Journal of Universal Mathematics
Vol.6 No.3-Supplement pp.62-83 (2023)
ISSN-2618-5660
DOI:10.33773/jum.1346319

SOME QUALITATIVE PROPERTIES OF SOLUTIONS OF CERTAIN NONLINEAR THIRD-ORDER STOCHASTIC DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. This study considered certain nonlinear third-order stochastic differential equations with delay. The third-order equation is reduced to an equivalent system of first-order differential equations and used to construct the desired complete Lyapunov-Krasovskiĭ functional. Standard conditions guaranteeing stability when the forcing term is zero, boundedness of solutions when the forcing term is non-zero, and lastly the existence and uniqueness of solutions are derived. The obtained results indicated that the adopted technique is effective in studying the qualitative behaviour of solutions. The obtained results are not only new but extend the frontier of knowledge of the qualitative behaviour of solutions of nonlinear stochastic differential with delay. Finally, two special cases are given to illustrate the derived theoretical results.

1. Introduction

In recent years, the studies of stability, boundedness, existence and uniqueness of solutions of a nonlinear third-order stochastic differential equations with delay have been discussed and still under intensive investigations by researchers. Some outstanding works on deterministic model with and without delay using the technique of Lyapunov, we refer to the papers in [7–10, 12, 14, 18, 25].

In this paper, we shall consider the third-order nonlinear stochastic differential equation with delay defined as

$$\dddot{x}(t) + a\ddot{x}(t) + g(\cdot) + h(x(t-\tau(t))) + \sigma x(t-\tau(t))\dot{\omega}(t) = p(\cdot), \tag{1.1}$$

where $g(\cdot) = g(x(t-\tau(t)), \dot{x}(t-\tau(t)))$, $p(\cdot) = p(t, x(t), \dot{x}(t), \ddot{x}(t))$, for simplicity we shall write $x(t) = x$, $y(t) = y$, and $z(t) = z$. Assign $y = \dot{x}$ and $z = \ddot{x}$ equation (1.1)

Date: **Received:** 2023-08-19; **Accepted:** 2023-10-14.
Key words and phrases. Existence of solution, Nonlinear stochastic differential equation, Third-order, Uniform stability, Uniform ultimate boundedness.

is equivalent to system of first order equations

$$\begin{aligned} &\dot{x} = y, \quad \dot{y} = z, \\ &\dot{z} = p(t,x,y,z) - h(x) - g(x,y) - az - \sigma\big[x - \int_{t-\tau(t)}^{t} y(s)ds\big]\dot{\omega}(t) \\ &+ \int_{t-\tau(t)}^{t} \big[g_x(x(s),y(s))y(s) + g_y(x(s),y(s))z(s) + h'(x(s))y(s)\big]ds, \end{aligned} \tag{1.2}$$

where the functions g, h, and p are continuous in their respective arguments on $\mathbb{R}^2$, $\mathbb{R}$, and $\mathbb{R}^+ \times \mathbb{R}^3$, respectively with $\mathbb{R}^+ = [0,\infty)$, $\mathbb{R} = (-\infty,\infty)$, and $\omega \in \mathbb{R}$ (a standard Wiener process, representing the noise) is defined on $\mathbb{R}^3$, $\tau(t)$ is a continuously differentiable function with $0 \le \tau(t) \le \tau_0$, τ_0, a, and σ are positive constants. The dots denote to differentiation with respect to the independent variable $t \in \mathbb{R}^+$, derivatives $h'(x), g_x(x,y)$, and $g_y(x,y)$ exist and are continuous. Moreover, the continuity of the functions g, h, and p is sufficient for the existences of the solutions and the local Lipschitz condition for system (1.2) to obtain a unique continuous solution represented by $(x(t), y(t), z(t))$.

Systematic investigations of differential equations of distinct orders, with and without delay and/or randomness, have been carried out by researchers. In particular, there are critical inspection on first order system of differential equations, we can mention the background books and papers in [15–17, 19–21, 24, 27–29]. In addition, researchers in [11] employed the direct method of Lyapunov to obtain standard criteria on stability and boundedness of solutions of a certain second-order non-autonomous stochastic differential equation

$$\ddot{x}(t) + f(x(t),\dot{x}(t))\dot{x}(t) + g(x(t)) + \gamma x(t)\dot{\omega}(t) = p(t,x(t),\dot{x}(t)),$$

where γ is a positive constant, $g \in C(\mathbb{R},\mathbb{R})$ $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, and $p \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R},\mathbb{R})$ are continuous functions. The function g is differentiable and continuous for all x.

Furthermore, authors in [2] considered stability of solutions of certain second-order stochastic delay differential equations

$$\ddot{x}(t)+b\dot{x}(t)+cx(t-\epsilon)+\gamma x(t)\dot{\omega}(t) = 0 \text{ and } \ddot{x}(t)+b\dot{x}(t)+f(x(t-\epsilon))+\gamma x(t-\psi_0)\dot{\omega}(t) = 0,$$

where b, c, γ are positive constants, ϵ and ψ_0 are positive constant delays, the function f is continuous with respect to x with $f(0) = 0$. What is more, article in [3] discussed new results on the stability and boundedness for solutions of second-order stochastic delay differential equation

$$\ddot{x}(t) + g(\dot{x}(t)) + bx(t-h) + \sigma x(t)\dot{\omega}(t) = p(t,x(t),\dot{x}(t),x(t-h)),$$

where b, σ are positive constants, h is a positive constant delay, g and p are continuous functions with $g(0) = 0$. In [5], a suitable Lyapunov functional is used to establish sufficient conditions guaranteeing the existence of stochastic asymptotic stability of the zero solution of the non-autonomous second-order stochastic delay differential equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)f(x(t-r)) + g(t,x)\dot{\omega}(t) = 0,$$

where $a(t)$ and $b(t)$ are two positive continuously differentiable functions on $[0,\infty)$, r is a positive constant delay, $f(x)$ and $g(t,x)$ are continuous functions defined on $\mathbb{R}$ and $\mathbb{R}^+ \times \mathbb{R}$ respectively with $f(0) = 0$. Researchers in [6] studied the stability and boundedness of solutions to certain nonlinear non autonomous second-order stochastic delay differential equations

$$\ddot{x}(t)+\psi(t)f(x(t),\dot{x}(t))\dot{x}(t)+g(x(t-\tau))+\sigma x(t)\dot{\omega}(t)=p(t,x(t),\dot{x}(t),x(t-\tau)),$$

where ψ, f, g, p are continuous functions in their respective arguments on $\mathbb{R}^+, \mathbb{R}^2, \mathbb{R}$, $\mathbb{R}^+\times\mathbb{R}^3$ respectively, $\sigma>0$ is a constant, and τ is a positive constant delay. No doubt, articles [2, 3, 5, 6, 11] are special cases of equation (1.1).

When $\tau(t)\equiv 0$, $g(\cdot)\equiv b\dot{x}(t)$, $h(x(t-\tau(t)))\equiv cx(t)$, and $x(t-\tau(t))\equiv x(t)$, equation (1.1) reduces to the third-order stochastic differential equation discussed in [1] namely

$$\dddot{x}(t)+a\ddot{x}(t)+b\dot{x}+cx(t)+\sigma x(t)\dot{\omega}(t)=p(t,x(t),\dot{x}(t),\ddot{x}(t)),$$

where $a>0, b>0, c>0$, $\sigma>0$ are constants, and $p(t,x,\dot{x},\ddot{x})$ is a continuous function. Authors in [4] investigated the asymptotic stability of the zero solution for the third-order stochastic delay differential equations given by

$$\dddot{x}(t)+a_1\ddot{x}(t)+g_1(\dot{x}(t-r_1(t)))+f_1(x(t))+\sigma_1 x(t)\dot{\omega}(t)=0$$

and

$$\dddot{x}(t)+a_2\ddot{x}(t)+f_2(x(t))(\dot{x}(t)+f_3(x(t-r_2(t)))+\sigma_2 x(t-h(t))\dot{\omega}(t)=0,$$

where a_1, a_2, σ_1, σ_2 are positive constants, γ_1, γ_2 are two positive constants such that $0\le r_1(t)\le\gamma_1$, $0\le r_2(t)\le\gamma_2$, $0\le h(t)$, $\sup h(t)=H$; g_1, f_1, f_2, and f_3 are continuous functions with $g_1(0)=f_1(0)=f_3(0)=0$. The two equations discussed in [4] are special cases of (1.1) since $g(\cdot)\equiv g_1(\dot{x}(t-\tau_1(t))$, $h(x(t-\tau(t)))\equiv f_1(x(t))$, $x(t-\tau(t))\equiv x(t)$, and $p(\cdot)\equiv 0$ in the first equation and $g(\cdot)\equiv f_2(x(t))\dot{x}(t)$ and $p(\cdot)\equiv 0$ in the second equation. Whenever $g(\cdot), x(t-\tau(t))$, and $\tau(t)$ are equivalent to $b\dot{x}(t), x(t)$, and $\tau>0$ a constant delay respectively then equation (1.1) is cut down to the third-order stochastic delay differential equations considered in [13] i.e.,

$$\dddot{x}(t)+a\ddot{x}(t)+b\dot{x}(t)+h(x(t-\tau))+\sigma x(t)\dot{\omega}(t)=p(t,x(t),\dot{x}(t),\ddot{x}(t)),$$

where the constants a, b, σ are positive, h,p are nonlinear continuous functions in their respective arguments and $h(0)=0, \tau>0$ is a delay constant.

