Hagia Sophia Journal of Geometry
e-ISSN 2687-4261
Vol. 5, No. 2, 31-40, 2023
Research Article

On the $\mathbb{Z}_3$-Graded Structures

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Abstract
After introducing some $\mathbb{Z}_3$-graded structures, we first give the definition of a $\mathbb{Z}_3$-graded quantum space and show that the algebra of functions on it, denoted by $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1|1})$, has a $\mathbb{Z}_3$-graded Hopf algebra structure. Later, we obtain a new $\mathbb{Z}_3$-graded quantum group, denoted by $\widetilde{\mathrm{GL}}_q(1|1)$, and show that the algebra of functions on this group is a $\mathbb{Z}_3$-graded Hopf algebra. Finally, we construct two non-commutative differential calculi on the algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ which are left covariant with respect to the $\mathbb{Z}_3$-graded Hopf algebra $\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1))$.

Keywords and 2020 Mathematics Subject Classification
Keywords: $\mathbb{Z}_3$-graded vector space — $\mathbb{Z}_3$-graded algebra — $\mathbb{Z}_3$-graded Hopf algebra — $\mathbb{Z}_3$-graded quantum group — $\mathbb{Z}_3$-graded differential calculus
MSC: 16E45, 17B37, 46L87, 58B32, 81R60

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Article History: : Received 14 August 2023; Accepted 08 November 2023

1. Introduction

Quantum groups were defined by Drinfeld [1] in the 1986s as certain non-commutative deformations of commutative Hopf algebras. In 1988, Manin [2] defined quantum spaces as linear groups acting on these spaces, making quantum groups still relevant today. A striking feature of the theory of quantum groups is its surprising relevance to many branches of mathematics and physics. Quantum groups have connections with many mathematical fields such as Lie groups, Lie algebras and representations, operator algebras, and noncommutative geometry [3]. From a physical point of view, quantum inverse scattering technique is closely related to topics such as integrable model theory, elementary particle physics, quantum field theories. Although there is no satisfactory general definition of a quantum group, they are commonly referred to as Hopf algebras [4]. Some examples are standard deformations of envelope Hopf algebras of semisimple Lie algebras and corresponding coordinate Hopf algebras of Lie groups [5].

From the point of view of mathematical physics, it is natural to study $\mathbb{Z}_2$-graded (super) versions of these structures [6]. For other grades (e.g. $\mathbb{Z}_N$) the problem is also mathematically interesting. In recent years, although not very many, $\mathbb{Z}_3$-graded structures have been considered as an extension of $\mathbb{Z}_2$-grade structures and thus a new field of study in Mathematics and Mathematical Physics has emerged. The first work on this subject was given in [7] on $1+1$-space. The quantum group of 2x2 matrices of $\mathbb{Z}_3$-graded and its properties are introduced in [8].

Starting from the fact that, through the work of Woronowicz [9], one can define a consistent differential calculus on the non-commutative space of a quantum group, two differential calculi covariant under the action of the quantum group $GL_q(n)$ were developed by Wess and Zumino [10]. In the light of these two works, many authors have subsequently developed many non-commutative differential calculi on $\mathbb{Z}_2$-graded spaces and groups (see for example, [11–17]), and also $\mathbb{Z}_3$-graded spaces and groups [18–23].

2. $\mathbb{Z}_3$-graded vector spaces

The theory of $\mathbb{Z}_3$-graded algebras starts with definition of the concept of $\mathbb{Z}_3$-graded vector space. A graded vector space is a vector space that is a decomposition of a vector space into a direct sum of vector subspaces and has the extra structure of a grading indexed by integers. Let us denote the set formed with the numbers 0, 1 and 2 by $\mathbb{Z}_3$.

Definition 1. *A $\mathbb{Z}_3$-graded vector space over a field $\mathbb{K}$ is a vector space V together with a decomposition into a direct sum of the form $V = V_0 \oplus V_1 \oplus V_2$, where V_0, V_1 and V_2 are three subspaces of V which are also vector spaces.*

Each subspace V_i is called the *i-grade part* of V, and its elements are of grade i. The grade of an element $v \in V$ is denoted by $\deg(v)$ and is equal to 0, 1 or 2. All elements of V are collectively said to be homogeneous.

Example 2. *If $V = \widetilde{\mathbb{C}}^{1|1|1}$, we express a vector $v \in \widetilde{\mathbb{C}}^{1|1|1}$ as*

$$v = \begin{pmatrix} x \\ \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix},$$

where $\deg(x) = 0$, $\deg(\theta) = 1$, *and* $\deg(\varphi) = 2$. *Note that* $\deg(v) = 0$ *(mod 3).*

Example 3. *We can express a vector v in a $\mathbb{Z}_3$-graded space $\widetilde{\mathbb{C}}^{0|2|1}$ as*

$$v = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \varphi \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix},$$

where $\deg(\theta_1) = 1 = \deg(\theta_2)$, *and* $\deg(\varphi) = 2$. *Note that* $\deg(v) = 1$ *(mod 3).*

Definition 4. *A linear map $f : U \longrightarrow V$ of $\mathbb{Z}_3$-graded vector spaces is called a $\mathbb{Z}_3$-graded vector space homomorphism if it preserves the grading of homogeneous elements, that is, it has the property $f(U_i) \subseteq V_{i+j}$ for all $i \in \mathbb{Z}_3$.*

3. $\mathbb{Z}_3$-graded algebras

A graded algebra is a graded vector space with a multiplication defined on its elements.

