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# $q_k$−Laplace Transform on Quantum Integral

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Abstract. In this study, we present $q_k$−Laplace transform by $q_k$−integral on quantum analogue. We give some properties of $q_k$−Laplace transform. The $q_k$−Laplace transforms of some common functions are calculated.

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## 1. Introduction

Quantum calculus is contemporary name for investigation of calculus without limits. Quantum calculus or $q$−calculus began with the studies of *Jackson* in 1909 and 1910 [19, 20]. But, this kind of calculus had already been worked out by *Euler* and *Jacobi* much earlier. Carmichael gave theory of linear $q$−difference equations in 1912 [3, 14] . Trjitzinsky studied analytic theory of linear $q$−difference equations [26] in 1933. In recent times, it attracts a lot of attention in mathematics that models quantum computing. $q$−calculus appeared as a link between mathematics and physics. It has a lot of applications in many areas such as number theory, combinatorics, orthogonal polynomials, hyper-geometric functions, quantum, mechanics and relativity theories. In a large number of essential aspects of quantum calculus are covered by a book written by Kac and Cheung [21]. In 2004, Bangerezako examined variational $q$−calculus [10] . Bohner and Hudson considered Euler-type BvP's in quantum calculus in 2007 [12]. Ahmad solved BvP's for non-linear 3.rd-order $q$−difference equations in 2011 [4]. In the same year, Cieśliński [15] improved $q$−exponential and $q$−trigonometric functions. Ahmad et. al. gave a study of 2.nd-order $q$−difference equations with boundary conditions [5]. Yu and Wang proved existence of solutions for nonlinear second-order $q$−difference equations with first-order $q$−derivatives [29]. Alp and Sarıkaya defined features of quantum integral which is called $q$−integral [9].

Quantum calculus has many fundamental aspects. It has been demonstrated that quantum calculus is a subfield of more comprehensive mathematical field of time scale calculus. This calculus provide a structure to examine dynamic equations on both discrete and continuous domains [11, 13, 28]. $q$−calculus is a special case of time scale calculus. Calculation of derivatives and integrals on the time scale and therefore the solution of differential equations is quite difficult. For this reason, the results obtained by examining the $q$−analysis, which has many applications in many fields, are extremely important.

$q_k$−analysis, on the other hand, is a much more general form of quantum analysis. Actually, Now let's talk about the historical development of this analysis. In 2003, Rajkovic [22] firstly defined the $q_a$−integral and later, Tariboon

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and Ntouyas [24] initiated the study of quantum calculus on finite intervals and defined $q_k-$derivative and $q_k-$integral of any function in 2013. They defined $q_k-$derivative of a function $f : [t_k, t_{k+1}] \to \mathbb{R}$ and its essential properties such as derivative of a sum, a product or a quotient of two functions. Furthermore, they defined $q_k-$integral and demonstrated its fundamental properties [25]. Similarly, Sudsutad [23] defined the $q_a-$derivative in 2015. Essentially, both derivative definitions and both integral definitions in [22–24] are the same. In 2015, Diaz and Teruel defined generalized Gamma and Beta Functions on $q_k-$ analysis [18]. Ahmad et. al. gave new concepts on impulsive IVPs and BVPs for $q_k-$analysis in 2016 [7]. In the same year, Ahmad et. al. considered nonlocal BVPs for impulsive fractional $q_k-$difference equations [6].

Very important work has been done in the classical case and $q-$analysis on the Laplace transform. Initial value problems are then solved by the Laplace transform [1, 2, 8, 16]. For example, Bohner and Guseinov devined $h-$ and $q-$ Laplace transformations in 2010 [11]. Ucar and Albayrak studied $q-$Laplace integral operators and applications in 2012 [27].

In this study, we introduce *Laplace transform* on $q_k-$integral. In our literature review, it was seen that the Laplace transform was not defined in the $q_k-$analysis and its properties were not examined. Here, we give some preliminaries that are necessary for understanding the Laplace transform on $q_k-$integral. In section of main results, we newly present Laplace transform on $q_k-$integral (or $q_k-$Laplace transform), $q_k-$binomial coefficients, some main definitions-theorems and some examples that are required to calculate $q_k-$Laplace of some essential functions.

## 2. Preliminaries

Here, let us recall some basic concepts of $q-$calculus, quantum calculus on finite intervals or $q_k-$ calculus and some related notions as usual Laplace and $q-$Laplace Transforms. Let us now state the definitions and theorems necessary to grasp the subject. The main part of our study depends on the understanding of this part.

**Definition 2.1** ( [17]). For $s \in \mathbb{C}$ with $Re(s) > 0$, *Gamma function* is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt. \tag{2.1}$$

By (2.1),

$$\Gamma(s+1) = s\Gamma(s), \qquad \Gamma(n+1) = n! \quad (n \in \mathbb{N}).$$

**Definition 2.2** ( [17]). For $t > 0$, *Laplace transform* of $f(t)$ is defined by

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{\alpha\to\infty} \int_0^\alpha e^{-st} f(t)dt. \tag{2.2}$$

One can say that (2.2) converges if the limit exists, and otherwise it diverges.

**Definition 2.3** ( [19, 20]). Let $q \in (0, 1)$ and $f(x)$ be an arbitrary function. The $q-$differential of $f$ is defined by

$$d_q f(x) = f(qx) - f(x),$$
$$d_q x = (q-1)x,$$

and $q-$derivative of $f$ on a subset of $\mathbb{R}$ is

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}, \qquad x \neq 0, \tag{2.3}$$
$$D_q f(0) = \lim_{x\to 0} D_q f(x),$$

where $qx$ and $x$ should be on the domain of $f$ and $D_q$ is $q-$difference operator.

It is easy to see that, if $f(x)$ is differentiable, it yields that

$$\lim_{q\to 1} D_q f(x) = \frac{df(x)}{dx} = f'(x).$$

The higher order $q-$derivatives are given by

$$D_q^n f(t) = D_q D_q^{n-1} f(t), \qquad n \in \mathbb{N}.$$

**Definition 2.4** ( [19, 20]). $q-$analogue of $n \in \mathbb{N}$ is defined by

$$[n]_q = \frac{q^n - 1}{q - 1} = q^{n-1} + ... + 1.$$

**Theorem 2.5** ( [19, 20]). *Let $f, g$ be $q-$ differentiable functions. Then, $D_q$ is a linear operator. In other words, for any constants $\alpha$ and $\beta$*

$$D_q \{\alpha f(t) + \beta g(t)\} = \alpha D_q \{f(t)\} + \beta D_q \{g(t)\}.$$

**Remark 2.6** ( [21]). Let $f, g$ be $q-$ differentiable functions. In this case, the following statements are true.

$$\begin{aligned} D_q \{f(t) g(t)\} &= f(qt) D_q g(t) + g(t) D_q f(t) \\ &= f(t) D_q g(t) + g(qt) D_q f(t), \\ D_q \left\{ \frac{f(t)}{g(t)} \right\} &= \frac{g(t) D_q f(t) - f(t) D_q g(t)}{g(qt) g(t)}. \end{aligned}$$

