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Yeni Bir Esnek Küme İşlemi: Esnek İkili Parçalı Simetrik Fark İşlemi

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ÖZ

Molodtsov tarafından geliştirilen esnek küme teorisi hem teorik hem de pratik olarak birçok alanda uygulanmıştır. Belirsizliği ele almak için yararlı bir matematiksel araçtır. Ortaya atıldığından bu yana çok sayıda esnek küme işlemi varyasyonu tanımlanmış ve kullanılmıştır. Bu çalışmada, esnek ikili parçalı simetrik fark işlemi adı verilen yeni bir esnek küme işlemi tanımlanıp, özellikleri klasik küme teorisinde var olan simetrik fark işleminin temel cebirsel özellikleri ile karşılaştırmalı olarak ele alınmış ve incelenmiştir. Ayrıca, esnek ikili parçalı simetrik fark işlemi ve kısıtlanmıs kesisim işlemleri ile birlikte sabit parametreye sahip tüm esnek kümelerin oluşturduğu cebirsel yapının, birimli ve değişmeli bir hemiring ve ayrıca Boole halkası olduğu gösterilmiştir.

A New Soft Set Operation: Soft Binary Piecewise Symmetric Difference Operation

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ABSTRACT

The soft set theory developed by Molodtsov has been applied both theoretically and practically in many fields. It is a useful piece of mathematics for handling uncertainty. Numerous variations of soft set operations have been described and used since its introduction. In this paper, a new soft set operation, called soft binary piecewise symmetric difference operation, is defined, its properties are considered and examined comparatively with the basic algebraic properties of symmetric difference operation existing in classical set theory. Moreover, we prove that the set of all the soft sets with a fixed parameter set together with the soft binary piecewise symmetric difference operation and the restricted intersection operation is a commutative hemiring with identity and also Boolean ring.

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INTRODUCTION

Due to the existence of some types of uncertainty, we are unable to effectively employ traditional ways to address issues in many domains, including engineering, environmental and health sciences, and economics. Molodtsov [1], in 1999, proposed Soft Set Theory as a mathematical method to deal with these uncertainties. Since then, this theory has been applied to a variety of fields, including information systems, decision-making, optimization theory, game theory, operations research, measurement theory, and some algebraic structures. The initial contributions to soft set operations were states by Maji et al. [2] and Pei and Miao [3]. Following this, Ali et al. [4] introduced and discussed several soft set operations, including restricted and extended soft set operations. Sezgin and Atagün [5] discussed the basic properties of soft set operations together with their interconnections. They also investigated and defined the idea of restricted symmetric difference of soft sets. A brand-new soft set operation called “extended difference of soft sets” was presented by Sezgin et al. [6]. Stojanovic [7] introduced the concept of "extended symmetric difference of soft sets" and its basic properties were investigated. Two main categories into which the operations of soft set theory fall, according to the research, are restricted soft set operations and extended soft set operations. Eren [8] created a brand-new class of soft difference operations, which we call as soft binary piecewise difference, and they also carefully analyzed the core characteristics of the operation. Other soft binary piecewise operations were defined Yavuz [9], who also carefully analyzed their core characteristics. Since the operations of soft sets are the fundamental concepts of soft set theory, soft set operations have been extensively studied since 2003. For more details, we refer to [10-28].

Semirings were initially described by Vandiver [29] in 1934 and consist of a set R together with the two associative binary operations addition "+" and multiplication "." such that distributes over "+" from both sides. Different researchers, including [30,31], have given a variety of theories and findings regarding semirings and some researchers have explored semirings with additive inverse [32-35]. Semirings have received extensive study more recently, particularly in respect to applications (see [36]). Semirings are very crucial in geometry, nonetheless, they are crucial for solving issues in a variety of applications of practical mathematics and information sciences, as well as being significant in pure mathematics. [37-42]. By a hemiring, we mean a special semiring with a zero and a commutative addition. In theoretical computer science, hemirings, are also crucial. Hemirings occurs naturally in several applications to the theory of formal languages, computer sciences and automata [42].

This paper contributes to the literature on soft set theory by describing a novel soft set operation, which we call "soft binary piecewise symmetric difference operation". This paper is arranged in the following manner. In Section 2, we recall preliminary concepts in soft set theory together with semirings and hemirings. In Section 3, definition and an example of soft binary piecewise symmetric difference operation are given and the full analysis of the algebraic properties of this new operation are handled comparatively with symmetric difference operation existing in classical set theory and we obtain very remarkable analogies. In the same section, it is proved that the set of all the soft sets with a fixed parameter set together with the soft binary piecewise symmetric difference operation and the soft restricted intersection operation is a commutative hemiring with identity and also Boolean ring. In the conclusion section, we put into focus the meaning of the study's findings and its potential influence on the field.