Definition 5. *An algebra $\mathscr{A}$ over $\mathbb{K}$ is called a $\mathbb{Z}_3$-graded algebra if it is a $\mathbb{Z}_3$-graded vector space over $\mathbb{K}$, with a bilinear map $\mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$ such that $\mathscr{A}_i \cdot \mathscr{A}_j \subset \mathscr{A}_{i+j}$ for $i, j \in \mathbb{Z}_3$.*

Definition 6. *(See, Example 2) Let $\mathbb{K}\{x, \theta, \varphi\}$ be a free associative algebra generated by x, θ, φ and I_q is a two-sided ideal generated by $x\theta - \theta x$, $x\varphi - q\varphi x$, $\theta\varphi - q^2\varphi\theta$, θ^3 and φ^3. The quantum space $\widetilde{\mathbb{C}}_q^{1|1|1}$ with the function algebra*

$$\mathscr{O}(\mathbb{C}_q^{1|1|1}) = \mathbb{K}\{x, \theta, \varphi\}/I_q$$

is called $\mathbb{Z}_3$-graded quantum space, where q is a cubic root of unity. In the $\mathbb{Z}_3$-graded algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1|1})$, the generator x is of degree zero, the generator θ is of degree 1 and the generator φ is of degree 2.

In accordance with Definition 6, we have [24]:

$$x\theta = \theta x, \quad x\varphi = q\varphi x, \quad \theta\varphi = q^2\varphi\theta, \quad \theta^3 = 0 = \varphi^3. \tag{1}$$

The associative algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1|1})$ is q-commutative and known as the algebra of polynomials over the $\mathbb{Z}_3$-graded space, where x, θ and φ are three coordinate functions. There is also a $\mathbb{Z}_3$-graded quantum group acting on the $\mathbb{Z}_3$-graded space $\widetilde{\mathbb{C}}_q^{1|1|1}$. The $\mathbb{Z}_3$-graded differential geometry of this space is studied in [24].

Example 7. *It is easy seen that the map $\rho : \mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1|1}) \to M_3(\mathbb{C})$ defined by*

$$\rho(x) = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^2 \end{pmatrix}, \quad \rho(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ q^2 - q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(\varphi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 - q^2 & 0 & 0 \end{pmatrix}$$

is a representation of the $\mathbb{Z}_3$-graded algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1|1})$, i.e. the matrices $\rho(.)$ preserve the relation (1). There exists also a map $\sigma : \mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1|1}) \to M_3(\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1|1}))$ is given by

$$\sigma(x) = \begin{pmatrix} qx & 0 & 0 \\ 0 & qx & 0 \\ 0 & 0 & q^2x \end{pmatrix}, \quad \sigma(\theta) = \begin{pmatrix} q\theta & 0 & 0 \\ (q^2-q)x & q\theta & 0 \\ 0 & 0 & q^2\theta \end{pmatrix}, \quad \sigma(\varphi) = \begin{pmatrix} q\varphi & 0 & 0 \\ 0 & q\varphi & 0 \\ (1-q^2)x & (q-1)\theta & q^2\varphi \end{pmatrix}.$$

The map σ is a $\mathbb{Z}_3$-graded $\mathbb{C}$-linear homomorphism such that

$$\sigma_{ij}(f\cdot g) = \sum_k q^{\deg(f)[\deg(x_j)-\deg(x_k)]}\sigma_{ik}(f)\cdot\sigma_{kj}(g), \quad \forall f,g \in \mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1|1}),$$

where $x_1 = x$, $x_2 = \theta$, $x_3 = \varphi$.

Definition 8. *Let $\mathscr{A}$ be a $\mathbb{Z}_3$-graded algebra and the map $L : \mathscr{A} \longrightarrow \mathscr{A}$ be a $\mathbb{Z}_3$-graded vector space homomorphism. If it satisfies the $\mathbb{Z}_3$-graded Leibniz rule*

$$L(ab) = L(a)b + q^{\deg(L)\deg(a)}aL(b), \qquad \forall a,b \in \mathscr{A},$$

where q is a cubic root of unity, then L is called a $\mathbb{Z}_3$-graded derivation (see, Definition 26).

4. Modules of $\mathbb{Z}_3$-graded algebras

Since a general algebra do not need to have invertible elements, modules do not need always to have bases. In the $\mathbb{Z}_3$-graded case, there is an extra requirement of compatible degree.

Definition 9. *Let $\mathscr{A}$ be a $\mathbb{Z}_3$-graded algebra and $\mathscr{M}$ be a $\mathbb{Z}_3$-graded vector space. If there exists a mapping $\mathscr{A}\times\mathscr{M} \to \mathscr{M}$, $(a,m) \mapsto am$ such that*

$$\deg(am) = \deg(a) + \deg(m) \quad \text{and} \quad a(bm) = (ab)m$$

for all $a,b \in \mathscr{A}$ and all $m \in \mathscr{M}$, then $\mathscr{M}$ is called a left $\mathbb{Z}_3$-graded $\mathscr{A}$-module. If there exists a homogeneous basis $\{e_1,\ldots,e_k,e_{k+1},\ldots,e_{k+l},e_{k+l+1},\ldots,e_{k+l+n}\}$ for $\mathscr{M}$, where $e_1,\ldots,e_k$ are elements of $\mathscr{M}_0$, $e_{k+1},\ldots,e_{k+l}$ are elements of $\mathscr{M}_1$ and $e_{k+l+1},\ldots,e_{k+l+n}$ are elements of $\mathscr{M}_2$. So, every element $m \in \mathscr{M}$ can uniquely be expressed as

$$m = \sum_{i=1}^{k+l+n} a^i e_i, \quad a^i \in \mathscr{A}.$$

For a $\mathbb{Z}_3$-graded algebra $\mathscr{A}$, there is a difference between left and right $\mathscr{A}$-modules. A left module can also be given the structure of a right module by defining the map

$$V\times\mathscr{A} \longrightarrow V, \quad (v,a) \mapsto q^{\deg(v)\deg(a)}av.$$

The set of $\mathbb{Z}_3$-graded derivations of $\mathscr{A}$ is an important example of a $\mathbb{Z}_3$-graded $\mathscr{A}$-module (see, Theorem 27).