**Definition 2.7** ( [16]). $q-$analogues of classical exponential function $e^x$ is defined by

$$\begin{aligned} e_q^x &= \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!}, \\ E_q^x &= \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j}{[j]_q!}. \end{aligned} \tag{2.4}$$

$q-$exponential functions satisfy the following relations:

$$e_q^x E_q^{-x} = E_q^x e_q^{-x} = 1, \qquad E_q^x = e_{\frac{1}{q}}^x, \qquad E_q^0 = 1.$$

Moreover, for $E_q^{-x} = \frac{1}{e_q^x}$, we have

$$\lim_{x \to \infty} E_q^{-x} = \lim_{x \to \infty} \frac{1}{e_q^x} = 0.$$

**Definition 2.8** ( [16]). $q-$analogues of classical trigonometric functions are

$$\begin{aligned} \sin_q x &= \frac{e_q^{ix} - e_q^{-ix}}{2i}, \qquad \mathrm{Sin}_q x = \frac{E_q^{ix} - E_q^{-ix}}{2i}, \\ \cos_q x &= \frac{e_q^{ix} + e_q^{-ix}}{2}, \qquad \mathrm{Cos}_q x = \frac{E_q^{ix} + E_q^{-ix}}{2}. \end{aligned}$$

Hyperbolic $q-$cosine, hyperbolic $q-$sine functions are defined by

$$\cosh_q ax = \frac{e^{ax} + e_q^{-ax}}{2} \text{ and } \sinh_q ax = \frac{e^{ax} - e_q^{-ax}}{2}.$$

**Definition 2.9** ( [21]). Jackson integral or $q-$integral of $f(x)$ is defined by

$$\int f(x) d_q x = (1 - q) x \sum_{j=0}^{\infty} q^j f\left(q^j x\right).$$

**Remark 2.10** ( [21]). From the above definition, we can easily derives a more general formula as

$$\begin{aligned} \int f(x) D_q g(x) d_q x &= (1 - q) x \sum_{j=0}^{\infty} q^j f\left(q^j x\right) D_q g\left(q^j x\right) \\ &= (1 - q) x \sum_{j=0}^{\infty} q^j f\left(q^j x\right) \frac{g\left(q^j x\right) - g\left(q^{j+1} x\right)}{(1 - q) q^j x}, \end{aligned}$$

or

$$\int f(x) d_q g(x) = \sum_{j=0}^{\infty} f\left(q^j x\right)\left(g\left(q^j x\right) - g\left(q^{j+1} x\right)\right).$$

**Definition 2.11** ( [19–21])**.** Let $0 < a < b$. The definite $q-$integral is defined as

$$\int_0^b f(x)\, d_q x = (1-q)\, b \sum_{j=0}^{\infty} q^j f\left(q^j b\right) \tag{2.5}$$

and

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x.$$

By (2.3) and (2.5), we get

$$\int_0^x D_q f(t)\, d_q t = f(x) - f(0).$$

Now, let's express the basic concepts of our study, $q_k-$derivative and $q_k-$integral and their properties.

**Definition 2.12** ( [24, 25])**.** Assume that $f : J_k \to \mathbb{R}$ is a continuous function, and $J_k = [t_k, t_{k+1}] \subset \mathbb{R}$, $t \in J_k$ and $0 < q_k < 1$. Then, the expression

$$\begin{aligned} D_{q_k} f(t) &= \frac{f(t) - f(q_k t + (1-q_k)\, t_k)}{(1-q_k)(t-t_k)}, \qquad t \neq t_k, \\ D_{q_k} f(t_k) &= \lim_{t \to t_k} D_{q_k} f(t), \end{aligned} \tag{2.6}$$

is called $q_k-$derivative of $f$ at $t$.

Underline the fact that if $t_k = 0$ and $q_k = q$ in (2.6), then $D_{q_k} f = D_q f$, where $D_q$ is $q-$derivative of $f(t)$ defined in (2.3).

**Definition 2.13** ( [24])**.** $q_k-$analogue of $n$ is defined by

$$[n]_{q_k} = \frac{1-q_k^n}{1-q_k}.$$

**Example 2.14** ( [24])**.** In $q_k-$calculus, $D_{q_k} (t-t_k)^n = [n]_{q_k} (t-t_k)^{n-1}$. As a matter of fact, $f(t) = (t-t_k)^n$, $t \in J_k$, where $J_k = [t_k, t_{k+1}] \subset \mathbb{R}$, then

$$\begin{aligned} D_{q_k} f(t) &= \frac{(t-t_k)^n - (q_k t + (1-q_k)\, t_k - t_k)^n}{(1-q_k)(t-t_k)} \\ &= \frac{(t-t_k)^n - q_k^n (t-t_k)^n}{(1-q_k)(t-t_k)} \\ &= [n]_{q_k} (t-t_k)^{n-1}. \end{aligned}$$

**Theorem 2.15** ( [24])**.** *Assume that $f, g : J_k \to \mathbb{R}$ are $q_k-$differentiable. Then:*

*(i) $f + g : J_k \to \mathbb{R}$ is $q_k-$differentiable with*

$$D_{q_k} (f(t) + g(t)) = D_{q_k} f(t) + D_{q_k} g(t),$$

*(ii) For any constant $\alpha$, $\alpha f : J_k \to \mathbb{R}$ is $q_k-$differentiable with*

$$D_{q_k} (\alpha f(t)) = \alpha D_{q_k} f(t),$$

*(iii) $fg : J_k \to \mathbb{R}$ is $q_k-$differentiable with*

$$\begin{aligned} D_{q_k} (fg)(t) &= f(t)\, D_{q_k} g(t) + g(q_k t + (1-q_k)\, t_k)\, D_{q_k} f(t) \\ &= g(t)\, D_{q_k} f(t) + f(q_k t + (1-q_k)\, t_k)\, D_{q_k} g(t), \end{aligned}$$

*(iv) If $g(t)\, g(q_k t + (1-q_k)\, t_k) \neq 0$, then $\frac{f}{g}$ is $q_k-$differentiable with*

$$D_{q_k} \left(\frac{f}{g}\right)(t) = \frac{g(t)\, D_{q_k} f(t) - f(t)\, D_{q_k} g(t)}{g(t)\, g(q_k t + (1-q_k)\, t_k)}.$$

**Definition 2.16** ( [24]). Assume that $f : J_k \to \mathbb{R}$ is a continuous function. Then, $q_k-$integral of $f$ is defined by

$$\int_{t_k}^{t} f(s)\, d_{q_k}s = (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f\left(q_k^n t + (1-q_k^n)t_k\right) \tag{2.7}$$

for $t \in J_k$. Furthermore, if $a \in (t_k, t)$, then definite $q_k-$integral is defined by

$$\begin{aligned}\int_{a}^{t} f(s)\, d_{q_k}s &= \int_{t_k}^{t} f(s)\, d_{q_k}s - \int_{t_k}^{a} f(s)\, d_{q_k}s \\ &= (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f\left(q_k^n t + \left(1-q_k^n\right)t_k\right) \\ &- (1-q_k)(a-t_k)\sum_{n=0}^{\infty} q_k^n f\left(q_k^n a + \left(1-q_k^n\right)t_k\right).\end{aligned}$$

It should be noted that if $t_k = 0$ and $q_k = q$, then (2.7) reduces to $q-$integral of $f(t)$, defined by

$$\int_{0}^{t} f(s)\, d_q s = (1-q)\, t\sum_{n=0}^{\infty} q^n f(q^n t),$$

for $t \in [0, \infty)$.

