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Shifted Fibonacci Numbers

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Abstract

Shifted Fibonacci numbers have been examined in the literature in terms of the greatest common divisor, but appropriate definitions and fundamental equations have not been worked on. In this article, we have obtained the Binet formula, which is a fundamental equation used to obtain the necessary element of the shifted Fibonacci number sequence. Additionally, we have obtained many well-known identities such as Cassini, Honsberger, and various other identities for this sequence. Furthermore, summation formulas for shifted Fibonacci numbers have been presented.

Kaymış Fibonacci Sayıları

Anahtar Kelimeler
Kaymış Fibonacci Sayıları; Binet Formülü; Özdeşlikler; Toplam Formülleri

Öz

Kaymış Fibonacci sayıları, literatürde, en büyük ortak bölen açısından incelenmiştir, ancak uygun tanım ve temel denklemler çalışılmamıştır. Bu makalede, kaymış Fibonacci sayı dizisinin gerekli elemanını elde etmek için kullanılan ve temel bir formül olan Binet formülünü verdik. Ayrıca, Cassini, Honsberger ve diğer birçok bilinen özdeşlikleri ve bu dizi için çok sayıda farklı özdeşlikler elde edilmiştir. Ayrıca, kaymış Fibonacci sayıları için toplama formülleri sunulmuştur.

1. Introduction

Shifted Fibonacci numbers firstly had been appeared in literature in 1973. Being one of the thousands of sequences in Sloane (1973) they attract nearly no attention. In Sloane and Plouffe (1995), which is the revised and expanded version of the book, the sequence did not get attention either. The first study focusing on the shifted Fibonacci numbers are done by Chen in (Chen 2011). In Chen (2011), the author aims to find greatest common divisors for the shifted Fibonacci numbers. There are several papers in the literature engaged with greatest common divisor and least common multiple of the sequences which involves terms of the shifted Fibonacci numbers such as (Dudley 1971, Hernandez and Luca 2003, Sanna 2020, Spilker et all. 2022).

In this paper, we give the shifted Fibonacci numbers' Binet formula and generating function. After these definitions, we give numerous identities and summation formulas for the sequence.

Some of the shifted Fibonacci numbers are defined as a sequence in 1973 as

1, 2, 2, 3, 4, 6, 9, 14, 22, 35, 56, 90, 145, 234, 378, 611, 988, 1598, 2585, 4182, 6766, 10947, 17712, 28658, 46369, 75026, 121394, 196419, 317812, 514230, 832041, …

with $f_n + 1$ where f_n is nth Fibonacci number.

As it is well known the Fibonacci numbers have the following recurrence relation for $n \geq 0$

$$f_{n+2} = f_{n+1} + f_n \tag{1}$$

where initial values of the sequence are $f_0 = 0$ and $f_1 = 1$.

To close this section, let us give the Binet formula of the Fibonacci numbers.

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2}$$

where the roots of the characteristic equation α and β are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. These roots are commonly referred as golden and silver ratios, respectively.

In the next section, we provide our main result which is to deeply study the shifted Fibonacci numbers' properties. Also, we present some identities and summation formulas for the shifted Fibonacci numbers.

2. Shifted Fibonacci Numbers

Shifted Fibonacci numbers are the generalization of Fibonacci numbers with different perspective. There are many generalization types in the literature, for k-Fibonacci numbers or h(x)-Fibonacci polynomials, see (Özkan et all. 2017, Nallı and Haukkanen 2009). Some authors choose to widen to complex field, quaternions and more (Halici and Karataş 2017, Horadam 1963). Others choose to generalize the sequence's initial values and recurrence relations (Falcon and Plaza 2007, Horadam 1961). Lately, Leonardo numbers defined which is alteration in recurrence relation and the sequence is attract attention from the community (Alp and Kocer 2021, Catarino and Borges 2019, Karataş 2022). The example papers can be extended but we confine ourselves with mentioned ones.

In order to study shifted Fibonacci numbers deeply, we start with their recurrence relation and initial values.

Definition 2.1 For $n \geq 0$, the nth shifted Fibonacci number is defined by

$$s_{n+2} = s_{n+1} + s_n - 1 \tag{3}$$

where initial values of the sequence are $s_0 = 1$ and $s_1 = 2$.

Note that using the definition (2.1), we can obtain new form of the recurrence relation as

$$s_{n+1} = 2s_n - s_{n-2} \tag{4}$$

where $n \geq 2$.

The Binet formula and generating function are invaluable tools for determining specific elements within the sequence. Due to their practical utility, they have been extensively investigated in studies concerning recurrence relations. In the following theorems we obtain both Binet formula and generating function for the shifted Fibonacci sequence.