In the case $g(\cdot), x(t-\tau(t))$, and $p(\cdot)$ are equivalent to $\phi(\dot{x}(t-r(t))), x(t-h)$, and 0 respectively then equation (1.1) is trim down to the third-order stochastic differential equation

$$\dddot{x}(t)+a\ddot{x}(t)+\phi(\dot{x}(t-r(t)))+\psi(x(t-r(t)))+\sigma x(t-h)\dot{\omega}(t)=0,$$

investigated in [22] where $a>0$ and $\sigma>0$ are constants, $h>0$ is a constant delay, $r(t)$ is a continuously differentiable function satisfying $0\le r(t)\le\beta_1$, $\beta_1>0$ a constant, ϕ and ψ are nonlinear continuous functions defined on $\mathbb{R}$ with $\phi(0)=\psi(0)=0$. Motivation for this work comes from the works in [1, 4, 13, 22], where Lyapunov functionals are exploited to acquire asymptotic stability, boundedness, existence and uniqueness of solutions of the equations considered. Section 2 presents definitions of terms and basic results used in this paper, stability of the trivial solutions are stated and proved in Section 3, boundedness and existence results are communicated in Section 4, and special cases of the theoretical results discussed in Sections 3 and 4 are presented as examples in Section 5.

2. Preliminary Results

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t>0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t>0}$ satisfying the usual conditions (i.e., it is right continuous and $\{\mathfrak{F}_0\}$ contains all $\mathbb{P}-$null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an $m-$dimensional Brownian motion defined on the probability space. Let $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $\|A\| = \sqrt{\text{trace } (A^T A)}$. Details can be seen [15] and [23]. Consider a non autonomous $n-$dimensional stochastic delay differential equation

$$dx(t) = F(t, x(t), x(t-\tau))dt + G(t, x(t), x(t-\tau))dB(t) \tag{2.1}$$

on $t > 0$ with initial data $\{x(\theta) : -\tau \le \theta \le 0\}, x_0 \in C([-\tau, 0], \mathbb{R}^n)$. Here $F : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and $G : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^{n\times m}$ are measurable functions. Suppose that the functions F, G satisfy the local Lipschitz condition, given any $b > 0$, $p \ge 2$, $F(t, 0, 0) \in C^1([0, b], \mathbb{R}^n)$, and $G(t, 0, 0) \in C^p([0, b], \mathbb{R}^{m\times n})$. Then there must be a stopping time $\beta = \beta(\omega) > 0$ such that equation (2.1) with $x_0 \in C^p_{\mathfrak{F}_{t_0}}$ [class of $\mathfrak{F}_t$-measurable $C([-\tau, 0], \mathbb{R}^n)$-valued random variables ξ_t and $E\|\xi_t\|^p < \infty$] has a unique maximal solution on $t \in [t_0, \beta)$ which is denoted by $x(t, x_0)$. Assume further that $F(t, 0, 0) = G(t, 0, 0) = 0$ for all $t \ge 0$. Hence, the stochastic delay differential equation admits zero solution $x(t, 0) \equiv 0$ for any given initial value $x_0 \in C([-\tau, 0], \mathbb{R}^n)$.

Definition 2.1. The zero solution of the stochastic differential equation (2.1) is said to be stochastically stable or stable in probability, if for every pair $\epsilon \in (0, 1)$ and $r > 0$, there exists a $\delta_0 = \delta_0(\epsilon, r) > 0$ such that $Pr\{\|x(t; x_0)\| < r \text{ for all } t \ge 0\} \ge 1 - \epsilon$ whenever $\|x_0\| < \delta_0$. Otherwise, it is said to be stochastically unstable.

Definition 2.2. The zero solution of the stochastic differential equation (2.1) is said to be stochastically asymptotically stable if it is stochastically stable and in addition if for every $\epsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $Pr\{\lim_{t\to\infty} x(t; x_0) = 0\} \ge 1 - \epsilon$ whenever $\|x_0\| < \delta$.

Definition 2.3. A solution $x(t, x_0)$ of the stochastic delay differential equation (2.1) is said to be stochastically bounded or bounded in probability, if it satisfies

$$E^{x_0}\|x(t, x_0)\| \le N(t_0, \|x_0\|), \ \forall\ t \ge t_0 \tag{2.2}$$

where E^{x_0} denotes the expectation operator with respect to the probability law associated with x_0, $N : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a constant function depending on t_0 and x_0.

Definition 2.4. The solutions $x(t_0, x_0)$ of the stochastic delay differential equation (2.1) is said to be uniformly stochastically bounded if N in (2.2) is independent of t_0.

Let $\mathbb{K}$ denote the family of all continuous non-decreasing functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho(0) = 0$ and $\rho(r) > 0$ if $r \ne 0$. In addition, $\mathbb{K}_\infty$ denotes the family of all functions $\rho \in \mathbb{K}$ with

$$\lim_{r\to\infty} \rho(r) = \infty.$$

Suppose that $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, denotes the family of all non negative functions $V = V(t, x_t)$ (Lyapunov functional) defined on $\mathbb{R}^+ \times \mathbb{R}^n$ which are twice continuously differentiable in x and once in t. By Itô's formula we have

$$dV(t, x_t) = LV(t, x_t)dt + V_x(t, x_t)G(t, x_t)dB(t),$$

where

$$LV(t,x_t) = \frac{\partial V(t,x_t)}{\partial t} + \frac{\partial V(t,x_t)}{\partial x_i}F(t,x(t)) + \frac{1}{2}\text{trace }[G^T(t,x_t)V_{xx}(t,x_t)G(t,x_t)] \tag{2.3}$$

with

$$V_{xx}(t,x_t) = \left(\frac{\partial^2 V(t,x_t)}{\partial x_i \partial x_j}\right)_{n\times n}, \; i,j = 1,\cdots,n$$

In this study we will use the diffusion operator $LV(t,x_t)$ defined in (2.3) to replace $V'(t,x(t)) = \frac{d}{dt}V(t,x(t))$. We now present the basic results that will be used in the proofs of the main results.

Lemma 2.5. *(See [15]) Assume that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, and $\eta \in \mathbb{K}$ such that*

(i) $V(t,0) = 0$, *for all* $t \geq 0$;
(ii) $V(t,x_t) \geq \eta(\|x(t)\|)$, $\eta(r) \to \infty$ *as* $r \to \infty$; *and*
(iii) $LV(t,x_t) \leq 0$ *for all* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the zero solution of stochastic delay differential equation (2.1) *is* stochastically stable. *If conditions (ii) and (iii) hold then* (2.1) *with* $x_0 \in C^p_{\mathfrak{F}_{t_0}}$ *has a unique global solution for* $t > 0$ *denoted by* $x(t;x_0)$.