5. $\mathbb{Z}_3$-graded Hopf algebras

The $\mathbb{Z}_3$-graded tensor product of two $\mathbb{Z}_3$-graded algebras $\mathscr{A}$ and $\mathscr{B}$ is a $\mathbb{Z}_3$-graded algebra $\mathscr{A}\otimes\mathscr{B}$ with a product rule determined in [25]:

Definition 10. *If $\mathscr{A}$ and $\mathscr{B}$ are two $\mathbb{Z}_3$-graded algebras, then the product rule in the $\mathbb{Z}_3$-graded algebra $\mathscr{A}\otimes\mathscr{B}$ is defined by*

$$(a_1\otimes b_1)(a_2\otimes b_2) = q^{\deg(b_1)\deg(a_2)}a_1a_2\otimes b_1b_2,$$

where a_i's and b_i's are homogeneous elements in the algebras $\mathscr{A}$ and $\mathscr{B}$, respectively.

Definition 11. *A $\mathbb{Z}_3$-graded Hopf algebra is a $\mathbb{Z}_3$-graded vector space $\mathscr{A}$ over $\mathbb{K}$ with three linear map Δ, ε and κ such that*

$$\begin{cases} (\Delta\otimes\mathrm{id})\circ\Delta &= (\mathrm{id}\otimes\Delta)\circ\Delta, \\ m\circ(\varepsilon\otimes\mathrm{id})\circ\Delta &= \mathrm{id} = m\circ(\mathrm{id}\otimes\varepsilon)\circ\Delta, \\ m\circ(\kappa\otimes\mathrm{id})\circ\Delta &= \eta\circ\varepsilon = m\circ(\mathrm{id}\otimes\kappa)\circ\Delta, \end{cases}$$

together with $\Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}$, $\varepsilon(\mathbf{1}) = 1$, $\kappa(\mathbf{1}) = \mathbf{1}$, where m is the product map, id is the identity map and $\eta : \mathbb{K} \longrightarrow \mathscr{A}$.

Remark 12. *The coproduct* Δ *is an algebra homomorphism from* $\mathscr{A}$ *to* $\mathscr{A} \otimes \mathscr{A}$ *which is multiplied in the way given in Definition 11 and the counit* ε *is an algebra homomorphism from* $\mathscr{A}$ *to* $\mathbb{K}$.

Remark 13. *Any element of a* $\mathbb{Z}_3$*-graded Hopf algebra* $\mathscr{A}$ *is expressed as a product on the generators and its antipode (coinverse)* κ *is calculated with the property*

$$\kappa(ab) = q^{\deg(a)\deg(b)}\kappa(b)\kappa(a), \qquad \forall a,b \in \mathscr{A}$$

in terms of antipode of the generators, where q *is a cubic root of unity.*

We define the extended $\mathbb{Z}_3$-graded quantum space to be the algebra containing $\widetilde{\mathbb{C}}_q^{1|1|1}$, the unit and the inverse of x, x^{-1}, which obeys $xx^{-1} = \mathbf{1} = x^{-1}x$. We will denote the unital extension of $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1|1})$ by $\mathscr{F}(\widetilde{\mathbb{C}}_q^{1|1|1})$.

Theorem 14. *The algebra* $\mathscr{F}(\widetilde{\mathbb{C}}_q^{1|1|1})$ *is a* $\mathbb{Z}_3$*-graded Hopf algebra. The definitions of a coproduct, a counit and a coinverse on the algebra* $\mathscr{F}(\widetilde{\mathbb{C}}_q^{1|1|1})$ *are as follows:*

i. The coproduct $\Delta : \mathscr{F}(\widetilde{\mathbb{C}}_q^{1|1|1}) \longrightarrow \mathscr{F}(\widetilde{\mathbb{C}}_q^{1|1|1}) \otimes \mathscr{F}(\widetilde{\mathbb{C}}_q^{1|1|1})$ *is defined by*

$$\Delta(x) = x \otimes x, \quad \Delta(\theta) = \theta \otimes x + \mathbf{1} \otimes \theta, \quad \Delta(\varphi) = \varphi \otimes \mathbf{1} + x \otimes \varphi.$$

ii. The counit $\varepsilon : \mathscr{F}(\widetilde{\mathbb{C}}_q^{1|1|1}) \longrightarrow \mathbb{C}$ *is given by*

$$\varepsilon(x) = 1, \quad \varepsilon(\theta) = 0, \quad \varepsilon(\varphi) = 0.$$

iii. If we extend the algebra $\mathscr{F}(\widetilde{\mathbb{C}}_q^{1|1|1})$ *by adding the inverse of* x *then the algebra* $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1|1})$ *admits a* $\mathbb{C}$*-algebra antihomomorphism (coinverse)* $\kappa : \mathscr{F}(\widetilde{\mathbb{C}}_q^{1|1|1}) \longrightarrow \mathscr{F}(\widetilde{\mathbb{C}}_{q^2}^{1|1|1})$ *defined by*

$$\kappa(x) = x^{-1}, \quad \kappa(\theta) = -\theta x^{-1}, \quad \kappa(\varphi) = -q^2\varphi x^{-1}.$$