**Theorem 2.17** ( [24]). *For $t \in J_k$, following formulas hold:*

*(i)* $D_{q_k}\int_{t_k}^{t} f(s)\, d_{q_k}s = f(t)$;

*(ii)* $\int_{t_k}^{t} D_{q_k} f(s)\, d_{q_k}s = f(t)$;

*(iii)* $\int_{a}^{t} D_{q_k} f(s)\, d_{q_k}s = f(t) - f(a)$ *for* $a \in (t_k, t)$;

*(iv)* $\int_{t_k}^{t} [f(s) + g(s)]\, d_{q_k}s = \int_{t_k}^{t} f(s)\, d_{q_k}s + \int_{t_k}^{t} g(s)\, d_{q_k}s$;

*(v)* $\int_{t_k}^{t} (\alpha f)(s)\, d_{q_k}s = \alpha\int_{t_k}^{t} f(s)\, d_{q_k}s$;

*(vi)* $\int_{t_k}^{t} f(s)\, D_{q_k} g(s)\, d_{q_k}s = (fg)(t) - \int_{t_k}^{t} g(q_k s + (1-q_k)t_k)\, D_{q_k} f(s)\, d_{q_k}s$,

*where* $f, g : J_k \to \mathbb{R}$ *are continuous functions,* $\alpha \in \mathbb{R}$.

In the next section, the $q_k-$Laplace transform will be defined and its properties will be given. The Laplace transform we will obtain is the general form of the following $q-$Laplace transform.

**Definition 2.18** ( [16, 19, 20]). $q-$Laplace transform of $f$ function is defined by

$$F(s) = L_q(f(t)) = \int_{0}^{\infty} E_q(-qst)\, f(t)\, d_q t \ \ (s > 0).$$

Then,

$$L_q(\alpha f(t) + \beta g(t)) = \alpha L_q(f(t)) + \beta L_q(g(t)),$$

where $\alpha$ and $\beta$ are constants.

**Definition 2.19** ( [16, 19, 20]). The $q-$extension of gamma function is defined by

$$\Gamma_q(t) = \int_{0}^{\infty} x^{t-1} E_q(-qx)\, d_q x \ \ (t > 1).$$

Thus,

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t), \; \Gamma_q(n+1) = [n]_q!.$$

## 3. Main Results

In the present section, we will define $q_k-$Laplace transform, $q_k-$binomial coefficients. Furthermore, some properties that are essential for comprehending $q_k-$ Laplace transform will be expressed. Finally, we will calculate $q_k-$Laplace transform of some common functions. First, let's calculate the $q_k-$ derivatives of $q_k-$ analogous of exponential functions that we will use in our main proofs.

**Lemma 3.1.** *For $s > 0$, $t, t_k \in J_k$,*

$$D_{q_k} E_{q_k}^{-s(t-t_k)} = -s E_{q_k}^{-q_k s(t-t_k)}.$$

*Proof.* By using $q_k-$derivative and (2.4), we get:

$$\begin{aligned}
D_{q_k} E_{q_k}^{-s(t-t_k)} =& D_{q_k}\left(\sum_{n=0}^{\infty} q_k^{\frac{n(n-1)}{2}} \frac{(-s(t-t_k))^n}{[n]_{q_k}!}\right)\\
=& D_{q_k}\left(1 + \sum_{n=1}^{\infty} \frac{q_k^{\frac{n(n-1)}{2}}(-1)^n}{[n]_{q_k}!} s^n (t-t_k)^n\right)\\
=& \sum_{n=1}^{\infty} \frac{q_k^{\frac{n(n-1)}{2}}(-1)^n}{[n]_{q_k}!} D_{q_k}(t-t_k)^n\\
=& \sum_{n=1}^{\infty} \frac{q_k^{\frac{n(n-1)}{2}}(-1)^n s^n (t-t_k)^{n-1}}{[n-1]_{q_k}!}\\
=& \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q_k^{\frac{n(n+1)}{2}} s^{n+1}(t-t_k)^n}{[n]_{q_k}!}\\
=& -s\sum_{n=0}^{\infty} \frac{(-1)^n q_k^{\frac{n(n-1)}{2}}}{[n]_{q_k}!} (q_k(t-t_k)s)^n\\
=& -sE_{q_k}^{-q_k s(t-t_k)}.
\end{aligned}$$

□

**Lemma 3.2.** *For $s > 0$, $t, t_k \in J_k$,*

$$D_{q_k} E_{q_k}^{-sq_k(t-t_k)} = -q_k s E_{q_k}^{-q_k^2 s(t-t_k)}.$$

*Proof.*

$$\begin{aligned}
D_{q_k} E_{q_k}^{-sq_k(t-t_k)} =& D_{q_k}\left(\sum_{n=0}^{\infty} q_k^{\frac{n(n-1)}{2}} \frac{(-s(t-t_k))^n}{[n]_{q_k}!}\right)\\
=& \sum_{n=1}^{\infty} \frac{q_k^{n(n-1)}(-1)^n s^n q_k^n D_{q_k}(t-t_k)^n}{[n]_{q_k}!}\\
=& \sum_{n=1}^{\infty} \frac{q_k^{n(n-1)}(-1)^n s^n q_k^n (t-t_k)^{n-1}}{[n-1]_{q_k}!}\\
=& \sum_{n=0}^{\infty} \frac{q_k^{\frac{n(n+1)}{2}}(-1)^{n+1} s^{n+1} q_k^{n+1}(t-t_k)^n}{[n]_{q_k}!}
\end{aligned}$$

$$= -q_k s \sum \frac{q_k^{\frac{n(n-1)}{2}} q_k^{2n} (-1)^n s^n (t-t_k)^n}{[n]_{q_k}!}$$
$$= -q_k s E_{q_k}^{-q_k^2 s(t-t_k)}.$$

□

**Definition 3.3.** $q_k-$Laplace transformation of $f$ is defined by

$$F(s) = L_{q_k}(f(t)) = \int_0^\infty f(t) E_{q_k}^{-q_k s(t-t_k)} d_{q_k} t.$$

Then,

$$L_{q_k}(\alpha f(t) + \beta g(t)) = \alpha L_{q_k}(f(t)) + \beta L_{q_k}(f(t)),$$

where $\alpha$ and $\beta$ are constants.

**Definition 3.4.** $q_k-$binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q_k} = \frac{[n]_{q_k}!}{[k]_{q_k}! [n-k]_{q_k}!},$$

where $[n]_{q_k}! = [n]_{q_k} [n-1]_{q_k} [n-2]_{q_k} ...[2][1]$, $[0]_{q_k}! = 1$ and $[n]_{q_k} = \frac{1-q_k^n}{1-q_k}$.