PRELIMINARIES

Definition 1. [1] Let U be the universal set, E be the parameter set, $P(U)$ be the power set of U and $A \subseteq E$. A pair (F, A) is called a soft set over U where F is a set-valued function such that $F: A \rightarrow P(U)$.

Throughout this paper, the set of all the soft sets over U is designated by $S_E(U)$. Let A be a fixed subset of E and $S_A(U)$ be the collection of all soft sets over U with the fixed parameter set A. Clearly $S_A(U)$ is a subset of $S_E(U)$. From now on, while soft set will be designated by SS and parameter set by PS; soft sets will be designated by SSs and parameter sets by PSs for the sake of ease.

Definition 2. [4] (K, W) is called a relative null SS (with regard to W), denoted by $\emptyset_W$, if $K(\omega) = \emptyset$ for all $\omega \in W$ and (K, W) is called a relative whole SS (with regard to W), denoted by U_W if $K(\omega) = U$ for all

$\omega \in W$. The relative whole SS U_E with regard to E is called the absolute SS over U. We shall denote by $\emptyset_\emptyset$ the unique soft set over U with an empty parameter set, which is called the empty soft set over U. Note that by $\emptyset_\emptyset$ and by $\emptyset_A$ are different soft sets over U [17].

Definition 3. [3] For two SSs (K,W) and $(T,Ş)$, (K,W) is a soft subset of $(T,Ş)$ and it is denoted by $(K,W) \tilde{\subseteq} (T,Ş)$, if $W \subseteq Ş$ and $K(\omega) \subseteq T(\omega)$, $\forall\, \omega \in W$. Two SSs (K,W) and $(T,Ş)$ are said to be soft equal if (K,W) is a soft subset of $(T,Ş)$ and $(T,Ş)$ is a soft subset of (K,W).

Definition 4. [4] The relative complement of a SS (K,W), denoted by $(K,W)^r$, is defined by $(K,W)^r = (K^r,W)$, where $K^r\colon W \to P(U)$ is a mapping given by $(K,W)^r = U\backslash W(\omega)$ for all $\omega \in W$. From now on, $U\backslash K(\omega)=[K(\omega)]'$ will be designated by $K'(\omega)$ for the sake of ease.

SS operations can be grouped into the following categories as a summary: If " Θ " is used to denote the set operations (Namely, Θ here can be $\cap$, $\cup$, $\backslash$, Δ), then the following soft set operations are defined as following:

Definition 5. [4,5] Let (K,W) and $(T,Ş)$ be SSs over U. The restricted Θ operation of (K,W) and $(T,Ş)$ is the SS (S,X), denoted by $(K,W)\Theta_R(T,Ş) = (B,X)$, where $X = W \cap Ş \neq \emptyset$ and $\forall\omega \in X$, $B(\omega) = K(\omega)\ \Theta\ T(\omega)$. Here note that if $W \cap Ş = \emptyset$, then $(K,W)\Theta_R(T,Ş) = \emptyset_\emptyset$ [17].

Definition 6. [3,4,6,7] Let (K,W) and $(T,Ş)$ be SSs over U. The extended Θ operation of (K,W) and $(T,Ş)$ is the SS (B,X), denoted by $(K,W)\Theta_\varepsilon(T,Ş) = (B,X)$, where $X = W \cup Ş$ and $\forall\omega \in X$

$$B(\omega) = \begin{cases} K(\omega), & \omega \in W\backslash Ş, \\ T(\omega), & \omega \in Ş\backslash W, \\ K(\omega)\ \Theta T(\omega), & \omega \in W \cap Ş. \end{cases}$$

Definition 7. [8,9] Let (K,W) and $(T,Ş)$ be SSs over U. The soft binary piecewise Θ operation of (K,W) and $(T,Ş)$ is the SS (B,W), denoted by, $(K,W)\tilde{\Theta}(T,Ş) = (B,W)$, where $\forall\omega \in W$,

$$B(\omega)=\begin{cases} K(\omega), & \omega \in W\backslash Ş \\ K(\omega)\ \Theta\ T\omega), & \omega \in W\cap Ş \end{cases}$$

In mathematics, a semiring is used in abstract algebra to describe an algebraic structure which is more general than ring. A semiring (R,+, $\cdot$) is an algebraic structure consisting of a non-empty set R together with two binary operations usually called addition and multiplication such that (R,+) is a semigroup, (R, $\cdot$) is a semigroup and multiplication is distributive over addition from both sides. If a semiring has identity with multiplication, then it is called semiring with identity and if it has commutative multiplication, then it is called a commutative semiring. If there exists an element $0 \in R$ such that $0 \cdot a = a \cdot 0 = 0$ and $0 + a = a + 0 = a$ for all $a \in R$, then 0 is called the zero of R. A semiring with commutative addition and zero element is called a hemiring. For more about semirings and hemirings, we refer to [29-42].