Lemma 2.6. *(See [15]) Suppose that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, and $\eta_0, \eta_1, \eta_2 \in \mathbb{K}$ such that*

(i) $V(t,0) = 0$, *for all* $t \geq 0$;
(ii) $\eta_0(\|x(t)\|) \leq V(t,x_t) \leq \eta_1(\|x(t)\|)$, $\eta_0(r) \to \infty$ *as* $r \to \infty$; *and*
(iii) $LV(t,x_t) \leq -\eta_2(\|x(t)\|)$ *for all* $(t,x_t) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the zero solution of stochastic delay differential equation (2.1) *is* uniformly stochastically asymptotically stable in the large

Assumption 2.7. *(See [21, 26]) Let $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, suppose that for any solutions $x(t_0, x_0)$ of stochastic delay differential equation* (2.1) *and for any fixed $0 \leq t_0 \leq T < \infty$, we have*

$$E^{x_0}\left\{\int_{t_0}^{T} V^2_{x_i}(t,x_t)G^2_{ik}(t,x_t)dt\right\} < \infty, \; 1 \leq i \leq n, \; 1 \leq k \leq m. \tag{2.4}$$

Assumption 2.8. *(See [21, 26]) A special case of the general condition* (2.4) *is the following condition. Assume that there exits a function $\rho(t)$ such that*

$$|V_{x_i}(t,x_t)G_{ik}(t,x_t)| < \rho(t), \; x \in \mathbb{R}^n, \; 1 \leq i \leq n, \; 1 \leq k \leq m, \tag{2.5}$$

for any fixed $0 \leq t_0 \leq T < \infty$,

$$\int_{t_0}^{T} \rho^2(t)dt < \infty. \tag{2.6}$$

Lemma 2.9. *(See [21, 26]) Assume there exists a Lyapunov function $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, satisfying Assumption 2.7, such that for all $(t,x_t) \in \mathbb{R}^+ \times \mathbb{R}^n$,*

(i) $\|x(t)\|^p \leq V(t,x_t) \leq \|x(t)\|^q$,
(ii) $LV(t,x_t) \leq -\alpha(t)\|x(t)\|^r + \psi(t)$,
(iii) $V(t,x_t) - V^{r/q}(t,x_t) \leq \mu$,

where $\alpha, \psi \in C(\mathbb{R}^+, \mathbb{R}^+)$, p, q, r *are positive constants,* $p \geq 1$, *and* μ *is a non negative constant. Then all solutions of stochastic delay differential equation* (2.1) *satisfy*

$$E^{x_0}\|x(t,x_0)\| \leq \left\{V(t_0,x_0)e^{-\int_{t_0}^t \alpha(s)ds} + A\right\}^{1/p}, \tag{2.7}$$

for all $t \geq t_0$, *where*

$$A := \int_{t_0}^t \Big(\mu\alpha(u) + \psi(u)\Big)e^{-\int_u^t \alpha(s)ds}du.$$

Lemma 2.10. *(See [21, 26]) Assume there exists a Lyapunov function* $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, *satisfying Assumption 2.7, such that for all* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$,

(i) $\|x(t)\|^p \leq V(t,x_t)$,
(ii) $LV(t,x_t) \leq -\alpha(t)V^q(t,x_t) + \psi(t)$,
(iii) $V(t,x_t) - V^q(t,x_t) \leq \mu$,

where $\alpha, \psi \in C(\mathbb{R}^+, \mathbb{R}^+)$, p, q *are positive constants,* $p \geq 1$, *and* μ *is a non negative constant. Then all solutions of stochastic delay differential equation* (2.1) *satisfy* (2.7) *for all* $t \geq t_0$.

Corollary 2.11. *(See [21, 26])*

(i) *Assume that hypotheses (i) to (iii) of Lemma 2.9 hold. In addition*

$$\int_{t_0}^t \Big(\mu\alpha(u) + \psi(u)\Big)e^{-\int_u^t \alpha(s)ds}du \leq M, \forall\ t \geq t_0 \geq 0, \tag{2.8}$$

for some positive constant M, *then all solution of stochastic delay differential equation* (2.1) *are* uniformly stochastically bounded.

(ii) *Assume the hypotheses (i) to (iii) of Lemma 2.10 hold. If condition* (2.8) *is satisfied, then all solutions of stochastic delay differential equation* (2.1) *are* stochastically bounded.

3. Stability of the Trivial Solution

We now present stability results of the trivial solution as follows. When $p(\cdot) \equiv 0$, (1.1) becomes

$$\dddot{x}(t)+a\ddot{x}(t)+g(x(t-\tau(t)),\ \dot{x}(t-\tau(t)))+h(x(t-\tau(t)))+\sigma x(t-\tau(t))\dot{\omega}(t) = 0. \tag{3.1}$$

As usual, by assigning $y = \dot{x}$ and $z = \ddot{x}$ equation (3.1) is stepped down to equivalent system of first order differential equations

$$\begin{aligned} &\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -h(x) - g(x,y) - az - \sigma\big[x - \int_{t-\tau(t)}^t y(s)ds\big]\dot{\omega}(t) \\ &+ \int_{t-\tau(t)}^t \big[g_x(x(s),\ y(s))y(s) + g_y(x(s),\ y(s))z(s) + h'(x(s))y(s)\big]ds, \end{aligned} \tag{3.2}$$

where the functions h and g are continuous in their respective arguments. For the purpose of this investigation, a continuously differential scalar functional constructed is defined as

$$\begin{aligned}V = V(t, X_t) &= \alpha \int_0^x h(s)ds + \frac{1}{2}\beta b x^2 + \frac{1}{2}(\alpha a + \gamma bc + c)y^2 + \frac{1}{2}(a+\gamma c)z^2 \\ &+ a^2 bxy + \gamma cyh(x) + \beta xz + \alpha yz + \int_{-\tau(t)}^0 \int_{t+s}^t (\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta))d\theta ds,\end{aligned} \tag{3.3}$$

where $a > 0$, $b > 0$, $c > 0$, $\alpha := a^2+ac+c^2$, $\beta := ab-c$, $\gamma := 1+b$ are constants, $h,\ g$ are continuous functions, positive constants λ_i $(i = 1, 2)$ will be verified latter, the function $\tau(t) \le \tau_0$ for $\tau_0 > 0$, and $X_t = x_t, y_t, z_t$. We have the following stability results.

Theorem 3.1. *In addition to the basic assumption on the functions g and h, suppose that $a, b, c, c_0, k_1, k_2, k_3, \beta_1$ are positive constants such that*

(i) $h(0) = 0,\ c_0 \le \dfrac{h(x)}{x}$ *for all* $x \ne 0$;

(ii) $g(0, 0) = 0,\ b \le \dfrac{g(x, y)}{y}$ *for all* x *and* $y \ne 0$;

(iii) $h'(x) \le c$ *for all* x, $ab - c > 0,\ \sigma^2 < \dfrac{2(ab-c)c_0}{a + (b+1)c}$;

(iv) $a^2b(a+c\gamma) > \alpha\beta$, $b\beta(a\alpha+c) > a^4b^2$, $(a\alpha+c)(a+c\gamma) > \alpha^2$, $a^2b\alpha > \beta(a\alpha+c)$, $b\beta\big[(a\alpha + c)(a + c\gamma) - \alpha^2\big] + \beta\big[a^2b\alpha - \beta(a\alpha + c)\big] > a^2b\big[a^2b(a + c\gamma) - \alpha\beta\big]$; *and*

(v) $|h'(x)| \le k_1$, $|g_x(x, y)| \le k_2$, $|g_y(x, y)| \le k_3$.

Then the trivial solution of system (3.2) *is asymptotically stable, provided that*

$$\beta_1 < \min\left\{\frac{2(ab-c)c_0 - (a+c\gamma)\sigma^2}{2B_3},\ \frac{(ab-c)c}{B_4},\ \frac{(ab-c)c}{B_5}\right\} \tag{3.4}$$

where

$B_3 := k_0(k_1 + k_2 + k_3) - (a + \gamma c)\sigma^2,$

$B_4 := \big[3k_0(k_1 + k_2) + (a + c\gamma)\sigma^2 + k_0(k_1 + k_2 + k_3)(1 - \beta_0)\big]/(1 - \beta_0),$

$B_5 := \big(3k_0k_3 + k_0(k_1 + k_2 + k_3)(1 - \beta_0)\big)/(1 - \beta_0),$ *and*

$k_0 := \max\{\alpha, \beta, (a + c\gamma)\}.$

Since asymptotic stability implies stability we have the following result.

Corollary 3.2. If all assumptions of Theorem 3.1 hold true, then the trivial solution of system (3.2) is stable if estimate (3.4) holds.

In what follows we present uniform asymptotic stability results.

Theorem 3.3. Further to the basic assumption on the functions g and h, suppose that $a,\ b,\ b_1,\ c,\ c_0,\ c_1,\ k_0,\ k_1,\ k_2$, and k_3 are positive constants such that

(i) $h(0) = 0,\ c_0 \le \dfrac{h(x)}{x} \le c_1$ for all $x \ne 0$;

(ii) $g(0, 0) = 0,\ b \le \dfrac{g(x, y)}{y} \le b_1$ for all x and $y \ne 0$;

(iii) $h'(x) \le c$ for all x, $ab - c > 0,\ \sigma^2 < \dfrac{2(ab-c)c_0}{a + (b+1)c}$;

(iv) $a^2b(a+c\gamma) > \alpha\beta$, $b\beta(a\alpha+c) > a^4b^2$, $(a\alpha+c)(a+c\gamma) > \alpha^2$, $a^2b\alpha > \beta(a\alpha+c)$, $b\beta\big[(a\alpha+c)(a+c\gamma)-\alpha^2\big]+\beta\big[a^2b\alpha-\beta(a\alpha+c)\big] > a^2b\big[a^2b(a+c\gamma)-\alpha\beta\big]$; and

(v) $|h'(x)| \leq k_1$, $|g_x(x,y)| \leq k_2$, $|g_y(x,y)| \leq k_3$;

Then the trivial solution of system (3.2) is uniformly asymptotically stable provided that inequality (3.4) holds.