**Lemma 3.5.** *$q_k-$Laplace transform of $f(t) = 1$ is*

$$L_{q_k}(1) = F(s) = \frac{1}{s} E_{q_k}^{st_k}. \tag{3.1}$$

*Proof.* From the definition of $q_k-$Laplace transform, it follows that

$$F(s) = L_{q_k}(1) = \int_0^\infty E_{q_k}^{-q_k s(t-t_k)} d_{q_k} t = \lim_{\alpha\to\infty} \int_0^\alpha E_{q_k}^{-q_k s(t-t_k)} d_{q_k} t.$$

By Lemma 3.2,

$$\int_0^\alpha E_{q_k}^{-q_k s(t-t_k)} d_{q_k} t = \int_0^\alpha \frac{D_{q_k} E_{q_k}^{-s(t-t_k)}}{-s} d_{q_k} t = -\frac{1}{s} \int_0^\alpha D_{q_k} E_{q_k}^{-s(t-t_k)} d_{q_k} t$$
$$= -\frac{1}{s} \left( E_{q_k}^{-s(t-t_k)} \right) \Big|_0^\alpha = -\frac{1}{s} \left( E_{q_k}^{-s(\alpha-t_k)} - E_{q_k}^{st_k} \right).$$

Taking limit when $\alpha \to \infty$, we get

$$\lim_{\alpha\to\infty} \left( -\frac{1}{s} \left( E_{q_k}^{-s(\alpha-t_k)} - E_{q_k}^{st_k} \right) \right) = -\frac{1}{s} \left( 0 - E_{q_k}^{st_k} \right) = \frac{1}{s} E_{q_k}^{st_k}.$$

□

If we take $q_k = q$ and $t_k = 0$, then (3.1) reduce to $L_q(1) = \frac{1}{s}$ that is defined in [16].

**Theorem 3.6.** *For $n \in \mathbb{R}$ with $n > -1$, $q_k-$Laplace transform $f(t) = (t-t_k)^n$ is*

$$L_{q_k}((t-t_k)^n) = \frac{(-1)^n}{s} (t_k)^n E_{q_k}^{st_k} + \frac{[n]_{q_k}}{s} L_{q_k}\left( (t-t_k)^{n-1} \right). \tag{3.2}$$

*Proof.* From definiton of $q_k-$Laplace transform, it follows that

$$L_{q_k}((t-t_k)^n) = \lim_{\alpha\to\infty} \int_0^\alpha (t-t_k)^n E_{q_k}^{-q_k s(t-t_k)} d_{q_k} t = \int_0^\infty (t-t_k)^n E_{q_k}^{-q_k s(t-t_k)} d_{q_k} t.$$

By Lemma 3.2 and Theorem 2.17,

$$\int_0^{\alpha} (t-t_k)^n E_{q_k}^{-q_k s(t-t_k)} d_{q_k}t = \int_0^{\alpha} (t-t_k)^n \frac{D_{q_k} E_{q_k}^{-s(t-t_k)}}{-s} d_{q_k}t$$
$$= -\frac{1}{s}\int_0^{\alpha} (t-t_k)^n D_{q_k} E_{q_k}^{-s(t-t_k)} d_{q_k}t$$
$$= -\frac{1}{s}\left[\left((t-t_k)^n E_{q_k}^{-s(t-t_k)}\right)\Big|_0^{\alpha} - \int_0^{\alpha} D_{q_k}(t-t_k)^n E_{q_k}^{-s(q_k t+(1-q_k)t_k-t_k)} d_{q_k}t\right]$$
$$= -\frac{1}{s}\left((t-t_k)^n E_{q_k}^{-s(t-t_k)}\right)\Big|_0^{\alpha} + \frac{1}{s}\int_0^{\alpha} D_{q_k}(t-t_k)^n E_{q_k}^{-q_k s(t-t_k)} d_{q_k}t$$
$$= -\frac{1}{s}\left((\alpha-t_k)^n E_{q_k}^{-s(\alpha-t_k)} - (-t_k)^n E_{q_k}^{st_k}\right) + \frac{[n]_{q_k}}{s}\int_0^{\alpha} (t-t_k)^{n-1} E_{q_k}^{-q_k s(t-t_k)} d_{q_k}t.$$

Taking limit when $\alpha \to \infty$, we get

$$L_{q_k}\left((t-t_k)^n\right) = \lim_{\alpha\to\infty}\int_0^{\alpha} (t-t_k)^n E_{q_k}^{-q_k s(t-t_k)} d_{q_k}t$$
$$= \lim_{\alpha\to\infty}\left[-\frac{1}{s}(\alpha-t_k)^n + E_{q_k}^{-s(\alpha-t_k)} + \frac{1}{s}(-1)^n (t_k) E_{q_k}^{st_k} + \frac{[n]_{q_k}}{s}\int_0^{\alpha} (t-t_k)^{n-1} E_{q_k}^{-q_k s(t-t_k)} d_{q_k}t\right]$$
$$= \frac{(-1)^n}{s}(t_k)^n E_{q_k}^{st_k} + \frac{[n]_{q_k}}{s}\int_0^{\infty} (t-t_k)^{n-1} E_{q_k}^{-q_k s(t-t_k)} d_{q_k}t$$
$$= \frac{(-1)^n}{s}(t_k)^n E_{q_k}^{st_k} + \frac{[n]_{q_k}}{s} L_{q_k}\left((t-t_k)^{n-1}\right).$$

□

**Theorem 3.7.** *Let $n \in \mathbb{N}$, then $q_k-$Laplace transform $f(t) = (t-t_k)^n$ is*

$$L_{q_k}\left((t-t_k)^n\right) = \sum_{i=0}^{n-1} \frac{(-1)^{n-i}}{s^{i+1}} (t_k)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}_{q_k} \left([i]_{q_k}\right)! E_{q_k}^{st_k} + \frac{[n]_{q_k}!}{s^{n+1}} E_{q_k.}^{st_k} \tag{3.3}$$

*Proof.* By using (3.2), it follows that

$$L_{q_k}\left((t-t_k)^n\right) = \frac{(-1)^n}{s}(t_k)^n E_{q_k}^{st_k} + \frac{[n]_{q_k}}{s} L_{q_k}\left((t-t_k)^{n-1}\right)$$
$$= \frac{(-1)^n}{s}(t_k)^n E_{q_k}^{st_k} + \frac{[n]_{q_k}}{s}\left\{\frac{(-1)^{n-1}}{s}(t_k)^{n-1} E_{q_k}^{st_k} + \frac{[n-1]_{q_k}}{s} L_{q_k}\left((t-t_k)^{n-2}\right)\right\}$$
$$= \frac{(-1)^n}{s}(t_k)^n E_{q_k}^{st_k} + \frac{[n]_{q_k}}{s}\frac{(-1)^{n-1}}{s}(t_k)^{n-1} E_{q_k}^{st_k} + \frac{[n]_{q_k}}{s}\frac{[n-1]_{q_k}}{s} L_{q_k}\left((t-t_k)^{n-2}\right)$$
$$= \frac{(-1)^n}{s}(t_k)^n E_{q_k}^{st_k} + \frac{[n]_{q_k}}{s}\frac{(-1)^{n-1}}{s}(t_k)^{n-1} E_{q_k}^{st_k}$$
$$+ \frac{[n]_{q_k}}{s}\frac{[n-1]_{q_k}}{s}\left\{\frac{(-1)^{n-2}}{s}(t_k)^{n-2} E_{q_k}^{st_k} + \frac{[n-2]_{q_k}}{s} L_{q_k}\left((t-t_k)^{n-3}\right)\right\}$$