MAIN RESULTS

Definition 8. Let $(\text{ɥ},Ƶ)$ and $(Ʊ,Ş)$ be SSs over U. The soft binary piecewise symmetric difference (Δ) operation of $(\text{ɥ},Ƶ)$ and $(Ʊ,Ş)$ is the SS $(\aleph,Ƶ)$, denoted by, $(\text{ɥ},Ƶ)\tilde{\Delta}(Ʊ,Ş) = (\aleph,Ƶ)$, where $\forall\omega \in Ƶ$,

$$\aleph(\omega)=\begin{cases} \text{ɥ}(\omega), & \omega \in Ƶ\backslash Ş \\ \text{ɥ}(\omega)\ \Delta Ʊ(\omega), & \omega \in Ƶ\cap Ş \end{cases}$$

Here note that, in [5], Sezgin and Atagün used "$\tilde{\Delta}$" for restricted symmetric difference; however, we prefer to use "Δ_R" for the restricted symmetric difference. Thus, in what follows, $\tilde{\Delta}$ will be used for the soft binary piecewise symmetric difference, not for restricted symmetric difference.

Example 9. Let $E=\{e_1,e_2,e_3,e_4\}$ be the PS, $Ƶ=\{e_1, e_3\}$ and $Ş=\{e_2, e_3, e_4\}$ be the subsets of E and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be universe set. Let $(ᴍ,Ƶ)$ and $(Ʊ,Ş)$ be SSs over U defined as following

$(ᴍ,Ƶ)=\{(e_1, \{h_2, h_5\}), (e_3,\{h_1,h_2,h_5\})\}$

$(Ʊ,Ş)=\{(e_2,\{h_1,h_4,h_5\}), (e_3,\{h_2,h_3,h_4\}),(e_4,\{h_3,h_5\}\})$.

Let $(ᴍ,Ƶ)\ \tilde{\Delta}\ (Ʊ,Ş)=(ℵ,Ƶ)$. Then,

$$ℵ(\omega)=\begin{cases} ᴍ(\omega), & \omega\in Ƶ\setminus Ş \\ ᴍ(\omega)\ \Delta Ʊ(\omega), & \omega\in Ƶ\cap Ş \end{cases}$$

Since $Ƶ=\{e_1, e_3\}$ and $Ƶ\setminus Ş=\{e_1\}$, so $ℵ(e_1)=ᴍ(e_1)=\{h_2, h_5\}$. And since $Ƶ\cap Ş=\{e_3\}$ so $ℵ(e_3)=ᴍ(e_3)\ \Delta Ʊ(e_3)=\{h_1, h_3, h_4,h_5\}$. Thus, $(ℵ,Ƶ)= (ᴍ, Ƶ)\ \tilde{\Delta}\ (Ʊ,Ş)=\{ (e_1,\{h_2,h_5\}), (e_3, \{h_1, h_3, h_4, h_5\})\}$.

The set of elements that are in either sets but not their intersection is known as the symmetric difference of two sets in classical theory. Namely, $Ƶ\Delta Ş=(Ƶ\cup Ş)\setminus(Ƶ\cap Ş)$. Now, we have:

Theorem 10. $(ᴍ, Ƶ)\ \tilde{\Delta}\ (Ʊ,Ş)] =[(ᴍ,Ƶ)\ \tilde{\cup}\ (Ʊ, Ş)]\ \tilde{\setminus}[(ᴍ,Ƶ) \cap_R (Ʊ, Ş)]$.