Next, the following corollary is immediate from Theorem 3.3.

Corollary 3.4. *If all assumptions of Theorem 3.3 hold, then the trivial solution of system* (3.2) *is uniformly stable provided that inequality* (3.4) *holds.*

To show that (3.3) is indeed a Lyapunov functional we need to state and prove two lemmas.

Lemma 3.5. Under the assumptions of Theorem 3.3 there exist positive constants E_1 and E_2 such that

$$E_1(x^2+y^2+z^2) \leq V(t,X_t) \leq E_2(x^2+y^2+z^2), \tag{3.5}$$

for all $t \geq 0$, x, y, and z. Moreover,

$$V(t,X_t) \to +\infty \text{ as } x^2+y^2+z^2 \to \infty. \tag{3.6}$$

Proof. To prove this lemma we shall show that $V(t,\mathbf{0}) = 0$ where $\mathbf{0} = (0,0,0)$, $V(t,X_t)$ is positive semi-definite, decrescent (or have an infinitesimal small upper-bound), and radially unbounded. To see these, equation (3.3) shows that

$$V(t,\mathbf{0}) = 0, \tag{3.7}$$

for all $t \geq 0$. Following, equation (3.3) can be represented in the form $V = \sum\limits_{j=1}^{3} V_j$ where

$$V_1 := \alpha\int_0^x h(s)ds + \frac{1}{2}bc\gamma y^2 + c\gamma y h(x);$$

$$V_2 := \frac{1}{2}b\beta x^2 + \frac{1}{2}\big[a\alpha+c\big]y^2 + \frac{1}{2}\big[a+c\gamma\big]z^2 + a^2bxy + \beta xz + \alpha yz; \text{ and}$$

$$V_3 := \int_{-\tau(t)}^{0}\int_{t+s}^{t}\big[\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)\big]d\theta ds.$$

Now the last two terms of V_1 can be represented as

$$\frac{1}{2}bc\gamma y^2 + c\gamma y h(x) = \frac{1}{2}bc\gamma\big[y+b^{-1}h(x)\big]^2 - \frac{1}{2}b^{-1}c\gamma h^2(x). \tag{3.8a}$$

Also, since $h^2(x) = 2\int_0^x h'(s)h(s)ds + h^2(0)$ and $h(0)=0$, it follows that

$$\begin{aligned}\alpha\int_0^x h(s)ds &= \alpha\int_0^x h(s)ds - \frac{1}{2}b^{-1}c\gamma h^2(x) + \frac{1}{2}b^{-1}c\gamma h^2(x)\\ &= \frac{1}{b}\int_0^x \big[b\alpha - c\gamma h'(s)\big]h(s)ds + \frac{1}{2}b^{-1}c\gamma h^2(x).\end{aligned} \tag{3.8b}$$

Adding equations (3.8a) and (3.8b) we have

$$V_1 = \frac{1}{b}\int_0^x \big[b\alpha - c\gamma h'(s)\big]h(s)ds + \frac{1}{2}bc\gamma\big[y+b^{-1}h(x)\big]^2.$$

Hypotheses (i) and (iii) of Theorem 3.3 result to

$$V_1 \geq \frac{1}{2b}c_0[b\alpha - c^2\gamma]x^2 + \frac{1}{2}bc\gamma[y + b^{-1}c_0x]^2 \geq \frac{1}{2b}c_0[b\alpha - c^2\gamma]x^2,$$

since $\frac{1}{2}bc\gamma[y + b^{-1}c_0x]^2 \geq 0$ for all x, y. The basic assumptions imply that

$$b\alpha - c^2\gamma = b(a^2 + ac + c^2) - (b+1)c^2 = a^2b + c\beta > 0.$$

Thus

$$V_1 \geq \frac{1}{2b}c_0[a^2b + c\beta]x^2, \text{ for all } x.$$

Next,

$$V_2 = \frac{1}{2}b\beta x^2 + \frac{1}{2}[a\alpha + c]y^2 + \frac{1}{2}[a + c\gamma]z^2 + a^2bxy + \beta xz + \alpha yz;$$

can be represented as $2V_2 := XAX^T$ where $X = \begin{pmatrix} x & y & z \end{pmatrix}$, X^T is the transpose of X, and

$$A := \begin{pmatrix} b\beta & a^2b & \beta \\ a^2b & a\alpha + c & \alpha \\ \beta & \alpha & a + c\gamma \end{pmatrix}.$$

We need to show that the determinant of the principal minors of matrix A (i.e., $|A_1|, |A_2|$, and $|A_3|$,) are positive. The basic assumptions indicate that

$$|A_1| := b\beta > 0.$$

Hypothesis (iv) gives raise to

$$|A_2| := \begin{vmatrix} b\beta & a^2b \\ a^2b & a\alpha + c \end{vmatrix} = b\beta(a\alpha + c) - a^4b^2 > 0,$$

and

$$|A_3| = |A| := \begin{vmatrix} b\beta & a^2b & \beta \\ a^2b & a\alpha + c & \alpha \\ \beta & \alpha & a + c\gamma \end{vmatrix} = b\beta\big[(a\alpha + c)(a + c\gamma) - \alpha^2\big] + \beta\big[a^2b\alpha - \beta(a\alpha + c)\big]$$
$$- a^2b\big[a^2b(a + c\gamma) - \alpha\beta\big] > 0.$$

Since all principal minors of matrix A are positive, then A is positive definite and a constant $\theta_1 = \theta_1(a, b, c) > 0$ exists such that

$$V_2 \geq \theta_1(x^2 + y^2 + z^2) \text{ for all } x, y, z.$$

Next, the double integrals in V_3 are obviously positive, thus there exist a constant $\mu > 0$ such that

$$V_3 = \int_{-\tau(t)}^{0}\int_{t+s}^{t}[\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)]d\theta ds \geq \mu(y^2 + z^2).$$

combining the $V_i (i = 1, 2, 3)$ there exists a positive constant θ_2 such that

$$V \geq \theta_2(x^2 + y^2 + z^2) \tag{3.9}$$

for all $t \geq 0,\ x,\ y,$ and z where

$$\theta_2 = \theta_1 \cdot \min\left\{\frac{1}{2b}c_0[a^2b + c\beta],\ \mu\right\}.$$

Inequality (3.9) establishes the lower inequality in (3.5) with θ_2 equivalent to E_1, hence by inequality (3.9), the function $V(t, X_t)$ is positive semi-definite.

Moreover, from inequality (3.9), we have the following relations

$$V(t,X_t)=0 \iff x^2+y^2+z^2=0, \tag{3.10a}$$

$$V(t,X_t)>0 \iff x^2+y^2+z^2\neq 0, \tag{3.10b}$$

it validly follows from equation (3.10a) and estimate (3.10b) that

$$V(t,X_t)\to +\infty \text{ as } x^2+y^2+z^2\to\infty, \tag{3.10c}$$

so that the function $V(t,X_t)$ is radially unbounded. In addition, assumptions (i) and (ii) of Theorem 3.3, the obvious inequality $2ab\le a^2+b^2$, the fact that $\dfrac{h(x)}{x}\le c_1$ for all $x\neq 0$, and since $\tau(t)\le\tau_0$, equation (3.3) becomes

$$V(t,X_t)\le \frac{1}{2}(c_1\alpha+b\beta+a^2b+cc_1\gamma+\beta)\|x\|^2+\frac{1}{2}(a\alpha+bc\gamma+c+a^2b+cc_1\gamma+\alpha$$
$$+\lambda_1\tau_0^2)\|y\|^2+\frac{1}{2}(a+c\gamma+\beta+\alpha+\lambda_1\tau_0^2)\|z\|^2.$$

In view of the last inequality, there exist a positive constant θ_3 such that

$$V(t,X_t)\le\theta_3(x^2+y^2+z^2) \tag{3.11}$$

for all $t\ge 0, x, y,$ and z where

$$\theta_3:=\frac{1}{2}\max\{c_1\alpha+b\beta+a^2b+cc_1\gamma+\beta,\ a\alpha+bc\gamma+c+a^2b+cc_1\gamma+\alpha$$
$$+\lambda_1\tau_0^2,\ a+c\gamma+\beta+\alpha+\lambda_1\tau_0^2\}.$$

Inequality (3.11) fulfils the upper inequality in (3.5) with θ_3 equivalent to E_2, thus the functional $V(t,X_t)$ has an infinitesimal small upper bound. This completes the prove of Lemma 3.5. □

The following lemma establishes the derivative of the functional $V(t,X_t)$ defined by (3.3), using Itô's formula defined by equation (2.3).