$$=\frac{(-1)^n}{s}(t_k)^n E_{q_k}^{st_k}+\frac{[n]_{q_k}}{s}\frac{(-1)^{n-1}}{s}(t_k)^{n-1}E_{q_k}^{st_k}+\frac{[n]_{q_k}}{s}\frac{[n-1]_{q_k}}{s}\frac{(-1)^{n-2}}{s}(t_k)^{n-2}E_{q_k}^{st_k}$$
$$+\frac{[n]_{q_k}}{s}\frac{[n-1]_{q_k}}{s}\frac{[n-2]_{q_k}}{s}L_{q_k}\left((t-t_k)^{n-3}\right)$$
$$\vdots$$
$$=\frac{(-1)^n}{s}(t_k)^n E_{q_k}^{st_k}+\frac{[n]_{q_k}}{s}\frac{(-1)^{n-1}}{s}(t_k)^{n-1}E_{q_k}^{st_k}$$
$$+\frac{[n]_{q_k}}{s}\frac{[n-1]_{q_k}}{s}\frac{(-1)^{n-2}}{s}(t_k)^{n-2}E_{q_k}^{st_k}+...+\frac{[n]_{q_k}!}{s^{n+1}}E_{q_k}^{st_k}$$
$$=\sum_{i=0}^{n-1}\frac{(-1)^{n-i}}{s^{i+1}}(t_k)^{n-i}\left[\begin{array}{c}n\\ i\end{array}\right]_{q_k}\left([i]_{q_k}\right)!.E_{q_k}^{st_k}+\frac{[n]_{q_k}!}{s^{n+1}}E_{q_k.}^{st_k}$$

□

If we take $q_k = q$ and $t_k = 0$, then (3.3) reduce to $L_q(t^n) = \frac{[n]_q!}{s^{n+1}}$ that is defined in [16].

**Theorem 3.8.** *$q_k$−Laplace transform of $f(t) = e_{q_k}^{a(t-t_k)}$ is*

$$L_{q_k}\left(e_{q_k}^{a(t-t_k)}\right)=\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{a^n(-1)^{n-j}(t_k)^{n-j}}{[n]_{q_k}!}+\frac{1}{s-a}E_{q_k}^{st_k},\qquad s>a. \tag{3.4}$$

*Proof.* From definition of $q_k$−Laplace transform, it follows that

$$L_{q_k}\left(e_{q_k}^{a(t-t_k)}\right)=\int_0^{\infty}e_{q_k}^{a(t-t_k)}E_{q_k}^{-q_k s(t-t_k)}d_{q_k}t$$
$$=\sum_{n=0}^{\infty}\frac{a^n}{[n]_{q_k}!}\int_0^{\infty}(t-t_k)^n E_{q_k}^{-q_k s(t-t_k)}d_{q_k}t$$
$$=\sum_{n=0}^{\infty}\frac{a^n}{[n]_{q_k}!}L_{q_k}\left((t-t_k)^n\right)$$
$$=\sum_{n=0}^{\infty}\frac{a^n}{[n]_{q_k}!}\left(\frac{(-1)^n}{s}(t_k)^n E_{q_k}^{st_k}+\frac{[n]_{q_k}}{s}\frac{(-1)^{n-1}}{s}(t_k)^{n-1}E_{q_k}^{st_k}+...+\frac{[n]_{q_k}!}{s^{n+1}}E_{q_k}^{st_k}\right)$$
$$=\frac{1}{s}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{a^n}{[n]_{q_k}!}(-1)^n(t_k)^n+\frac{1}{s^2}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{a^n}{[n]_{q_k}!}(-1)^{n-1}(t_k)^{n-1}$$
$$+...+\frac{1}{s^n}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{a^n}{[n]_{q_k}!}+\frac{1}{s-a}E_{q_k}^{st_k}$$
$$=\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{a^n(-1)^{n-j}(t_k)^{n-j}}{[n]_{q_k}!}+\frac{1}{s-a}E_{q_k}^{st_k}.$$

□

If we take $q_k = q$ and $t_k = 0$, then (3.4) reduce to $L_q(e_q(at)) = \frac{1}{s-a}$, $s > a$ that is defined in [16].

**Theorem 3.9.** *$q_k$−Laplace transform of $f(t) = E_{q_k}^{a(t-t_k)}$ is*

$$L_{q_k}\left(E_{q_k}^{a(t-t_k)}\right)=\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}\frac{a^n}{[n]_{q_k}!}q_k^{\frac{n(n-1)}{2}}(-1)^{n-j}(t_k)^{n-j}E_{q_k}^{st_k}+\frac{1}{s}\sum_{n=0}^{\infty}\frac{a^n}{s^n}E_{q_k}^{st_k}q_k^{\frac{n(n-1)}{2}},\quad s>0\ .$$