Proof. It is not difficult to verify the properties of the costructures given in Definition 11. As an example, let's show that Δ and κ preserve the third relation given in (1). Since

$$\begin{aligned}\Delta(\theta\varphi) = \Delta(\theta)\Delta(\varphi) &= \theta\varphi \otimes x + \theta x \otimes x\varphi + q^2\varphi \otimes \theta + x \otimes \theta\varphi \\ &= q^2\varphi\theta \otimes x + qx\theta \otimes \varphi x + q^2\varphi \otimes \theta + q^2 x \otimes \varphi\theta\end{aligned}$$

and

$$\Delta(\varphi\theta) = \varphi\theta \otimes x + \varphi \otimes \theta + q^2 x\theta \otimes \varphi x + x \otimes \varphi\theta,$$

we have $\Delta(\theta\varphi - q^2\varphi\theta) = 0$. Since $\kappa(\theta\varphi) = q^2\kappa(\varphi)\kappa(\theta) = q\varphi\theta x^{-2}$ and $\kappa(\varphi\theta) = q^2\kappa(\theta)\kappa(\varphi) = \theta\varphi x^{-2}$, we have $\kappa(\theta\varphi - q^2\varphi\theta) = 0$, as expected. ■

6. $\mathbb{Z}_3$-graded matrices

If $\mathscr{A}$ is a $\mathbb{Z}_3$-graded algebra, $\mathbb{Z}_3$-graded matrices with entries in $\mathscr{A}$ define even homomorphisms of free $\mathbb{Z}_3$-graded $\mathscr{A}$-modules in terms of particular bases.

Definition 15. *An* $n \times n$ *matrix* T *over a* $\mathbb{Z}_3$*-graded algebra* $\mathscr{A}$ *is a* $\mathbb{Z}_3$*-graded matrix whose entries are elements of* $\mathscr{A}$ *and which has the form* $T = T_0 + T_1 + T_2$*, where* T_0*,* T_1 *and* T_2 *are of grade 0, 1 and 2, respectively.*

7. A new $\mathbb{Z}_3$-graded quantum group $\widetilde{\mathrm{GL}}_q(1|1)$

In this section, we introduce two $\mathbb{Z}_3$-graded quantum spaces and a new $\mathbb{Z}_3$-graded quantum group, denoted by $\widetilde{\mathrm{GL}}_q(1|1)$.

Definition 16. *[8,21] Let $\mathscr{O}(\widetilde{\mathbb{C}}_q^{0|1|1}) := \mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ be the $\mathbb{Z}_3$-graded algebra with θ and φ obeying the relations*

$$\theta\varphi = q^2\varphi\theta, \quad \theta^3 = 0 = \varphi^3, \tag{2}$$

where $\deg(\theta) = 2$, $\deg(\varphi) = 1$, and $q^3 = 1$. We call $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ the $\mathbb{Z}_3$-graded function algebra of the $\mathbb{Z}_3$-graded quantum plane $\widetilde{\mathbb{C}}_q^{1|1}$.

Definition 17. *Let $\Lambda(\widetilde{\mathbb{C}}_q^{1|1})$ be the $\mathbb{Z}_3$-graded algebra with y and η obeying the relations*

$$y\eta = q\eta y, \quad \eta^3 = 0, \tag{3}$$

where $\deg(y) = 0$, $\deg(\eta) = 2$, and $q^3 = 1$. We call $\Lambda(\widetilde{\mathbb{C}}_q^{1|1})$ the exterior algebra of the $\mathbb{Z}_3$-graded quantum plane $\widetilde{\mathbb{C}}_q^{1|1}$.

Let a,β,γ,d be the generators of a $\mathbb{Z}_3$-graded algebra, where the generators a and d are of grade 0, the generators β and γ are of grade 1 and 2, respectively. We denote the polynomial algebra $\mathbb{K}[a,\beta,\gamma,d]$ by $\mathscr{O}(\widetilde{M}_q(1|1))$ and we write a point (a,β,γ,d) of $\mathscr{O}(\widetilde{M}_q(1|1))$ as a $\mathbb{Z}_3$-graded matrix of the form $T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$. We write $\Theta' = T\Theta$ and $\hat{\Theta}' = T\hat{\Theta}$, where $\Theta = (\theta,\varphi)^t$ and $\hat{\Theta} = (y,\eta)^t$. Then we have

Theorem 18. *The generators of $\mathscr{O}(\widetilde{M}_q(1|1))$ satisfy the following relations*

$$\begin{cases} a\beta = q\beta a, & a\gamma = q\gamma a, \quad d\beta = \beta d, \quad d\gamma = q\gamma d, \\ \beta\gamma = q^2\gamma\beta, & \beta^3 = 0 = \gamma^3, \quad ad = da + (q^2-1)\beta\gamma, \end{cases} \tag{4}$$

if and only if Θ' and $\hat{\Theta}'$ satisfy the relations (2) and (3), respectively.

Remark 19. *Strictly speaking, the demand that Θ' and $\hat{\Theta}'$ satisfy relations (2) and (3), respectively, gives two commutation relations $a\beta = k_1\beta a$ for $k_1 \in \{1,q\}$ and $d\gamma = k_2\gamma d$ for $k_2 \in \{q,q^2\}$. Here $k_1 = q = k_2$ is chosen. Also, although $\beta^3 = 0$ is not obtained from the operations, it is chosen to be appropriate for the $\mathbb{Z}_3$-graded case.*

A note that the relations obtained in Theorem 18 are quite different from the relations given in [8] and/or [21]. This is a consequence of Definitions 16 and 17.