Lemma 3.6. Under the assumption of Theorem 3.1 there exists a positive constant E_3 such that along the solution path of system (3.2)

$$LV(t,X_t)\le -E_3(x^2+y^2+z^2),\ \forall\ x,y,z. \tag{3.12}$$

Proof. The first partial derivative of the functional $V(t,X_t)$ along the solution path of (3.2) is

$$\begin{aligned}LV_{(3.2)}(t,X_t)&=-\frac{1}{2}V_4-V_5+V_6+(\lambda_1y^2+\lambda_2z^2)\tau(t)\\&\quad-(1-\tau'(t))\int_{t-\tau(t)}^{t}[\lambda_1y^2(\theta)+\lambda_2z^2(\theta)]d\theta,\end{aligned} \tag{3.13}$$

where

$$V_4=\left(\beta\frac{h(x)}{x}-\frac{a+c\gamma}{2}\sigma^2\right)x^2+\left(\alpha\frac{g(x,y)}{y}-(a^2b+c\gamma h'(x))\right)y^2+\left(a(a+c\gamma)-\alpha\right)z^2;$$

$$V_5=\frac{1}{2}\left(\beta\frac{h(x)}{x}-\frac{a+c\gamma}{2}\sigma^2\right)x^2+\frac{1}{2}\left(\alpha\frac{g(x,y)}{y}-(a^2b+c\gamma h'(x))\right)y^2+\frac{1}{2}\left(a(a+c\gamma)-\alpha\right)z^2$$

$$+\beta\left(\frac{g(x,y)}{y}-b\right)xy+\left(a\beta+a-a^2b\right)xz+\left[(a+c\gamma\frac{g(x,y)}{y}-\beta-bc\gamma-c)\right]yz;\text{ and}$$

$$V_6=\left[\beta x+\alpha y+(a+c\gamma)z\right]\int_{t-\tau(t)}^{t}[h'(\cdot)y(s)+g_x(\cdot)y(s)+g_y(\cdot)z(s)]ds$$

$$-\sigma^2(a+c\gamma)x\int_{t-\tau(t)}^{t}y(s)ds+\frac{1}{2}\sigma^2\int_{t-\tau(t)}^{t}y^2(s)ds.$$

Hypotheses (i) to (iii) of Theorem 3.1 the following inequalities hold:

$$\begin{aligned}&\beta\frac{h(x)}{x}-\frac{a+c\gamma}{2}\sigma^2\geq c_0\beta-\frac{a+c\gamma}{2}\sigma^2;\\ &\alpha\frac{g(x,y)}{y}-(a^2b+c\gamma h'(x))\geq(a^2+ac+c^2)b-a^2b-bc^2-c^2=abc-c^2;\\ &a(a+c\gamma)-\alpha\geq a^2+ac(b+1)-(a^2+ac+c^2)=abc-c^2.\end{aligned}\tag{3.14}$$

Estimate (3.14) gives rise to

$$V_4\geq\left[(ab-c)c_0-\frac{a+c\gamma}{2}\sigma^2\right]x^2+\left(abc-c^2\right)y^2+\left(abc-c^2\right)z^2,$$

for all $t\geq0,x,y,z$. Let $V_5=\sum\limits_{i=1}^{3}V_{5i}$ where

$$V_{51}:=\frac{1}{4}\left(\beta\frac{h(x)}{x}-\frac{(a+c\gamma)}{2}\sigma^2\right)x^2+\beta\left(\frac{g(x,y)}{y}-b\right)xy+\frac{1}{4}\left(\frac{\alpha g(x,y)}{y}\right.$$

$$\left.-(a^2b+c\gamma h'(x))\right)y^2;$$

$$V_{52}:=\frac{1}{4}\left(\beta\frac{h(x)}{x}-\frac{(a+c\gamma)}{2}\sigma^2\right)x^2+(a\beta+a-a^2b)xz+\frac{1}{4}\left(a(a+c\gamma)-\alpha\right)z^2;\text{ and}$$

$$V_{53}:=\frac{1}{4}\left(\alpha\frac{g(x,y)}{y}-(a^2b+c\gamma h'(x))\right)y^2+\left((a+c\gamma)\frac{g(x,y)}{y}-\beta-bc\gamma-c\right)yz$$

$$+\frac{1}{4}\left(a(a+c\gamma)-\alpha\right)z^2.$$

Note that V_{5i} $(i=1,2,3)$ is a quadratic function with coefficients of x^2,y^2, and z^2 positive, using Hessian matrix, we obtain

$$\left(\beta\frac{h(x)}{x}-\frac{(a+c\gamma)}{2}\sigma^2\right)\left(\alpha\frac{g(x,y)}{y}-(a^2b+c\gamma h'(x))\right)>4\beta^2\left(\frac{g(x,y)}{y}-b\right)^2;$$

$$\left(\beta\frac{h(x)}{x}-\frac{(a+c\gamma)}{2}\sigma^2\right)\left(a(a+c\gamma)-\alpha\right)>4\left(a\beta+a-a^2b\right)^2;\text{ and}$$

$$\left(\alpha\frac{g(x,y)}{y}-(a^2b+c\gamma h'(x))\right)\left(a(a+c\gamma)-\alpha\right)>4\left((a+c\gamma)\frac{g(x,y)}{y}-\beta-bc\gamma-c\right)^2.$$

Applying these estimates in V_{5i} to give the following inequalities:

$$V_{51} \geq \left[\sqrt{\frac{1}{4}\big(\beta\frac{h(x)}{x} - \frac{(a+c\gamma)}{2}\sigma^2\big)}|x| + \sqrt{\frac{1}{4}\big(\alpha\frac{g(x,y)}{y} - (a^2b + c\gamma h'(x))\big)}|y|\right]^2$$
$$\geq 0, \forall\ t \geq 0, x, y;$$
$$V_{52} \geq \left[\sqrt{\frac{1}{4}\big(\beta\frac{h(x)}{x} - \frac{(a+c\gamma)}{2}\sigma^2\big)}|x| + \sqrt{\frac{1}{4}\big(a(a+c\gamma) - \alpha\big)}|z|\right]^2 \geq 0, \forall\ t \geq 0, x, z; \text{ and}$$
$$V_{53} \geq \left[\sqrt{\frac{1}{4}\big(\alpha\frac{g(x,y)}{y} - (a^2b + c\gamma h'(x))\big)}|y| + \sqrt{\frac{1}{4}\big(a(a+c\gamma) - \alpha\big)}|z|\right]^2 \geq 0, \forall\ t \geq 0, y, z.$$

These last three inequalities assure

$$V_5 \geq 0,\ \forall\ t \geq 0, x, y, z.$$

Apply the following inequality $2xy \leq 2|xy| \leq x^2+y^2$ and hypothesis (iv) of Theorem 3.1 give.

$$V_6 \leq \frac{k_0(k_1+k_2+k_3)}{2}(x^2+y^2+z^2)\tau(t) + \frac{1}{2}\int_{t-\tau(t)}^{t}\big[3k_0(k_1+k_2) + (a+c\gamma)\sigma^2\big]y^2(s)ds$$
$$+ \frac{3k_0k_3}{2}\int_{t-\tau(t)}^{t} z^2(s)ds + \frac{1}{2}(a+c\gamma)\sigma^2x^2\tau(t),$$

where $k_0 := \max\{\alpha, \beta, (a+c\gamma)\}$. Utilizing inequalities V_4, V_5, and V_6 in equation (3.12) we obtain

$$\begin{aligned} LV_{(3.2)}(t,X_t) \leq & -\frac{1}{2}\Bigg[(ab-c)c_0 - \frac{a+c\gamma}{2}\sigma^2 - \Big(k_0(k_1+k_2+k_3) \\ & + (a+c\gamma)\sigma^2\Big)\tau(t)\Bigg]x^2 - \frac{1}{2}\Big[(ab-c)c - \big(2\lambda_1 + k_0(k_1+k_2+k_3)\big)\tau(t)\Big]y^2 \\ & - \frac{1}{2}\Big[(ab-c)c - \Big(2\lambda_2 + k_0(k_1+k_2+k_3)\Big)\tau(t)\Big]z^2 \\ & - \Big(\lambda_1[1-\tau'(t)] - \frac{3}{2}k_0(k_1+k_2) - \frac{1}{2}(a+c\gamma)\sigma^2\Big)\int_{t-\tau(t)}^{t} y^2(s)ds \\ & - \Big(\lambda_2[1-\tau'(t)] - \frac{3}{2}k_0k_3\Big)\int_{t-\tau(t)}^{t} z^2(s)ds. \end{aligned} \tag{3.15}$$