*Proof.* From definition of $q_k-$Laplace transform, it follows that

$$
\begin{aligned}
L_{q_k}\left(E_{q_k}^{a(t-t_k)}\right) &= \int_0^\infty E_q^{a(t-t_k)} E_q^{-q_k s(t-t_k)} d_{q_k}t \\
&= \int_0^\infty E_q^{a(t-t_k)} \sum_{n=0}^{\infty} q_k^{\frac{n(n-1)}{2}} \frac{a^n (t-t_k)^n}{[n]_{q_k}!} d_{q_k}t \\
&= \sum_{n=0}^{\infty} \frac{a^n}{[n]_{q_k}!} q_k^{\frac{n(n-1)}{2}} \int_0^\infty E_{q_k}^{-q_k s(t-t_k)} (t-t_k)^n d_{q_k}t \\
&= \sum_{n=0}^{\infty} \frac{a^n}{[n]_{q_k}!} q_k^{\frac{n(n-1)}{2}} \Big( \frac{(-1)^n}{s} (t_k)^n E_{q_k}^{st_k} + \frac{[n]_{q_k}}{s} \frac{(-1)^{n-1}}{s} (t_k)^{n-1} E_{q_k}^{st_k} \\
&+ \frac{[n]_{q_k}}{s} \frac{[n-1]_{q_k}}{s} \frac{(-1)^{n-2}}{s} (t_k)^{n-2} E_{q_k}^{st_k} + \ldots + \frac{[n]_{q_k}!}{s^{n+1}} E_{q_k}^{st_k} \Big) \\
&= \sum_{n=0}^{\infty} \frac{a^n}{[n]_{q_k}!} q_k^{\frac{n(n-1)}{2}} \frac{(-1)^n}{s} (t_k)^n E_{q_k}^{st_k} + \sum_{n=0}^{\infty} \frac{a^n}{[n]_{q_k}!} q_k^{\frac{n(n-1)}{2}} \frac{[n]_{q_k}}{s} \frac{(-1)^{n-1}}{s} (t_k)^{n-1} E_{q_k}^{s} \\
&+ \sum_{n=0}^{\infty} \frac{a^n}{[n]_{q_k}!} q_k^{\frac{n(n-1)}{2}} \frac{[n]_{q_k}}{s} \frac{[n-1]_{q_k}}{s} \frac{(-1)^{n-2}}{s} (t_k)^{n-2} E_{q_k}^{st_k} \\
&+ \ldots + \sum_{n=0}^{\infty} \frac{a^n}{[n]_{q_k}!} \frac{[n]_{q_k}!}{s^{n+1}} E_{q_k}^{st_k} q_k^{\frac{n(n-1)}{2}} \\
&= \frac{1}{s} \sum_{n=0}^{\infty} \frac{a^n}{[n]_{q_k}!} q_k^{\frac{n(n-1)}{2}} (-1)^n (t_k)^n E_{q_k}^{st_k} + \frac{1}{s^2} \frac{a^n}{[n]_{q_k}!} q_k^{\frac{n(n-1)}{2}} (-1)^{n-1} (t_k)^{n-1} E_{q_k}^{st_k} \\
&+ \frac{1}{s^3} \sum_{n=0}^{\infty} \frac{a^n}{[n]_{q_k}!} q_k^{\frac{n(n-1)}{2}} (-1)^{n-2} (t_k)^{n-2} E_{q_k}^{st_k} + \ldots + \sum_{n=0}^{\infty} \frac{a^n}{s^{n+1}} E_{q_k}^{st_k} q_k^{\frac{n(n-1)}{2}} \\
&= \sum_{j=0}^{n-1} \frac{1}{s^{j+1}} \sum_{n=0}^{\infty} \frac{a^n}{[n]_{q_k}!} q_k^{\frac{n(n-1)}{2}} (-1)^{n-j} (t_k)^{n-j} E_{q_k}^{st_k} + \frac{1}{s} \sum_{n=0}^{\infty} \frac{a^n}{s^n} E_{q_k}^{st_k} q_k^{\frac{n(n-1)}{2}}.
\end{aligned}
$$

□

**Theorem 3.10.** *$q_k-$Laplace transform of $q_k-$cosine, $q_k-$sine, $q_k-$Cosine, $q_k-$Sine functions are*

$$L_{q_k}\left\{\cos_{q_k} a(t-t_k)\right\} = \sum_{j=0}^{n-1} \frac{1}{s^{j+1}} \sum_{n=0}^{\infty} E_{q_k}^{st_k} \frac{a^{2n} (-1)^{3n-j} (t_k)^{2n-j}}{[2n]_{q_k}!} + E_{q_k}^{st_k} \frac{s}{s^2+a^2}, \tag{3.5}$$

$$L_{q_k}\left\{\sin_{q_k} a(t-t_k)\right\} = \sum_{j=0}^{n-1} \frac{1}{s^{j+1}} \sum_{n=0}^{\infty} E_{q_k}^{st_k} \frac{a^{2n+1} (-1)^{3n+1-j} (t_k)^{2n+1-j}}{[2n+1]_{q_k}!} + E_{q_k}^{st_k} \frac{a}{s^2+a^2}, \tag{3.6}$$

$$
\begin{aligned}
L_{q_k}\left\{Cos_{q_k} a(t-t_k)\right\} &= L_{q_k}\left\{\frac{E_{q_k}^{ia(t-t_k)} + E_{q_k}^{-ia(t-t_k)}}{2}\right\} \\
&= \sum_{j=0}^{n-1} \frac{1}{s^{j+1}} \sum_{n=0}^{\infty} \frac{a^{2n} q_k^{n(2n-1)} (-1)^{3n-j} (t_k)^{2n-j} E_{q_k}^{st_k}}{[2n]_{q_k}!} \\
&+ \frac{1}{s} \sum_{n=0}^{\infty} E_{q_k}^{st_k} q_k^{n(2n-1)} (-1)^n \left(\frac{a}{s}\right)^{2n},
\end{aligned}
$$

$$\begin{aligned} L_{q_k}\left\{S\,in_{q_k}a\,(t-t_k)\right\} &= L_{q_k}\left\{\frac{E_{q_k}^{ia(t-t_k)}-E_{q_k}^{-ia(t-t_k)}}{2i}\right\} \\ &= \sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}\frac{a^{2n+1}q_k^{n(2n+1)}(-1)^{3n+1-j}(t_k)^{2n+1-j}E_{q_k}^{st_k}}{[2n+1]_{q_k}!} \\ &+\frac{1}{s}\sum_{n=0}^{\infty}E_{q_k}^{st_k}q_k^{n(2n+1)}(-1)^n\left(\frac{a}{s}\right)^{2n+1}. \end{aligned}$$

*Proof.* Consider following definitions of $q_k-$cosine, $q_k-$sine, $q_k-$Cosine, $q_k-$Sine functions:

$$\begin{aligned} \cos_{q_k}a\,(t-t_k) &= \tfrac{e_{q_k}^{ia(t-t_k)}+e_{q_k}^{-ia(t-t_k)}}{2} \quad \text{and} \quad \sin_{q_k}a\,(t-t_k) = \tfrac{e_{q_k}^{ia(t-t_k)}-e_{q_k}^{-ia(t-t_k)}}{2i}, \\ \mathrm{Cos}_{q_k}a\,(t-t_k) &= \tfrac{E_{q_k}^{ia(t-t_k)}+E_{q_k}^{-ia(t-t_k)}}{2} \quad \text{and} \quad \mathrm{Sin}_{q_k}a\,(t-t_k) = \tfrac{E_{q_k}^{ia(t-t_k)}-E_{q_k}^{-ia(t-t_k)}}{2i}. \end{aligned}$$

Then, by using linearity of $q_k-$Laplace transform,

$$\begin{aligned} L_{q_k}\left\{\cos_{q_k}a\,(t-t_k)\right\} &= L_{q_k}\left\{\frac{e_{q_k}^{ia(t-t_k)}+e_{q_k}^{-ia(t-t_k)}}{2}\right\} \\ &= \frac{1}{2}\left(L_{q_k}\left\{e_{q_k}^{ia(t-t_k)}\right\}+L_{q_k}\left\{e_{q_k}^{-ia(t-t_k)}\right\}\right) \\ &= \frac{1}{2}(\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{(ia)^n(-1)^{n-j}(t_k)^{n-j}}{[n]_{q_k}!}+\frac{1}{s-ia}E_{q_k}^{st_k} \\ &+\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{(-ia)^n(-1)^{n-j}(t_k)^{n-j}}{[n]_{q_k}!}+\frac{1}{s+ia}E_{q_k}^{st_k}) \\ &= \frac{1}{2}\left(\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{[1+(-1)^n](ia)^n(-1)^{n-j}(t_k)^{n-j}}{[n]_{q_k}!}\right) \\ &+\frac{1}{2}\left(\frac{1}{s-ia}E_{q_k}^{st_k}+\frac{1}{s+ia}E_{q_k}^{st_k}\right) \\ &= \sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{a^{2n}(-1)^{3n-j}(t_k)^{2n-j}}{[2n]_{q_k}!}+E_{q_k}^{st_k}\frac{s}{s^2+a^2}. \end{aligned}$$

and in the same manner we have

$$L_{q_k}\left\{\sin_{q_k}a\,(t-t_k)\right\} = \sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{a^{2n+1}(-1)^{3n+1-j}(t_k)^{2n+1-j}}{[2n+1]_{q_k}!}+E_{q_k}^{st_k}\frac{a}{s^2+a^2}.$$

If we take $q_k = q$ and $t_k = 0$, then (3.5) and (3.6) reduce to $L_q(\cos_q(at)) = \frac{s}{s^2+a^2}$ and $L_q(\sin_q(at)) = \frac{a}{s^2+a^2}$ that are defined in [16], respectively.