The $\mathbb{Z}_3$-graded quantum determinant of the matrix T is given by

$$\mathscr{D}_q(T) := \mathscr{D}_q = ad - q^2\beta\gamma = da - \beta\gamma. \tag{5}$$

Remark 20. *The $\mathbb{Z}_3$-graded quantum determinant $\mathscr{D}_q$ is not central element of the $\mathbb{Z}_3$-graded algebra $\mathscr{O}(\widetilde{M}_q(1|1))$ and it satisfies the following commutation relations with the generators $\mathscr{O}(\widetilde{M}_q(1|1))$*

$$a\cdot\mathscr{D}_q = \mathscr{D}_q\cdot a, \quad \beta\cdot\mathscr{D}_q = q^2\mathscr{D}_q\cdot\beta, \quad \gamma\cdot\mathscr{D}_q = q\mathscr{D}_q\cdot\gamma, \quad d\cdot\mathscr{D}_q = \mathscr{D}_q\cdot d. \tag{6}$$

The proofs of the following two theorems can be done similarly to the proofs given in [8].

Theorem 21. *There exists a unique bialgebra structure on the algebra $\mathscr{O}(\widetilde{M}_q(1|1))$ with the costructures*

$$\Delta : \mathscr{O}(\widetilde{M}_q(1|1)) \longrightarrow \mathscr{O}(\widetilde{M}_q(1|1)) \otimes \mathscr{O}(\widetilde{M}_q(1|1)), \quad \Delta(t_{ij}) = \sum_{k=1}^{2} t_{ik}\otimes t_{kj},$$

$$\varepsilon : \mathscr{O}(\widetilde{M}_q(1|1)) \longrightarrow \mathbb{C}, \quad \varepsilon(t_{ij}) = \delta_{ij}$$

where $t_{11} = a$, $t_{12} = \beta$, $t_{21} = \gamma$, $t_{22} = d$. In addition, we have $\Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}$ and $\varepsilon(\mathbf{1}) = 1$.

Using the quantum determinant $\mathscr{D}_q$, we can define a $\mathbb{Z}_3$-graded Hopf algebra by adding the inverse $\mathscr{D}_q^{-1}$ to $\mathscr{O}(\widetilde{M}_q(2))$. Let $\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1))$ be the quotient of the algebra $\mathscr{O}(\widetilde{M}_q(1|1))$ by the two-sided ideal generated by the element $t\mathscr{D}_q - 1$. For short we write

$$\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1)) := \mathscr{O}(\widetilde{M}_q(2))[t]/\langle t\mathscr{D}_q - 1\rangle.$$

Then the algebra $\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1))$ is again a bialgebra:

Theorem 22. *The bialgebra $\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1))$ is a $\mathbb{Z}_3$-graded Hopf algebra. The coinverse κ of $\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1))$ is given by*

$$\kappa(a) = d\,\mathscr{D}_q^{-1}, \quad \kappa(\beta) = -q^2\beta\,\mathscr{D}_q^{-1}, \quad \kappa(\gamma) = -q^2\gamma\mathscr{D}_q^{-1}, \quad \kappa(d) = a\,\mathscr{D}_q^{-1}.$$

In addition, we have $\kappa(\mathbf{1}) = \mathbf{1}$.

Definition 23. *The $\mathbb{Z}_3$-graded Hopf algebra $\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1))$ is called the coordinate algebra of the $\mathbb{Z}_3$-graded quantum group $\widetilde{\mathrm{GL}}_q(1|1)$.*

Remark 24. *The cubes of the generators of the algebra $\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1))$ are central elements of $\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1))$.*

The $\mathbb{Z}_3$-graded algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ is a (left) quantum space for the $\mathbb{Z}_3$-graded quantum group $\widetilde{\mathrm{GL}}_q(1|1)$, where its action δ_L is given by

$$\delta_L(\theta) = a \otimes \theta + \beta \otimes \varphi, \quad \delta_L(\varphi) = \gamma \otimes \theta + b \otimes \varphi. \tag{7}$$

More precisely, δ_L has to be extended to an algebra homomorphism of $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ into $\mathscr{O}(\widetilde{GL}_q(1|1)) \otimes \mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$.

The proof of the theorem given below can be done similarly to the proof given in [8].

Theorem 25. *The algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ is a left comodule algebra of the bialgebra $\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1))$ with left coaction δ_L given by (7).*

8. $\mathbb{Z}_3$-graded de Rham Complexes of $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$

In this section, we will set up two noncommutative differential calculi, de Rham complexes, on the associative unital algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ which are left-covariant with respect to the $\mathbb{Z}_3$-graded Hopf algebra $\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1))$. A $\mathbb{Z}_3$-graded de Rham complex on $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ generated by two generators, their first and second order differentials and quadratic and cubic relations. So we first need to find possible commutation relations between the generators θ, φ and their differentials for first order differential calculus.

For information, let's say that all the relations obtained in this section are completely different from the relations obtained in [21].

8.1 $\mathbb{Z}_3$-graded first order differential calculi on $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$

Let us start with the definition of the $\mathbb{Z}_3$-graded first-order differential calculus on a $\mathbb{Z}_3$-graded algebra.