Let $\tau'(t) \leq \beta_0$, $\beta_0 \in (0,1)$, $\tau(t) \leq \beta_1$, suppose $\lambda_1 := \big[3k_0(k_1+k_2)+(a+c\gamma)\sigma^2\big]\big[2(1-\beta_0)\big]^{-1} > 0$, and $\lambda_2 := 3k_0k_3\big[2(1-\beta_0)\big]^{-1} > 0$ so estimate (3.15) becomes

$$\begin{aligned} LV_{(3.2)}(t,X_t) \leq & -\frac{1}{2}\Bigg[(ab-c)c_0 - \frac{a+c\gamma}{2}\sigma^2 \\ & - \Big(k_0(k_1+k_2+k_3) - (a+\gamma c)\sigma^2\Big)\beta_1\Bigg]x^2 \\ & - \frac{1}{2}\Bigg[(ab-c)c - \Bigg(\bigg[\frac{3k_0(k_1+k_2)+(a+c\gamma)\sigma^2}{1-\beta_0}\bigg] + k_0(k_1+k_2+k_3)\Bigg)\beta_1\Bigg]y^2 \\ & - \frac{1}{2}\Bigg[(ab-c)c - \Bigg(\bigg(\frac{3k_0k_3}{1-\beta_0}\bigg) + k_0(k_1+k_2+k_3)\Bigg)\beta_1\Bigg]z^2. \end{aligned} \tag{3.16}$$

Inequalities (3.4) and (3.16) invoke the existence of a positive constant k_4 such that

$$LV_{(3.2)}(t, X_t) \leq -k_4(x^2 + y^2 + z^2) \tag{3.17}$$

for all $t \geq 0, x, y,$ and z where

$$\begin{aligned} k_4 := &\frac{1}{2} \min \Bigg\{ (ab - c)c_0 - \frac{a + c\gamma}{2}\sigma^2 - \Big(k_0(k_1 + k_2 + k_3) - (a + \gamma c)\sigma^2 \Big) \beta_1, \\ &(ab - c)c - \left(\left[\frac{3k_0(k_1 + k_2) + (a + c\gamma)\sigma^2}{1 - \beta_0} \right] + k_0(k_1 + k_2 + k_3) \right) \beta_1, \\ &(ab - c)c - \left(\left(\frac{3k_0k_3}{1 - \beta_0} \right) + k_0(k_1 + k_2 + k_3) \right) \beta_1 \Bigg\}. \end{aligned}$$

Inequality (3.17) satisfies estimate (3.12) with k_4 equivalent to E_3, hence Lemma 3.6 is proved. □

Proof of Theorems 3.1. Suppose (X_t) is any solution of (3.2), the functional $V(t, X_t)$ defined in (3.3) satisfies equation (3.7), estimates (3.9), (3.10c), and (3.17), so that conditions (i), (ii), and (iii) of the Lemma 2.5 are satisfied, hence by Lemma 2.5 the solution of (3.2) is stochastically asymptotically stable. □

Proof of Theorems 3.3. Given that (X_t) is any solution of (3.2) and the functional $V(t, X_t)$ defined in (3.3) satisfies equation (3.7), estimates (3.9), (3.10c), (3.11), and (3.17), fulfil assumptions (i), (ii), and (iii) of the Lemma 2.6, hence by Lemma 2.6 the solution of (3.2) is uniformly stochastically asymptotically stable. □

4. Boundedness and Existence Results

Furthermore, if $p(t, x, y, z) \neq 0$ in system (1.2), we have the following boundedness and ultimate boundedness results

Theorem 4.1. Suppose conditions (i) to (iv) and inequality (3.4) of Theorem 3.1 hold and in addition, if $|p(t, x, y, z)| \leq P_0$ where P_0 is a finite constant, then the solutions (X_t) of system (1.2) are not only stochastically bounded but also stochastically ultimately bounded.

Proof. Let (X_t) be any solution of system (1.2), by applying the Itô's formula on the functional defined in (3.3), along the solution path of (1.2), results to

$$LV_{(1.2)}(t, X_t) = LV_{(3.2)}(t, X_t) + \big[\beta x + \alpha y + (a + c\gamma)z\big] p(t, x, y, z).$$

Now from estimate (3.17) we find that

$$LV_{(1.2)}(t, X_t) \leq -k_4(x^2 + y^2 + z^2) + k_5(|x| + |y| + |z|)|p(t, x, y, z)|$$

where $k_5 = \max\{\beta, \alpha, (a + c\gamma)\}$. Since $|p(t, x, y, z)| \leq P_0$ for all $t \geq 0, x, y,$ and z, it follows that

$$\begin{aligned} LV_{(1.2)}(t, X_t) \leq -\frac{1}{2}k_4(x^2 + y^2 + z^2) + P_0k_4^{-1}k_5^2 - \frac{1}{2}k_4P_0 \Big[&(|x| - k_4^{-1}k_5)^2 \\ &+ (|y| - k_4^{-1}k_5)^2 + (|z| - k_4^{-1}k_5)^2 \Big] \end{aligned}$$

$\forall\ t \geq 0, x, y$, and z. Since k_4 and P_0 are positive constants and $(|x| - k_4^{-1}k_5)^2 + (|y| - k_4^{-1}k_5)^2 + (|z| - k_4^{-1}k_5)^2 \geq 0$ for all x, y, and z. Therefore there exist positive constants k_6 and k_7 such that

$$LV_{(1.2)}(t, X_t) \leq -k_6(x^2 + y^2 + z^2) + k_7 \tag{4.1}$$

where $k_6 = \frac{1}{2}k_4$ and $k_7 = P_0 k_4^{-1} k_5^2$ for all $t \geq 0, x, y$, and z. Estimate (3.11) implies that $\theta_3^{-1} V(t, X_t) \leq (x^2 + y^2 + z^2)$ for all $t \geq 0, x, y$, and z. The last estimate and inequality (4.1) result to

$$LV_{(1.2)}(t, X_t) \leq -k_8 V(t, X_t) + k_7 \tag{4.2}$$

for all $t \geq 0, x, y$, and z where $k_8 := k_6 \theta_3^{-1}$. Inequality (4.2) fulfills condition (ii) of Lemma 2.10 with $\alpha(t) = k_8, \psi(t) = k_7, q = 1$.

Furthermore, the lower inequality (3.5) (or estimate (3.9)) satisfies hypothesis (i) of Lemma 2.10. Now by estimate (4.2), we have $q = 1$, this implies that $\mu = 0$, so that hypothesis (iii) of Lemma 2.10 holds. Substituting the values of α, ψ, and μ in (2.8), to find that

$$\int_{t_0}^{t} (\mu\alpha(u) + \psi(u)) e^{-\int_u^t \alpha(s)ds} du = k_7 k_6^{-1}[1 - e^{-k_6(t-t_0)}] \leq k_7 k_6^{-1} \tag{4.3}$$

for all $t \geq t_0 \geq 0$, inequality (4.3) satisfies estimate (2.8) of Corollary 2.11 with $M = k_7 k_6^{-1} > 0$.