Proof: Since the PS of the SSs of both hand side is Ƶ, the first condition for the soft equality is satisfied. Now let $(ᴍ, Ƶ)\ \tilde{\cup}(Ʊ,Ş)=(ℵ,Ƶ)$ where $\forall\omega\in Ƶ$;

$$ℵ(\omega)=\begin{cases} ᴍ(\omega), & \omega\in Ƶ\setminus Ş \\ ᴍ(\omega)\cup Ʊ(\omega), & \omega\in Ƶ\cap Ş \end{cases}$$

Let $(ᴍ, Ƶ) \cap_R(Ʊ,Ş)=(M,Ƶ\cap Ş)$, where $\forall\omega\in Ƶ\cap Ş$; $M(\omega)=ᴍ(\omega)\cap Ʊ(\omega)$. Let $(ℵ,Ƶ)\tilde{\setminus}\ (M,Ƶ\cap Ş)=(S,Ƶ)$, where for $\forall\omega\in Ƶ$

$$S(\omega)=\begin{cases} ℵ(\omega), & \omega\in Ƶ\setminus(Ƶ\cap Ş)=Ƶ\setminus Ş \\ ℵ(\omega)\setminus M(\omega), & \omega\in Ƶ\cap(Ƶ\cap Ş)=Ƶ\cap Ş \end{cases}$$

Thus,

$$S(\omega)=\begin{cases} ᴍ(\omega), & \omega\in(Ƶ\setminus Ş)\setminus Ş=Ƶ\setminus Ş \\ ᴍ(\omega)\cup Ʊ(\omega), & \omega\in(Ƶ\cap Ş)\setminus Ş=\emptyset \\ ᴍ(\omega)\setminus(ᴍ(\omega)\cap Ʊ(\omega)), & \omega\in(Ƶ\setminus Ş)\cap Ş=\emptyset \\ [ᴍ(\omega)\cup Ʊ(\omega)]\setminus(ᴍ(\omega)\cap Ʊ(\omega), & \omega\in(Ƶ\cap Ş)\cap Ş=Ƶ\cap Ş \end{cases}$$

Thus,

$$S(\omega)=\begin{cases} ᴍ(\omega), & \omega\in Ƶ\setminus Ş \\ [ᴍ(\omega)\cup Ʊ(\omega)]\setminus(ᴍ(\omega)\cap Ʊ(\omega)), & \omega\in Ƶ\cap Ş \end{cases}$$

Hence,

$$\begin{cases} ᴍ(\omega), & \omega\in Ƶ\setminus Ş \end{cases}$$

$$S(\omega)= \quad ᴍ(\omega)\ \Delta Ʊ(\omega), \qquad \omega\in Ƶ\cap Ş$$

Thus, $(S,Ƶ)= (ᴍ, Ƶ)\ \tilde{\Delta}\ (Ʊ,Ş)$.

In classical theory, $Ƶ\ \Delta Ş=(Ƶ\setminus Ş)\cup(Ş\setminus Ƶ)$. Now, we have:

Theorem 11. $(ᴍ, Ƶ)\ \tilde{\Delta}\ (Ʊ,Ş) =[(ᴍ,Ƶ)\ \tilde{\setminus}(Ʊ, Ş)]\ \tilde{\cup}[(Ʊ,Ş)\tilde{\setminus}(ᴍ, Ƶ)]$.

Proof: Since the PS of the SSs of both hand side is Ƶ, the first condition for the soft equality is satisfied. Now let $(ᴍ, Ƶ)\ \tilde{\setminus}(Ʊ,Ş)=(\aleph,Ƶ)$ where $\forall\omega\in Ƶ$;

$$\aleph(\omega)=\begin{cases} ᴍ(\omega), & \omega\in Ƶ\setminus Ş \\ ᴍ(\omega)\setminus Ʊ(\omega), & \omega\in Ƶ\cap Ş \end{cases}$$

Let $(Ʊ,Ş)\ \tilde{\setminus}(ᴍ,Ƶ)=(K,Ş)$ where $\forall\omega\in Ƶ$;

$$K(\omega)=\begin{cases} Ʊ(\omega), & \omega\in Ş\setminus Ƶ \\ Ʊ(\omega)\setminus ᴍ(\omega), & \omega\in Ş\cap Ƶ \end{cases}$$

Let $(\aleph,Ƶ)\tilde{\cup}\ (K,Ş)=(S,Ƶ)$, where for $\forall\omega\in Ƶ$;

$$S(\omega)=\begin{cases} \aleph(\omega), & \omega\in Ƶ\setminus Ş \\ \aleph(\omega)\cup K(\omega), & \omega\in Ƶ\cap Ş \end{cases}$$