Definition 26. *Let $\mathscr{A}$ be a $\mathbb{Z}_3$-graded associative algebra with unity and $\mathbf{d} : \mathscr{A} \longrightarrow \Omega^1$ is a linear mapping of degree one such that*

$$\mathbf{d}(f \cdot g) = (\mathbf{d}f) \cdot g + q^{p(f)} f \cdot (\mathbf{d}g)$$

for all homogeneous element $f \in \mathscr{A}$ and all $g \in \mathscr{A}$, where q is a primitive cubic root of unity. Then a pair $(\Omega^1, \mathbf{d})$ is called a $\mathbb{Z}_3$-graded differential calculus over $\mathscr{A}$, where $\Omega^1 = \mathrm{Lin}\{a \cdot \mathbf{d}b \cdot c : a, b, c \in \mathscr{A}\}$.

To obtain the commutation relations between the generators of the algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ and their first order differentials, we consider the generators θ, φ together with $\mathbf{d}\theta$, $\mathbf{d}\varphi$, which we consider as elements of Ω^1, a space of 1-forms. We allow the first order differentials in Ω^1 to be multiplied from the left and right by the generators of $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$, so that by definition of multiplication the resulting 1-forms belong to Ω^1. This means that Ω^1 is an $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$-bimodule.

Theorem 27. *There exist two $\mathbb{Z}_3$-graded first-order differential calculi $\Omega^1(\widetilde{\mathbb{C}}_q^{1|1})$ over the $\mathbb{Z}_3$-graded algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ which are covariant with respect to the $\mathbb{Z}_3$-graded Hopf algebra $\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1))$ such that $\{\mathbf{d}\theta, \mathbf{d}\varphi\}$ is the free $\mathscr{F}(\widetilde{\mathrm{GL}}_q(1|1))$-module base of $\Omega^1(\widetilde{\mathbb{C}}_q^{1|1})$. The bi-module structure of these calculi is determined by*

$$\begin{cases} \theta \cdot \mathbf{d}\theta &= P\,\mathbf{d}\theta \cdot \theta, \quad \theta \cdot \mathbf{d}\varphi &= -q(q+P)\,\mathbf{d}\varphi \cdot \theta + (1-qP)\,\mathbf{d}\theta \cdot \varphi, \\ \varphi \cdot \mathbf{d}\varphi &= qP\,\mathbf{d}\varphi \cdot \varphi, \quad \varphi \cdot \mathbf{d}\theta &= -(q+P)\,\mathbf{d}\theta \cdot \varphi + (1-P)\,\mathbf{d}\varphi \cdot \theta, \end{cases} \tag{8}$$

where $P \in \{1, q^2\}$.

Proof. Let Ω^1 be a free right module over $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ spanned by the elements of the set $\{\mathbf{d}\theta, \mathbf{d}\varphi\}$. In general the coordinates will not commute with their differentials. Therefore, we assume that the possible commutation relations of the generators with their first order differentials are of the form

$$\begin{cases} \theta \cdot \mathbf{d}\theta &= P\,\mathbf{d}\theta \cdot \theta, \quad \theta \cdot \mathbf{d}\varphi = A_1\,\mathbf{d}\varphi \cdot \theta + A_2\,\mathbf{d}\theta \cdot \varphi, \\ \varphi \cdot \mathbf{d}\varphi &= Q\,\mathbf{d}\varphi \cdot \varphi, \quad \varphi \cdot \mathbf{d}\theta = A_3\,\mathbf{d}\theta \cdot \varphi + A_4\,\mathbf{d}\varphi \cdot \theta, \end{cases} \tag{9}$$

where constants P, Q and A_j are possibly dependent on q. That is, let the left $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$-module structure on Ω^1 be completely defined by the above relations. We will perform the proof in three steps.

- **Step** 1. If we extend the operator δ_L given in (7) to the space Ω^1, we can write

$$\delta_L(\mathbf{d}\theta) = a \otimes \mathbf{d}\theta + q\beta \otimes \mathbf{d}\varphi, \quad \delta_L(\mathbf{d}\varphi) = q^2\gamma \otimes \mathbf{d}\theta + d \otimes \mathbf{d}\varphi. \tag{10}$$

  Now if we apply the operator δ_L to both sides of the equalities given in (9) and use the relations (4), we see that after long operations, $Q = qP$, $A_3 = q(P - A_2)$ and $A_4 = q(P - A_1)$.

- **Step** 2. We now apply the differential operator $\mathbf{d}$ to both sides of $\theta^3 = 0$ using the property of $\mathbf{d}$ in Definition 26. Then, we can write

$$0 = \mathbf{d}(\theta^3) = \mathbf{d}\theta \cdot \theta^2 + q^2\theta \cdot \mathbf{d}(\theta^2) = \mathbf{d}\theta \cdot \theta^2 + q^2\theta \cdot (\mathbf{d}\theta \cdot \theta + q^2\theta \cdot \mathbf{d}\theta) = (1 + q^2P + qP^2)\,\mathrm{d}\theta \cdot \theta^2.$$

  Therefore $1 + q^2P + qP^2$ must be equal to zero so that the solution of this equation gives either $P = 1$ or $P = q^2$.

- **Step** 3. Finally, when we apply the differential operator $\mathbf{d}$ from the left to both sides of the relation $\theta \cdot \varphi - q^2\varphi \cdot \theta = 0$ and compare the resulting case with the relations (9), we get $A_1 = -q(q + P)$ and $A_2 = 1 - qP$.