Also, to verify inequalities (2.5) and (2.6) of Assumption 2.8 (a special case of Assumption 2.7). System (1.2) and the Lyapunov functional (3.3) result to

$$|V_{xi}(t, X_t) G_{ik}(t, X_t)| \leq \frac{1}{2}\sigma \Big\{ [2\beta + \alpha + (a + c\gamma) + k_5\beta_1]\|x\|^2 + [\alpha + k_5\beta_1]\|y\|^2 + [(a + c\gamma) + k_5\beta_1]\|z\|^2 + \frac{3}{4}k_5\sigma\beta_1^2\|y\|^2 \Big\}.$$

In view of the above inequality there exists a positive constant k_9 such that

$$|V_{xi}(t, X_t) G_{ik}(t, X_t)| \leq k_9(x^2 + y^2 + z^2),$$

and for $0 \leq t_0 \leq T < \infty$ and

$$\int_{t_0}^{T} \rho^2(s) ds < \infty$$

where $\rho(t) := k_9(x^2 + y^2 + z^2)(t)$ and $k_9 := \frac{1}{2}\sigma \max\Big\{ \alpha + 2\beta + a + c\gamma + k_5\beta_1, \alpha + k_5\beta_1 + \frac{3}{4}k_5\sigma\beta_1^2, a + c\gamma + k_5\beta_1 \Big\}$. Thus, Assumption 2.7 is satisfied, i.e.,

$$E^{x_0}\Big\{ \int_{t_0}^{T} V_{xi}^2(t, X_t) G_{ik}^2(t, X_t) dt \Big\} < \infty. \tag{4.4}$$

Hypotheses (i) to (iii) of Lemma 2.10 and estimate (2.8) hold true so that Corollary 2.11 (ii) follows, hence by Corollary 2.11 (ii) all solutions of (1.2) are not only bounded but also ultimately stochastically bounded. □

Next theorem presents uniform stochastic boundedness and uniform ultimate stochastic boundedness of solutions of system (1.2).

Theorem 4.2. Suppose that conditions (i) to (v) of Theorem 3.3 and inequality (3.4) are satisfied and in addition $|p(t,x,y,z)| < P_0$ where P_0 is a finite constant, then the solutions (X_t) of system (1.2) are not only uniform stochastically bounded but also uniformly ultimately stochastically bounded.

Proof. Given that (X_t) is any solution of the system (1.2) and the functional (3.3) satisfy inequalities (3.9), (3.11), (4.1) so that hypotheses (i) and (ii) of Lemma 2.9 hold. Also with $p = q = r = 2$ we have $\mu = 0$ so that hypothesis (iii) of Lemma 2.9 holds. In addition, the inequalities (4.3) and (4.4) together with Lemma 2.9 satisfy the hypothesis of Corollary 2.11(i), hence by Corollary 2.11(i) the solutions of system (1.2) are not only uniform stochastically bounded, but also uniform ultimately stochastically bounded. □

Next, we shall state and prove an existence and uniqueness theorem as follows.

Theorem 4.3. If assumptions of Theorem 4.1 are satisfied, then there exists a unique solution of system (1.2).

Proof. Let (X_t) be any solution of (1.2), the functional defined in (3.3) satisfy the following estimates (3.9), (3.10c), and (3.17), these inequalities successfully satisfy all assumptions of Lemma 2.5 thus by Lemma 2.5 solution of system (1.2) exists and unique. Hence, the proof of Theorem 4.3 is completed. □

Next, we shall consider arbitrary third-order stochastic differential equations with delay and show that all assumptions of Theorems 3.1, 3.3, 4.1, 4.2, and 4.3 hold true.

5. Examples

Example 5.1. Consider the third-order stochastic differential equation

$$\begin{aligned} &\dddot{x}(t) + a\ddot{x}(t) + \left[3x\dot{x}(t-\tau(t)) + \left(\frac{\dot{x}(t-\tau(t))}{2 + x^2(t-\tau(t)) + \dot{x}^2(t-\tau(t))}\right)\right] \\ &+ \left[x(t-\tau(t)) + \left(\frac{x(t-\tau(t))}{1 + x^2(t-\tau(t))}\right)\right] + \sigma x(t-\tau(t))\dot{\omega}(t) = 0. \end{aligned} \tag{5.1}$$

Assign $y = \dot{x}$ and $z = \ddot{x}$ equation (5.1) is equivalent to system of first order equations

$$\begin{aligned} &\dot{x} = y, \quad \dot{y} = z, \\ &\dot{z} = -\left(\frac{2x + x^3}{1 + x^2}\right) - \left[\frac{3(2y + x^2y + y^3) + y}{2 + x^2 + y^2}\right] - az - \sigma\left[x - \int_{t-\tau(t)}^{t} y(s)ds\right] \\ &+ \int_{t-\tau(t)}^{t} \left\{\left[1 + \frac{1}{1 + x^2(s)} - \frac{2x^2(s)}{(1 + x^2(s))^2}\right]y(s) - \frac{2x(s)y^2(s)}{(2 + x^2(s) + y^2(s))^2}\right. \\ &\left. + \left[3 + \frac{1}{2 + x^2(s) + y^2(s)} - \frac{2y^2(s)}{(2 + x^2(s) + y^2(s))^2}\right]z(s)\right\}ds. \end{aligned} \tag{5.2}$$

Now, comparing equations (3.2) with (5.2) the following relations hold:

(i) The function $h(x) := \dfrac{x(2 + x^2)}{1 + x^2} = x + \dfrac{x}{1 + x^2}$, clearly $h(0) = 0$ and $\dfrac{h(x)}{x} = 1 + \dfrac{1}{1 + x^2}$. Since $1 + x^2 \geq 1$ for all x, it follows $0 < \dfrac{1}{1 + x^2} \leq 1$ for all x.

Further simplification of the last inequality gives

$$1 = c_0 \leq \frac{h(x)}{x} \leq c_1 = 2 \ \forall \ x \neq 0.$$

(ii) The derivative of $h = h(x)$ with respect to x is defined as $h'(x) := 1 + \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2}$. Since $2x^2(1+x^2)^{-2} \geq 0$ for all x and by (i) to find that

$$h'(x) \leq c = 2, \ \forall \ x. \tag{5.3}$$

Moreover,

$$|h'(x)| \leq k_1 = 2, \ \forall \ x. \tag{5.4}$$

See Figure 1 for the coincide bounds on $h'(x)$ and $|h'(x)|$. Inequalities (5.3) and (5.4) hold true for all $x \in \mathbb{R}$.

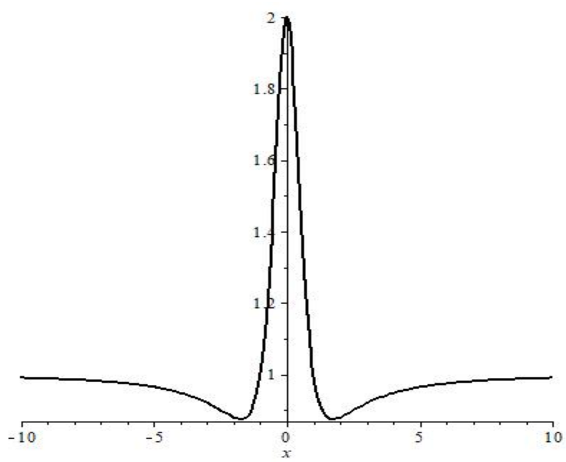

FIGURE 1. Upper bound on the functions $h'(x)$ and $|h'(x)|$ for $x \in [-10, 10]$

(iii) The function $g = g(x, y)$ is defined as $g(x, y) := 3y + \frac{y}{2+x^2+y^2}$. Obviously, $g(0, 0) = 0$ and that $\frac{g(x,y)}{y} = 3 + \frac{1}{2+x^2+y^2}$. It is not difficult to show that $3 = b \leq \frac{g(x,y)}{y} \leq b_1 = 3\frac{1}{2} \ \forall \ x, y \neq 0$.

(iv) The first partial derivatives of g with respect to x and y are given by $g_x(x, y) := \frac{-2xy}{(2+x^2+y^2)^2}$ and $g_y(x, y) := 3 + \frac{1}{2+x^2+y^2} - \frac{2y^2}{(2+x^2+y^2)^2}$ respectively, and is easy to see that

$$|g_x(x, y)| \leq k_2 = 0.12 \tag{5.5a}$$

for all x, y, and

$$|g_y(x, y)| \leq k_3 = 3.5 \tag{5.5b}$$

for all x, y. Figures 2 and 3 confirm estimates (5.5a) and (5.5b) respectively for $-2 \leq x, y \leq 2$.