Now, we achieve $q_k-$Laplace transform of $q_k-$Cosine and $q_k-$Sine functions

$$
\begin{aligned}
L_{q_k}\left\{\mathrm{Cos}_{q_k} a\,(t-t_k)\right\} &= L_{q_k}\left\{\frac{E_{q_k}^{ia(t-t_k)}+E_{q_k}^{-ia(t-t_k)}}{2}\right\} \\
&= \frac{1}{2}\left(L_{q_k}\left\{E_{q_k}^{ia(t-t_k)}\right\}+L_{q_k}\left\{E_{q_k}^{-ia(t-t_k)}\right\}\right) \\
&= \frac{1}{2}\left(\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}\frac{(ia)^n\, q_k^{\frac{n(n-1)}{2}}\,(-1)^{n-j}\,(t_k)^{n-j}\,E_{q_k}^{st_k}}{[n]_{q_k}!}\right. \\
&+ \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{ia}{s}\right)^n E_{q_k}^{st_k} q_k^{\frac{n(n-1)}{2}} \\
&+ \sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}\frac{(-ia)^n\, q_k^{\frac{n(n-1)}{2}}\,(-1)^{n-j}\,(t_k)^{n-j}\,E_{q_k}^{st_k}}{[n]_{q_k}!} \\
&\left.+\frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{ia}{s}\right)^n E_{q_k}^{st_k} q_k^{\frac{n(n-1)}{2}}\right) \\
&= \frac{1}{2}\left(\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}\frac{\left[1+(-1)^n\right](ia)^n\, q_k^{\frac{n(n-1)}{2}}\,(-1)^{n-j}\,(t_k)^{n-j}\,E_{q_k}^{st_k}}{[n]_{q_k}!}\right) \\
&+ \frac{1}{2}\left(\frac{1}{s}\sum_{n=0}^{\infty}\left[1+(-1)^n\right]\left(\frac{ia}{s}\right)^n E_{q_k}^{st_k} q_k^{\frac{n(n-1)}{2}}\right) \\
&= \sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}\frac{a^{2n} q_k^{n(2n-1)}\,(-1)^{3n-j}\,(t_k)^{2n-j}\,E_{q_k}^{st_k}}{[2n]_{q_k}!} \\
&+ \frac{1}{s}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{s}\right)^{2n} E_{q_k}^{st_k} q_k^{\frac{n(n-1)}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
L_{q_k}\left\{\mathrm{Sin}_{q_k} a\,(t-t_k)\right\} &= L_{q_k}\left\{\frac{E_{q_k}^{ia(t-t_k)}-E_{q_k}^{-ia(t-t_k)}}{2i}\right\} \\
&= \frac{1}{2i}\left(L_{q_k}\left\{E_{q_k}^{ia(t-t_k)}\right\}-L_{q_k}\left\{E_{q_k}^{-ia(t-t_k)}\right\}\right) \\
&= \frac{1}{2i}\left(\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}\frac{(ia)^n\, q_k^{\frac{n(n-1)}{2}}\,(-1)^{n-j}\,(t_k)^{n-j}\,E_{q_k}^{st_k}}{}\right. \\
&+ \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{ia}{s}\right)^n E_{q_k}^{st_k} q_k^{\frac{n(n-1)}{2}} \\
&- \sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}\frac{(-ia)^n\, q_k^{\frac{n(n-1)}{2}}\,(-1)^{n-j}\,(t_k)^{n-j}\,E_{q_k}^{st_k}}{[n]_{q_k}!} \\
&\left.-\frac{1}{s}\sum_{n=0}^{\infty}\left(-\frac{ia}{s}\right)^n E_{q_k}^{st_k} q_k^{\frac{n(n-1)}{2}}\right)
\end{aligned}
$$

$$= \frac{1}{2i}\left(\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}\frac{\left[1-(-1)^n\right](ia)^n q_k^{\frac{n(n-1)}{2}}(-1)^{n-j}(t_k)^{n-j}E_{q_k}^{st_k}}{[n]_{q_k}!}\right)$$
$$-\frac{1}{2i}\left(\frac{1}{s}\sum_{n=0}^{\infty}\left[1-(-1)^n\right]\left(\frac{ia}{s}\right)^n E_{q_k}^{st_k}q_k^{\frac{n(n-1)}{2}}\right)$$
$$=\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}\frac{a^{2n+1}q_k^{(2n+1)n}(-1)^{3n+1-j}(t_k)^{2n+1-j}E_{q_k}^{st_k}}{[2n+1]_{q_k}!}$$
$$+\frac{1}{s}\sum_{n=0}^{\infty}E_{q_k}^{st_k}q_k^{n(2n+1)}(-1)^n\left(\frac{a}{s}\right)^{2n+1}.$$

So, the proof is completed. □

**Theorem 3.11.** *$q_k$−Laplace transform of hyperbolic q−cosine, hyperbolic q−sine functions are*

$$L_{q_k}\left\{\cosh_{q_k} a(t-t_k)\right\} = \sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{a^{2n}(-1)^{2n-j}(t_k)^{2n-j}}{[2n]_{q_k}!} + E_{q_k}^{st_k}\frac{s}{s^2-a^2}, \tag{3.7}$$

$$L_{q_k}\left\{\sinh_{q_k} a(t-t_k)\right\} = \sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{a^{2n+1}(-1)^{2n+1-j}(t_k)^{2n+1-j}}{[2n+1]_{q_k}!} + E_{q_k}^{st_k}\frac{a}{s^2-a^2}. \tag{3.8}$$

*Proof.* Hyperbolic $q_k$−cosine, hyperbolic $q_k$−sine functions are defined by

$$\cosh_{q_k} a(t-t_k) = \frac{e_{q_k}^{a(t-t_k)}+e_{q_k}^{-a(t-t_k)}}{2},$$
$$\sinh_{q_k} a(t-t_k) = \frac{e_{q_k}^{a(t-t_k)}-e_{q_k}^{-a(t-t_k)}}{2}.$$