Theorem 2.2 For $n \geq 0$,

$$s_n = \frac{\alpha^n + \alpha - \beta^n - \beta}{\alpha - \beta} \tag{5}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Proof. The proof of the theorem is very straightforward using the fact that $s_n = f_n + 1$ and Binet formula of Fibonacci numbers.

$$s_n = f_n + 1 = \frac{\alpha^n - \beta^n}{\alpha - \beta} + 1 = \frac{\alpha^n + \alpha - \beta^n - \beta}{\alpha - \beta}. \square$$

Theorem 2.3 The generating function for the shifted Fibonacci sequence is

$$g(t) = \frac{1-2t^2}{t^3 - 2t + 1} \tag{6}$$

for $t^3 - 2t + 1 \neq 0$.

Proof. The power series representation of the generating function $g(t)$ is

$$= s_0 + s_1 t + s_2 t^2 + \sum_{n=3}^{\infty} s_n t^n$$

$$= 1 + 2t + 2t^2 + \sum_{n=3}^{\infty} (2s_{n-1} - s_{n-3}) t^n$$

$$= 1 + 2t^2 + 2t \sum_{n=0}^{\infty} s_n t^n - t^3 \sum_{n=0}^{\infty} s_n t^n$$

$$= 1 - 2t^2 + 2tg(t) - t^3 g(t).$$

To obtain $g(t)$ we can rearrange the equation and get the desired result.

$g(t) = \frac{1-2t^2}{t^3-2t+1}$. □

In the following theorem, we give the Cassini identity. After that, we give the Honsberger identity for the shifted Fibonacci numbers.

Theorem 2.4 For $n \geq 1$, the Cassini identity for shifted Fibonacci numbers is

$$s_n^2 - s_{n-1}\, s_{n+1} = -(f_{n-3} - (-1)^n\,). \tag{7}$$

Proof. The proof of the identity can be made using both Binet formula and properties of the shifted Fibonacci numbers. We choose to occupy Binet formula to prove.

$$s_n^2 - s_{n-1}s_{n+1} = \left(\frac{\alpha^n + \alpha - \beta^n - \beta}{\alpha - \beta}\right)^2 - \left(\frac{\alpha^{n-1} + \alpha - \beta^{n-1} - \beta}{\alpha - \beta}\right)\left(\frac{\alpha^{n+1} + \alpha - \beta^{n+1} - \beta}{\alpha - \beta}\right).$$

Using the basic properties of golden and silver ratios such as $\alpha\beta = -1$ and $\alpha + \beta = 1$ with needed relations we have

$$s_n^2 - s_{n-1}s_{n+1} = -(f_{n-3} - (-1)^n)$$

which is desired. □

Theorem 2.5 For $k \geq 1$, the Honsberger identity for shifted Fibonacci numbers is

$$s_{k-1}s_n + s_k s_{n+1} = f_{k+n} + s_{k+1} + s_{n+2}. \tag{8}$$

Proof. Let us prove the identity with Binet formula and properties of the sequence. We can restate the identity as

$$s_{k-1}s_n - s_k s_{n+1} = \left(\frac{\alpha^{k-1} + \alpha - \beta^{k-1} - \beta}{\alpha - \beta}\right)\left(\frac{\alpha^n + \alpha - \beta^n - \beta}{\alpha - \beta}\right) + \left(\frac{\alpha^k + \alpha - \beta^k - \beta}{\alpha - \beta}\right)\left(\frac{\alpha^{n+1} + \alpha - \beta^{n+1} - \beta}{\alpha - \beta}\right).$$

Taking account that $\alpha = \frac{1+\sqrt{5}}{2}$and $\beta = \frac{1-\sqrt{5}}{2}$ the following is clear

$s_{k-1}s_n + s_k s_{n+1} = f_{k+n} + s_{k+1} + s_{n+2}$. □

As we can see from the Cassini and Honsberger identities, Fibonacci and shifted Fibonacci sequences both appear in the equations. From the ability of using Fibonacci numbers' properties in shifted ones, we will give numerous identities for them.

Theorem 2.6 For $n \geq 0$, the following identity holds true

$$s_n s_{n+3} - s_{n+1}s_{n+2} = f_n - (-1)^n. \tag{9}$$

Proof. In order to prove the identity, we use definition 2.1 and needed identities.

$$s_n s_{n+3} - s_{n+1}s_{n+2} = \left(\frac{\alpha^n + \alpha - \beta^n - \beta}{\alpha - \beta}\right)\left(\frac{\alpha^{n+3} + \alpha - \beta^{n+3} - \beta}{\alpha - \beta}\right) - \left(\frac{\alpha^{n+1} + \alpha - \beta^{n+1} - \beta}{\alpha - \beta}\right)\left(\frac{\alpha^{n+2} + \alpha - \beta^{n+2} - \beta}{\alpha - \beta}\right).$$

After using exact values of α and β we get the desired result as

$s_n s_{n+3} - s_{n+1}s_{n+2} = f_n - (-1)^n$. □

In the next theorem, we give a sum of two adjacent and squared shifted Fibonacci numbers with non-negative indices.