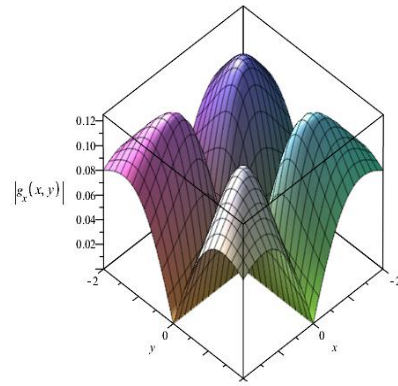

FIGURE 2. Bound on the function $|g_x(x, y)|$ for $x, y \in [-2, 2]$

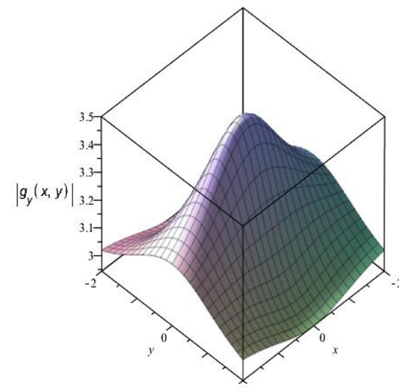

FIGURE 3. Bound on the function $|g_y(x, y)|$ for $x, y \in [-2, 2]$

Since $b = 3, c = 2$, and $c_0 = 1$ it follows from the inequality $ab - c > 0$ that $a > 2/3 \approx 0.7$, we choose $a = 0.8$ and $\sigma^2 < \dfrac{2(ab-c)c_0}{a+(b+1)c}$ so that $\sigma < 0.3$ we choose $\sigma = 0.29$. The following assumptions are verified as

$\alpha := a^2 + ac + c^2 = 6.24 > 0,\ \beta := ab - c = 0.4 > 0,$ and $\gamma := 1 + b = 4 > 0,$

$k_0 = \max\{6.24,\ 0.4,\ 8.8\} = 8.8 > 0,\ a^2b(a + c\gamma) - \alpha\beta = 14.4 > 0,$

$b\beta(a\alpha + c) - a^4b^2 = 4.704 > 0, (a\alpha + c)(a + c\gamma) - \alpha^2 = 22.592 > 0,$

$a^2b\alpha - \beta(a\alpha + c) = 9.184 > 0,\ b\beta\big[(a\alpha + c)(a + c\gamma) - \alpha^2\big] + \beta\big[a^2b\alpha - \beta(a\alpha + c)\big]$
$- a^4b^2(a + c\gamma) + a^2b\alpha\beta = 3.136 > 0,\ B_3 := k_0(k_1 + k_2 + k_3) - (a + \gamma c)\sigma^2 = 48.7159 > 0.$

Next, since $0 < \beta_0 < 1$ two cases are to be considered:

Case 1: When $\beta_0 = 0.001$, we have the following estimates:

$B_4 := \big[3k_0(k_1 + k_2) + (a + c\gamma)\sigma^2 + k_0(k_1 + k_2 + k_3)(1 - \beta_0)\big]/(1 - \beta_0) = 106.2208 > 0,$ and
$B_5 := \big(3k_0k_3 + k_0(k_1 + k_2 + k_3)(1 - \beta_0)\big)/(1 - \beta_0)141.9485.$

In this case the inequality (3.4) yields

$$
\begin{aligned}
\beta_1 &< \min\left\{\frac{2(ab-c)c_0-(a+(b+1)c)\sigma^2}{2B_3},\ \frac{(ab-c)c}{B_4},\ \frac{(ab-c)c}{B_5}\right\} \\
&= \min\{6.1\times 10^{-4},\ 7.5\times 10^{-3},\ 5.6\times 10^{-3}\} \\
&= 6.1\times 10^{-4}
\end{aligned} \tag{5.6}
$$

Case 2: When $\beta_0 = 0.999$ we have the following estimates:

$B_4 := \big[3k_0(k_1+k_2)+(a+c\gamma)\sigma^2+k_0(k_1+k_2+k_3)(1-\beta_0)\big]/(1-\beta_0) = 56757.536 > 0,$ and
$B_5 := \big(3k_0k_3+k_0(k_1+k_2+k_3)(1-\beta_0)\big)/(1-\beta_0) = 92449.456 > 0.$

In this case inequality (3.4) yields

$$
\beta_1 < \min\{6.1\times 10^{-4},\ 1.4\times 10^{-5},\ 8.6\times 10^{-6}\} = 8.6\times 10^{-6} \tag{5.7}
$$

Thus in both cases β_1 is positive, hence for system (5.2) we have the following remark

Remark 5.2. If there exist positive constants $0.12, 1, 2, 3,$ and 3.5 such that

(i) $h(0)=0,\ 1=c_0\le \dfrac{h(x)}{x}\le c_1=2$ for all $x\neq 0$;
(ii) $g(0,0)=0,\ 3=b\le \dfrac{g(x,y)}{y}\le b_1=3.5$ for all x and $y\neq 0$;
(iii) $h'(x)\le c=2$ for all x, $ab-c=0.4>0$, $\sigma=0.29>0$;
(iv) $a^2b(a+c\gamma)-\alpha\beta=14.4>0$, $b\beta(a\alpha+c)-a^4b^2=4.704>0$, $(a\alpha+c)(a+c\gamma)-\alpha^2=22.592>0$, $a^2b\alpha-\beta(a\alpha+c)=9.184>0$, $b\beta\big[(a\alpha+c)(a+c\gamma)-\alpha^2\big]+\beta\big[a^2b\alpha-\beta(a\alpha+c)\big]-a^2b\big[a^2b(a+c\gamma)-\alpha\beta\big]=3.136>0$; and
(v) $|h'(x)|\le k_1=2$, $|g_x(x,y)|\le k_2=0.12$, $|g_y(x,y)|\le k_3=3.5$;

Then the trivial solution of system (5.2) is stochastically stable, asymptotically stochastically stable, uniformly stochastically stable, and uniform asymptotically stochastically stable provided that $8.6\times 10^{-6}\le\beta_1\le 6.1\times 10^{-4}$.

Finally, we shall consider the case $p(\cdot)\neq 0$.

Example 5.3. Consider the third-order stochastic differential equation

$$
\begin{aligned}
&\dddot{x}(t)+a\ddot{x}(t)+\left[3x\dot{x}(t-\tau(t))+\left(\frac{\dot{x}(t-\tau(t))}{2+x^2(t-\tau(t))+\dot{x}^2(t-\tau(t))}\right)\right] \\
&+\left[x(t-\tau(t))+\left(\frac{x(t-\tau(t))}{1+x^2(t-\tau(t))}\right)\right]+\sigma x(t-\tau(t))\dot{\omega}(t) \\
&=\frac{1}{10+t^2+x^2+\dot{x}^2+\ddot{x}^2},
\end{aligned} \tag{5.8}
$$

Assign $y = \dot{x}$ and $z = \ddot{x}$ equation (5.1) is equivalent to system of first order equations

$$\begin{aligned}
&\dot{x} = y, \quad \dot{y} = z,\\
&\dot{z} = -\left(\frac{2x + x^3}{1 + x^2}\right) - \left[\frac{3(2y + x^2y + y^3) + y}{2 + x^2 + y^2}\right] - az - \sigma\left[x - \int_{t-\tau(t)}^{t} y(s)ds\right]\\
&+ \int_{t-\tau(t)}^{t} \left\{\left[1 + \frac{1}{1 + x^2(s)} - \frac{2x^2(s)}{(1 + x^2(s))^2}\right]y(s) - \frac{2x(s)y^2(s)}{(2 + x^2(s) + y^2(s))^2}\right.\\
&\left. + \left[3 + \frac{1}{2 + x^2(s) + y^2(s)} - \frac{2y^2(s)}{(2 + x^2(s) + y^2(s))^2}\right]z(s)\right\}ds\\
&+ \frac{1}{10 + t^2 + x^2 + y^2 + z^2}.
\end{aligned} \tag{5.9}$$

Now comparing (1.2) with (5.9) items (i) to (v) of Remark 5.2 hold. In addition $p(t, x, y, z) := \frac{1}{10 + t^2 + x^2 + y^2 + z^2}$. Since $10 + t^2 + x^2 + y^2 + z^2 \geq 10$ for all $t \geq 0, x, y$, and z there exists a finite constant P_0 such that $|p(t, x, y, z)| < P_0 = \frac{1}{10}$ for all $t \geq 0, x, y$, and z.

Remark 5.4. If in addition to the hypotheses of Theorem 5.2, there exists a finite constant 1/10 such that $|p(t, x, y, z)| < P_0 = \frac{1}{10}$ for all $t \geq 0, x, y$, and z, then the conclusions of Theorems 4.1, 4.2, and 4.3 hold true for all β_1 in the close interval $[8.6 \times 10^{-6},\ 6.1 \times 10^{-4}]$.

6. Conclusion

This paper presents some qualitative properties of solutions to certain third-order nonlinear nonautonomous stochastic differential equations with variable delay. Novel and outstanding results obtained in this paper compliment and extend many outstanding existing results in literature.

7. Acknowledgments

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The authors declared that they have not received any financial support for the research, authorship or publication of this study

The Declaration of Conflict of Interest/ Common Interest
The author(s) declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

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The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s)

declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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