Then, by using linearity of $q_k$−Laplace transform,

$$L_{q_k}\left\{\cosh_{q_k} a(t-t_k)\right\} = L_{q_k}\left\{\frac{e_{q_k}^{a(t-t_k)}+e_{q_k}^{-a(t-t_k)}}{2}\right\}$$
$$=\frac{1}{2}\left(L_{q_k}\left\{e_{q_k}^{a(t-t_k)}\right\}+L_{q_k}\left\{e_{q_k}^{a(t-t_k)}\right\}\right)$$
$$=\frac{1}{2}\left(\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{a^n(-1)^{n-j}(t_k)^{n-j}}{[n]_{q_k}!}\right.$$
$$\left.+\frac{1}{s-a}E_{q_k}^{st_k}+\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{(-a)^n(-1)^{n-j}(t_k)^{n-j}}{[n]_{q_k}!}+\frac{1}{s+a}E_{q_k}^{st_k}\right)$$
$$=\frac{1}{2}\left(\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{\left[1+(-1)^n\right]a^n(-1)^{n-j}(t_k)^{n-j}}{[n]_{q_k}!}\right)$$
$$+\frac{1}{2}\left(\frac{1}{s-a}E_{q_k}^{st_k}+\frac{1}{s+a}E_{q_k}^{st_k}\right)$$
$$=\sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{a^{2n}(-1)^{2n-j}(t_k)^{2n-j}}{[2n]_{q_k}!}+E_{q_k}^{st_k}\frac{s}{s^2-a^2}$$

and in the same manner we get

$$L_{q_k}\left\{\sinh_{q_k} a\left(t-t_k\right)\right\} = \sum_{j=0}^{n-1}\frac{1}{s^{j+1}}\sum_{n=0}^{\infty}E_{q_k}^{st_k}\frac{a^{2n+1}\left(-1\right)^{2n+1-j}\left(t_k\right)^{2n+1-j}}{[2n+1]_{q_k}!} + E_{q_k}^{st_k}\frac{a}{s^2-a^2}.$$

□

If we take $q_k = q$ and $t_k = 0$, then (3.7) and (3.8) reduce to $L_q(\cosh_{q_k}(at)) = \frac{s}{s^2-a^2}$ and $L_q(\sin_q(at)) = \frac{a}{s^2-a^2}$ that are defined in [16], respectively.

**Theorem 3.12.** *If $f$, $D_q f$, $D_q^2 f$,...,$D_q^{n-1} f$ are continuous and $D_q^n f$ is piecewise continuous on $(0,\infty)$ and are of exponential order, then we have*

$$L_{q_k}\left\{D_{q_k}^n f(t)\right\} = s^n L_{q_k}\{f(t)\} - E_{q_k}^{q_k t_k s}\sum_{i=0}^{n-1}s^{n-1-i}D_{q_k}^i f(0). \tag{3.9}$$

*Proof.* $f$ is exponential order $c$ if there exist $c$, $K>0$ and $T>0$ such that

$$|f(t)| < Ke^{ct},\ \text{for all } t<T.$$

Therefore, we have

$$\lim_{t\to\infty}E_{q_k}^{-q_k s(t-t_k)}f(t) = 0. \tag{3.10}$$

Then, by using (3.10) we can write

$$\begin{aligned}
L_{q_k}\left\{D_{q_k}f(t)\right\} &= \int_0^\infty E_{q_k}^{-q_k s(t-t_k)}D_{q_k}f(t)\,d_{q_k}t \\
&= E_{q_k}^{-q_k s(t-t_k)}f(t)\Big|_0^\infty + q_k s\int_0^\infty f\left(q_k t+(1-q_k)t_k\right)E_{q_k}^{-q_k^2 s(t-t_k)}d_{q_k}t \\
&= E_{q_k}^{-q_k s(t-t_k)}f(t)\Big|_0^\infty + s\int_0^\infty q_k f\left(q_k t+(1-q_k)t_k\right)E_{q_k}^{-q_k^2 s(t-t_k)}d_{q_k}t \\
&= E_{q_k}^{-q_k s(t-t_k)}f(t)\Big|_0^\infty + s\int_0^\infty f(u)\,E_{q_k}^{-q_k s(u-t_k)}d_{q_k}u \\
&= \lim_{t\to\infty}E_{q_k}^{-q_k s(t-t_k)}f(t) - E_{q_k}^{-q_k s(0-t_k)}f(0) + sL_{q_k}\{f(t)\} \\
&= 0 - E_{q_k}^{q_k st_k}f(0) + sL_{q_k}\{f(t)\} \\
&= sL_{q_k}\{f(t)\} - E_{q_k}^{q_k st_k}f(0).
\end{aligned}$$

If we replace $f(t)$ by $D_{q_k}f(t)$ we have

$$\begin{aligned}
L_{q_k}\left\{D_{q_k}^2 f(t)\right\} &= \int_0^\infty E_{q_k}^{-q_k s(t-t_k)}D_{q_k}^2 f(t)\,d_{q_k}t \\
&= E_{q_k}^{-q_k s(t-t_k)}D_{q_k}f(t)\Big|_0^\infty + q_k s\int_0^\infty D_{q_k}f\left(q_k t+(1-q_k)t_k\right)E_{q_k}^{-q_k^2 s(t-t_k)}d_{q_k}t \\
&= E_{q_k}^{-q_k s(t-t_k)}D_{q_k}f(t)\Big|_0^\infty + s\int_0^\infty D_{q_k}f(u)\,E_{q_k}^{-q_k s(u-t_k)}d_{q_k}u
\end{aligned}$$

$$= \lim_{t\to\infty} E_{q_k}^{-q_k s(t-t_k)} D_{q_k} f(t) - E_{q_k}^{-q_k s(0-t_k)} D_{q_k} f(0) + sL_{q_k}\left\{D_{q_k} f(t)\right\}$$
$$= sL_{q_k}\left\{D_{q_k} f(t)\right\} - E_{q_k}^{q_k s t_k} D_{q_k} f(0)$$
$$= s\left\{sL_{q_k}\{f(t)\} - E_{q_k}^{q_k s t_k} f(0)\right\} - E_{q_k}^{q_k s t_k} D_{q_k} f(0).$$

If we proceed with this way, we have

$$L_{q_k}\left\{D_{q_k}^n f(t)\right\} = s^n L_{q_k}\{f(t)\} - E_{q_k}^{q_k t_k s} \sum_{i=0}^{n-1} s^{n-1-i} D_{q_k}^i f(0).$$

and proof is completed. □

If we take $q_k = q$ and $t_k = 0$, then (3.9) reduce to $L_q(D_q^n f(t)) = s^n L_q(f(t)) - \sum_{i=0}^{n-1} s^{n-1-i} D_{q_k}^i f(0)$ that is defined in [16].

## 4. Conclusions

In this study, the $q_k-$Laplace transform is defined and its properties are expressed. A generalization of the $q-$Laplace transform, whose important results were given earlier, is obtained. Some common functions are not defined in this analysis and images of these functions under the $q_k-$Laplace transform are obtained. In some special selections, it has been observed that the results coincide with the $q-$Laplace transform. Using these results, initial value problems of this type with $q_k-$derivatives can be solved.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

## Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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