■

Remark 28. *Since the set $\{\mathbf{d}\theta, \mathbf{d}\varphi\}$ is homogeneous base of the space $\Omega^1(\widetilde{\mathbb{C}}_q^{1|1})$, a map $\sigma : \mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1}) \to M_2(\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1}))$ can be defined by the formulas*

$$f \cdot \mathbf{d}\theta_j = q^{\deg(f)\deg(\mathbf{d}\theta_j)} \sum_{i=1}^{2} \mathbf{d}\theta_i \cdot \sigma_{ij}(f), \quad \forall f \in \mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1}),$$

where $\theta_1 = \theta$ and $\theta_2 = \varphi$ and

$$\sigma(\theta) = \begin{pmatrix} P\theta & (q^2 - P)\varphi \\ 0 & -(q+P)\theta \end{pmatrix}, \quad \sigma(\varphi) = \begin{pmatrix} -(q+P)\varphi & 0 \\ (1-P)\theta & q^2P\varphi \end{pmatrix}.$$

Theorem 29. *The map σ is a $\mathbb{Z}_3$-graded $\mathbb{C}$-linear homomorphism such that*

$$\sigma_{ij}(f \cdot g) = \sum_{k=1}^{2} q^{\deg(f)[\deg(\theta_k) - p(\theta_j)]} \sigma_{ik}(f) \cdot \sigma_{kj}(g)$$

for all $f, g, \theta_i \in \mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$, and it preserves the relations (2) given in Definition 16.

8.2 High-order $\mathbb{Z}_3$-graded differential calculi on $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$

The appearance of higher-order differentials is a peculiar feature of the $\mathbb{Z}_3$-graded calculus. The second order differentials $\mathbf{d}^2\theta$, $\mathbf{d}^2\varphi$ are considered as elements of a space Ω^2 of 2-forms, which is in fact also a bimodule.

Definition 30. *Let $\mathscr{A}$ be a $\mathbb{Z}_3$-graded associative algebra with unity and $\Omega = \bigoplus_{i \in \mathbb{Z}_3} \Omega^i(\mathscr{A})$ be a $\mathbb{Z}$-graded algebra with $\Omega^0(\mathscr{A}) = \mathscr{A}$. A pair $(\Omega, \mathbf{d})$ is called a $\mathbb{Z}_3$-graded differential calculus over $\mathscr{A}$ if $\mathbf{d} : \Omega \to \Omega$ is a linear map of degree one which satisfies*

i. $\mathbb{Z}_3$-graded Leibniz rule of second order:

$$\mathbf{d}^2(f \cdot g) = \mathbf{d}^2 f \cdot g + (q^{\deg(f)} + q^{\deg(\mathbf{d}f)})\mathbf{d}f \wedge \mathbf{d}g + q^{-\deg(f)} f \cdot \mathbf{d}^2 g$$

for all homogeneous $f \in \mathscr{A}$ and all $g \in \mathscr{A}$, where q is a primitive cubic root of unity, *ii.* $\mathbf{d}^3 = 0$.

It follows from this definition that in order to arrive at a higher order differential calculus on $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$, we need to find the commutation relations between the generators of $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ and their second order differentials, as well as the commutation relations between both the first order and second order differentials. If we take y and η in Definition 17 as the differentials of the generators of $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$, i.e. $\mathbf{d}\theta = y$ and $\mathbf{d}\varphi = \eta$, we can write the commutation relations between the first order differentials of the generators of $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ as

$$\mathbf{d}\theta\wedge\mathbf{d}\varphi = q\,\mathbf{d}\varphi\wedge\mathbf{d}\theta, \quad \mathbf{d}\varphi\wedge\mathbf{d}\varphi\wedge\mathbf{d}\varphi = 0. \tag{11}$$

Now we want to find the relations between the generators of the algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ and their second order differentials. But, since $\mathbf{d}^2 \neq 0$ we cannot obtain them by applying the operator $\mathbf{d}$ to the relations (8). However, we can consider that the relations with the second order differentials of the generators of the algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ are of the form

$$\begin{cases} \theta\cdot\mathbf{d}^2\theta = P'\,\mathbf{d}^2\theta\cdot\theta, & \theta\cdot\mathbf{d}^2\varphi = A_1'\,\mathbf{d}^2\varphi\cdot\theta + A_2'\,\mathbf{d}^2\theta\cdot\varphi, \\ \varphi\cdot\mathbf{d}^2\varphi = Q'\,\mathbf{d}^2\varphi\cdot\varphi, & \varphi\cdot\mathbf{d}^2\theta = A_3'\,\mathbf{d}^2\theta\cdot\varphi + A_4'\,\mathbf{d}^2\varphi\cdot\theta, \end{cases} \tag{12}$$

where constants P', Q' and A_j' are possibly dependent on q. To find the constants appearing in these relations, we compare the results by applying the operator $\mathbf{d}$ twice to the relations (8) and once to the relations (12). In this case, we see that they are $P' = q$, $A_1' = q$, $A_2' = q - q^2$, $A_3' = q$, $A_4' = 0$ and $Q' = 1$. On the other hand, when we apply the operator $\mathbf{d}$ to relations (8), we see that, for example,

$$\theta\cdot\mathbf{d}^2\varphi = -(1+qP)\,\mathbf{d}^2\varphi\cdot\theta + q(1-qP)\,\mathbf{d}^2\theta\cdot\varphi + q^2(P-1)\,\mathbf{d}\varphi\wedge\mathbf{d}\theta$$

such that relations are not homogeneous with respect to the generator and their second-order differentials. To make them homogeneous, we compare such relations with relations (12) written in place of their respective constants. In this case, we see the following dependencies emerge:

$$\begin{cases} \mathbf{d}\theta\wedge\mathbf{d}\theta = -\mathbf{d}^2\theta\cdot\theta, & \mathbf{d}\theta\wedge\mathbf{d}\varphi = q(\mathbf{d}^2\theta\cdot\varphi + \mathbf{d}^2\varphi\cdot\theta), \\ \mathbf{d}\varphi\wedge\mathbf{d}\theta = \mathbf{d}^2\theta\cdot\varphi + \mathbf{d}^2\varphi\cdot\theta, & \mathbf{d}\varphi\wedge\mathbf{d}\varphi = -q\,\mathbf{d}^2\varphi\cdot\varphi. \end{cases} \tag{13}$$

