruled surface M_1 generated by unit tangent vector $\mathbf{T}$ in $\mathbb{E}^4$ is

$$\beta_{\mathbf{T}}(s) = \left(-s\cos s+\sin s, s\sin s+\cos s, -s\cos 2s+\frac{1}{2}\sin 2s, s\sin 2s+\frac{1}{2}\cos 2s\right).$$

The striction curve of the ruled surface M_2 generated by the quasi-normal vector $\mathbf{N}_q$ in $\mathbb{E}^4$ is

$$\beta_{\mathbf{N}_q}(s) = \left(-s\cos s+\sin s, s\sin s+\cos s, -s\cos 2s+\frac{1}{2}\sin 2s+\frac{1}{\sqrt{5}}\cos 2s, s\sin 2s+\frac{1}{2}\cos 2s-\frac{1}{\sqrt{5}}\sin 2s\right).$$

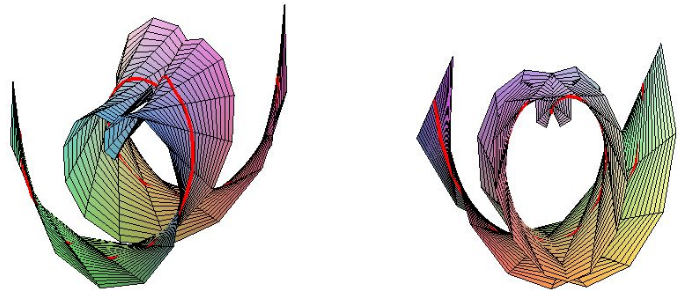

Fig. 3. (a) the surface in xzt space (b) the surface in yzt space and striction curves

The striction curve of the ruled surface M_3 generated by the second quasi-binormal vector $\mathbf{C}_q$ in $\mathbb{E}^4$ is

$$\begin{aligned}\beta_{\mathbf{C}_q}(s) &= \Big(-s\cos s+\sin s-\frac{3s(5+9s^2)(-3s\cos s+2\sin s)}{(81s^4+9s^2+25)\sqrt{5}}, s\sin s+\cos s-\frac{3s(5+9s^2)(3s\sin s+2\cos s)}{(81s^4+9s^2+25)\sqrt{5}},\\ &\quad -s\cos 2s+\frac{1}{2}\sin 2s+\frac{3s(5+9s^2)\sin 2s}{(81s^4+9s^2+25)\sqrt{5}}, s\sin 2s+\frac{1}{2}\cos 2s+\frac{3s(5+9s^2)\cos 2s}{(81s^4+9s^2+25)\sqrt{5}}\Big).\end{aligned}$$

On the other hand, the parametric expression of the ruled surface M_1 generated by the unit tangent vector field **T** *in* 4*-dimensional Euclidean space is*

$$\varphi(s,u)=\left(-s\cos s+\left(1-\frac{u}{\sqrt{5}}\right)\sin s, s\sin s+\left(1+\frac{u}{\sqrt{5}}\right)\cos s, -s\cos 2s+\left(\frac{1}{2}+\frac{2u}{\sqrt{5}}\right)\sin 2s, s\sin 2s+\left(\frac{1}{2}+\frac{2u}{\sqrt{5}}\right)\cos 2s\right).$$

Thus, the first fundamental form coefficients of the surface M_1 are

$$E=5s^2+\frac{17u^2}{5},\quad F=\sqrt{5}s,\quad G=1 \text{ and } W=\frac{17u^2}{5}.$$

Hence, the Gaussian curvature and mean curvature of the ruled surface M_1 are found as

$$K_{M_1}=\frac{25s^2}{u^2(25s^2+17u^2)}$$

and

$$H_{M_1}=\frac{5(4515s^2u^2+2125s^2-850\sqrt{5}s^2u-578\sqrt{5}u^3-425u^2+2087u^4)+3u^4}{1156u^4(25s^2+17u^2)}$$

and

$$\frac{H_{M_1}}{K_{M_1}}=\frac{5(4515s^2u^2+2125s^2-850\sqrt{5}s^2u-578\sqrt{5}u^3-425u^2+2087u^4)+3u^4}{28900u^2s^2}.$$

All the figures in this study were created by using Maple programme.

4. Conclusions

In this study, we examine the ruled surfaces generated by the quasi-vectors using parametrization (6). We calculate the striction curves, the Gaussian curvatures, and the mean curvatures of these ruled surfaces, and establish their respective relationships. To validate and exemplify the significant findings, we provide an illustrative example plotted in projection spaces. For future works, we will investigate how to extend these other ambient spaces with different dimensions and using other quasi-frame vectors.

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