Hence,

$$S(\omega)=\begin{cases} ᴍ(\omega), & \omega\in(Ƶ\setminus Ş)\setminus Ş=Ƶ\setminus Ş \\ ᴍ(\omega)\setminus Ʊ(\omega), & \omega\in(Ƶ\cap Ş)\setminus Ş=\emptyset \\ ᴍ(\omega)\cup Ʊ(\omega), & \omega\in(Ƶ\setminus Ş)\cap(Ş\setminus Ƶ)=\emptyset \\ ᴍ(\omega)\cup(Ʊ(\omega)\setminus ᴍ(\omega)), & \omega\in(Ƶ\setminus Ş)\cap(Ş\cap Ƶ)=\emptyset \\ (ᴍ(\omega)\setminus Ʊ(\omega))\cup Ʊ(\omega), & \omega\in(Ƶ\cap Ş)\cap(Ş\setminus Ƶ)=\emptyset \\ [ᴍ(\omega)\setminus Ʊ(\omega)]\cup[Ʊ(\omega)\setminus ᴍ(\omega)], & \omega\in(Ƶ\cap Ş)\cap(Ş\cap Ƶ)=Ƶ\cap Ş \end{cases}$$

Thus,

$$S(\omega)=\begin{cases} ᴍ(\omega), & \omega\in Ƶ\setminus Ş \\ [ᴍ(\omega)\setminus Ʊ(\omega)]\cup[Ʊ(\omega)\setminus ᴍ(\omega)], & \omega\in Ƶ\cap Ş \end{cases}$$

Therefore,

$$S(\omega)=\begin{cases} ᴍ(\omega), & \omega\in Ƶ\setminus Ş \\ ᴍ(\omega)\ \Delta Ʊ(\omega), & \omega\in Ƶ\cap Ş \end{cases}$$

Hence, $(S,Ƶ)= (ᴍ, Ƶ)\ \tilde{\Delta}\ (Ʊ,Ş)$.

Theorem 12.

1) $S_E(U)$ is closed under $\tilde{\Delta}$. Namely, when (ᴍ,Ƶ) and (Ʊ,Ç) are two SSs over U, then so is (ᴍ,Ƶ) $\tilde{\Delta}$(Ʊ, Ç) as $\tilde{\Delta}$ is a binary operation in $S_E(U)$. $S_Ƶ(U)$ is closed under $\tilde{\Delta}$, too.

In classical theory, (MΔL)ΔN= MΔ(LΔN). As an analogy, we have:

2) [(ᴍ, Ƶ) $\tilde{\Delta}$ (Ʊ,Ƶ)] $\tilde{\Delta}$(ℵ,Ƶ)=(ᴍ,Ƶ) $\tilde{\Delta}$[(Ʊ, Ƶ) $\tilde{\Delta}$(ℵ, Ƶ)].

Proof: Let (ᴍ, Ƶ) $\tilde{\Delta}$(Ʊ,Ƶ)=(T,Ƶ), where ∀ω∈Ƶ;

$$T(\omega)=\begin{cases} ᴍ(\omega), & \omega\in Ƶ\setminus Ƶ=\emptyset \\ ᴍ(\omega)\ \Delta Ʊ(\omega), & \omega\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Let (T,Ƶ) $\tilde{\Delta}$(ℵ,Ƶ) =(M,Ƶ), where ∀ω∈Ƶ;

$$M(\omega)=\begin{cases} T(\omega), & \omega\in Ƶ\setminus Ƶ=\emptyset \\ T(\omega)\ \Delta ℵ(\omega), & \omega\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Thus,

$$M(\omega)=\begin{cases} T(\omega), & \omega\in Ƶ\setminus Ƶ=\emptyset \\ [ᴍ(\omega)\ \Delta Ʊ(\omega)]\Delta ℵ(\omega), & \omega\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Let (Ʊ,Ƶ) $\tilde{\Delta}$ (ℵ,Ƶ)=(K,Ƶ), where ∀ω∈Ƶ;

$$K(\omega)=\begin{cases} Ʊ(\omega), & \omega\in Ƶ\setminus Ƶ=\emptyset \\ Ʊ(\omega)\ \Delta ℵ(\omega), & \omega\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Let (ᴍ,Ƶ) $\tilde{\Delta}$(K,Ƶ) =(N,Ƶ), where ∀ω∈Ƶ;

$$N(\omega)=\begin{cases} ᴍ(\omega), & \omega\in Ƶ\setminus Ƶ=\emptyset \\ ᴍ(\omega)\ \Delta K(\omega), & \omega\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Thus,

$$N(\omega)=\begin{cases} ᴍ(\omega), & \omega\in Ƶ\setminus Ƶ=\emptyset \\ ᴍ(\omega)\ \Delta[Ʊ(\omega)\Delta ℵ(\omega)], & \omega\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

It is seen that (M,Ƶ)=(N,Ƶ).