Theorem 2.7 The following identity holds true

$$s_n^2 + s_{n+1}^2 = f_{2n+1} + 2s_{n+2}. \quad (10)$$

Proof. If we use definition of shifted Fibonacci numbers, we get

$$s_n^2 + s_{n+1}^2 = \left(\frac{\alpha^n+\alpha-\beta^n-\beta}{\alpha-\beta}\right)^2 + \left(\frac{\alpha^{n+1}+\alpha-\beta^{n+1}-\beta}{\alpha-\beta}\right)^2$$

Using necessary identities, we compute the desired result as

$$s_n^2 + s_{n+1}^2 = f_{2n+1} + 2s_{n+2}. \square$$

Theorem 2.8 For $m, n \geq 1$,

$$s_{m+1}s_{n+1} - s_{m-1}s_{n-1} = f_{m+n} + f_m + f_n. \quad (11)$$

Proof. For the prove we use definition of shifted Fibonacci numbers as

$$s_{m+1}s_{n+1} - s_{m-1}s_{n-1} = \left(\frac{\alpha^{m+1} + \alpha - \beta^{m+1} - \beta}{\alpha - \beta}\right)\left(\frac{\alpha^{n+1} + \alpha - \beta^{n+1} - \beta}{\alpha - \beta}\right) - \left(\frac{\alpha^{m-1} + \alpha - \beta^{m-1} - \beta}{\alpha - \beta}\right)\left(\frac{\alpha^{n-1} + \alpha - \beta^{n-1} - \beta}{\alpha - \beta}\right).$$

Occupying identities and arranging the equation we get the desired result as

$$s_{m+1}s_{n+1} - s_{m-1}s_{n-1} = f_{m+n} + f_m + f_n. \square$$

Theorem 2.9 For $n \geq 2$, the following identity holds true

$$s_n^2 - s_{n-2}s_{n+2} = (-1)^n - f_n. \quad (12)$$

Proof. Using definition of shifted Fibonacci numbers in the identity we get

$$s_n^2 - s_{n-2}s_{n+2} = \left(\frac{\alpha^n + \alpha - \beta^n - \beta}{\alpha - \beta}\right)^2 - \left(\frac{\alpha^{n-2} + \alpha - \beta^{n-2} - \beta}{\alpha - \beta}\right)\left(\frac{\alpha^{n+2} + \alpha - \beta^{n+2} - \beta}{\alpha - \beta}\right).$$

Using necessary properties, we get the desired result.

$$s_n^2 - s_{n-2}s_{n+2} = (-1)^n - f_n. \square$$

Theorem 2.10 The d'Ocagne identity for shifted Fibonacci numbers is

$$s_m s_{n+1} - s_{m+1}s_n = (-1)^n f_{m-n} + s_{m-1} + s_{n-1} \quad (13)$$

where $m > n$ and $, n \geq 1$.

Proof. For the proof of the equation (13) we use the definition of shifted Fibonacci numbers.

$$s_m s_{n+1} - s_{m+1}s_n = \left(\frac{\alpha^m + \alpha - \beta^m - \beta}{\alpha - \beta}\right)\left(\frac{\alpha^{n+1} + \alpha - \beta^{n+1} - \beta}{\alpha - \beta}\right) - \left(\frac{\alpha^{m+1} + \alpha - \beta^{m+1} - \beta}{\alpha - \beta}\right)\left(\frac{\alpha^n + \alpha - \beta^n - \beta}{\alpha - \beta}\right)$$

Occupying identities and arranging the equation we get the desired result as

$$s_{m+1}s_{n+1} - s_{m-1}s_{n-1} = f_{m+n} + f_m + f_n. \square$$

Now, we give some addition formulas for the shifted Fibonacci numbers.

Theorem 2.11 For $n \geq 0$, summation formulas of the shifted Fibonacci numbers are

$$\sum_{i=0}^{n} s_i = s_{n+2} + n - 1 \quad (14)$$

$$\sum_{i=0}^{n} s_{2i} = s_{2n+1} + n - 1 \quad (15)$$

$$\sum_{i=0}^{n} s_{2i+1} = s_{2n+2} + n \quad (16)$$

Proof. We only prove the formula the (16). The other formulas can be proven accordingly.

$\sum_{i=0}^{n} s_{2i+1} = \sum_{i=0}^{n}(f_{2i+1} + 1) = n + \sum_{i=0}^{n} f_{2i+1}$.

After the expansion above, the result is clear. □

3. Conclusion

In this study, we defined the shifted Fibonacci numbers and give their fundamental properties and identities. With this research we filled a gap in the literature. Also, this study paves the road for numerous papers which can be about additional identities or generalizations for the shifted Fibonacci numbers.

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