These relations are obtained assuming $P \neq 1$. However, it is very interesting that even if $P = 1$, all the relations (14)-(16) given in the following theorem are still valid. Now we can collectively give all the relations in the following theorem:

Theorem 31. *There exist two $\mathbb{Z}_3$-graded differential calculi $\Omega(\widetilde{\mathbb{C}}_q^{1|1})$ over the $\mathbb{Z}_3$-graded algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ which are covariant with respect to the $\mathbb{Z}_3$-graded Hopf algebra $\mathscr{O}(\widetilde{\mathrm{GL}}_q(1|1))$ such that $\{\mathbf{d}\theta, \mathbf{d}\varphi, \mathbf{d}^2\theta, \mathbf{d}^2\varphi\}$ is the free $\mathscr{F}(\widetilde{\mathrm{GL}}_q(1|1))$-module base of $\Omega(\widetilde{\mathbb{C}}_q^{1|1})$. The bi-module structure of these calculi is determined by:*

i. The generators of $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ with first order differentials satisfy the following commutation relations

$$\begin{cases} \theta\cdot\mathbf{d}\theta = P\,\mathbf{d}\theta\cdot\theta, & \theta\cdot\mathbf{d}\varphi = -q(q+P)\,\mathbf{d}\varphi\cdot\theta + (1-qP)\,\mathbf{d}\theta\cdot\varphi, \\ \varphi\cdot\mathbf{d}\varphi = qP\,\mathbf{d}\varphi\cdot\varphi, & \varphi\cdot\mathbf{d}\theta = -(q+P)\,\mathbf{d}\theta\cdot\varphi + (1-P)\,\mathbf{d}\varphi\cdot\theta, \end{cases}$$

where $P \in \{1, q^2\}$.

ii. The relations of the second order differentials with the generators of $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ are of the form

$$\begin{cases} \theta\cdot\mathbf{d}^2\theta = q\,\mathbf{d}^2\theta\cdot\theta, & \theta\cdot\mathbf{d}^2\varphi = q\,\mathbf{d}^2\varphi\cdot\theta + (q-q^2)\,\mathbf{d}^2\theta\cdot\varphi, \\ \varphi\cdot\mathbf{d}^2\varphi = \mathbf{d}^2\varphi\cdot\varphi, & \varphi\cdot\mathbf{d}^2\theta = q\,\mathbf{d}^2\theta\cdot\varphi. \end{cases} \tag{14}$$

iii. The relations between the first order differential are of the form

$$\mathbf{d}\theta\wedge\mathbf{d}\varphi = q\,\mathbf{d}\varphi\wedge\mathbf{d}\theta, \quad \mathbf{d}\varphi\wedge\mathbf{d}\varphi\wedge\mathbf{d}\varphi = 0.$$

iv. The relations between the first order differential with the second order differentials are of the form

$$\begin{cases} \mathbf{d}\theta\wedge\mathbf{d}^2\theta = q^2\,\mathbf{d}^2\theta\wedge\mathbf{d}\theta, & \mathbf{d}\theta\wedge\mathbf{d}^2\varphi = q\,\mathbf{d}^2\varphi\wedge\mathbf{d}\theta + (q^2-1)\,\mathbf{d}^2\theta\wedge\mathbf{d}\varphi, \\ \mathbf{d}\varphi\wedge\mathbf{d}^2\varphi = \mathbf{d}^2\varphi\wedge\mathbf{d}\varphi, & \mathbf{d}\varphi\wedge\mathbf{d}^2\theta = q^2\,\mathbf{d}^2\theta\wedge\mathbf{d}\varphi. \end{cases} \tag{15}$$

v. The relations between the second order differentials are of the form

$$\mathbf{d}^2\theta \wedge \mathbf{d}^2\varphi = \mathbf{d}^2\varphi \wedge \mathbf{d}^2\theta, \quad \mathbf{d}^2\varphi \wedge \mathbf{d}^2\varphi \wedge \mathbf{d}^2\varphi = 0. \tag{16}$$

Remark 32. *To check the covariance of relations (14)-(16), we extend the definition of δ_L to $\Omega(\widetilde{\mathbb{C}}_q^{1|1})$. In this case we can write*

$$\delta_L(\mathbf{d}^2\theta) = a \otimes \mathbf{d}^2\theta + q^2\beta \otimes \mathbf{d}^2\varphi, \quad \delta_L(\mathbf{d}^2\varphi) = q\gamma \otimes \mathbf{d}^2\theta + d \otimes \mathbf{d}^2\varphi. \tag{17}$$

Thus we have obtained all the relations and so we have completed the Rham complex of the $\mathbb{Z}_3$-graded algebra $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$.

9. Conclusions

In this paper, we define a $\mathbb{Z}_3$-graded quantum space, denoted by $\widetilde{\mathbb{C}}_q^{1|1|1}$, and show that the algebra of functions on this space has a $\mathbb{Z}_3$-graded Hopf algebra structure. We also define a new $\mathbb{Z}_3$-graded quantum plane, denoted by $\widetilde{\mathbb{C}}_q^{1|1}$, obtain a new $\mathbb{Z}_3$-graded quantum group of matrices acting on it, and construct two covariant differential calculi on $\mathscr{O}(\widetilde{\mathbb{C}}_q^{1|1})$ with respect to this group.

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