Namely, for the SSs whose PSs are the same, $\tilde{\Delta}$ is associative. Here's what we have right now:

3) [(ᴍ, Ƶ) $\tilde{\Delta}$(Ʊ,Ç)] $\tilde{\Delta}$(ℵ,Ö) = (ᴍ,Ƶ) $\tilde{\Delta}$[(Ʊ,Ç) $\tilde{\Delta}$(ℵ,Ö)] where Ƶ∩Ç'∩Ö=∅.

Proof: Let (ᴍ, Ƶ) $\tilde{\Delta}$(Ʊ,Ç)=(T,Ƶ), where ∀ω∈Ƶ;

ᴍ(ω), ω∈Ƶ\Ç

$T(\omega)=$

$$\text{ϻ}(\omega)\ \Delta\text{Ʊ}(\omega)\ , \qquad \omega\in\text{Ƶ}\cap\text{Ç}$$

Let (T ,Ƶ) $\tilde{\Delta}$(ℵ,Ö) =(M,Ƶ), where $\forall\omega\in$Ƶ;

$$M(\omega)=\begin{cases} T(\omega), & \omega\in\text{Ƶ}\backslash\text{Ö} \\ T(\omega)\ \Delta\aleph(\omega)\ , & \omega\in\text{Ƶ}\cap\text{Ö} \end{cases}$$

Thus,

$$M(\omega)=\begin{cases} \text{ϻ}(\omega), & \omega\in(\text{Ƶ}\backslash\text{Ç})\backslash\text{Ö}=\text{Ƶ}\cap\text{Ç}'\cap\text{Ö}' \\ \text{ϻ}(\omega)\ \Delta\text{Ʊ}(\omega), & \omega\in(\text{Ƶ}\cap\text{Ç})\backslash\text{Ö}=\text{Ƶ}\cap\text{Ç}\cap\text{Ö}' \\ \text{ϻ}(\omega)\ \Delta\aleph(\omega), & \omega\in(\text{Ƶ}\backslash\text{Ç})\cap\text{Ö}=\text{Ƶ}\cap\text{Ç}'\cap\text{Ö} \\ [\ \text{ϻ}(\omega)\ \Delta\text{Ʊ}(\omega)]\ \Delta\aleph(\omega), & \omega\in(\text{Ƶ}\cap\text{Ç})\ \cap\text{Ö}=\text{Ƶ}\cap\text{Ç}\cap\text{Ö} \end{cases}$$

Let (Ʊ,Ç) $\tilde{\Delta}$ (ℵ,Ö)=(K,Ç), where $\forall\omega\in$Ç;

$$K(\omega)=\begin{cases} \text{Ʊ}(\omega), & \omega\in\text{Ç}\backslash\text{Ö} \\ \text{Ʊ}(\omega)\ \Delta\aleph(\omega)\ , & \omega\in\text{Ç}\cap\text{Ö} \end{cases}$$

Let (ϻ,Ƶ) $\tilde{\Delta}$(K,Ç) =(S,Ƶ), where $\forall\omega\in$Ƶ;

$$S(\omega)=\begin{cases} \text{ϻ}(\omega), & \omega\in\text{Ƶ}\backslash\text{Ç} \\ \text{ϻ}(\omega)\ \Delta K(\omega)\ , & \omega\in\text{Ƶ}\cap\text{Ç} \end{cases}$$

Thus,

$$S(\omega)=\begin{cases} \text{ϻ}(\omega), & \omega\in\text{Ƶ}\backslash\text{Ç} \\ \text{ϻ}(\omega)\ \Delta\text{Ʊ}(\omega), & \omega\in\text{Ƶ}\cap(\text{Ç}\backslash\text{Ö})=\text{Ƶ}\cap\text{Ç}\cap\text{Ö}' \\ \text{ϻ}(\omega)\ \Delta\ [\text{Ʊ}(\omega)\ \Delta\aleph(\omega)], & \omega\in\text{Ƶ}\cap(\text{Ç}\cap\text{Ö})=\text{Ƶ}\cap\text{Ç}\cap\text{Ö} \end{cases}$$

Here, let's consider $\omega\in$Ƶ\Ç in the second equation. Since Ƶ\Ç= Ƶ∩Ç', if $\omega\in$Ç', then $\omega\in$Ö\Ç or $\omega\in$(Ç∪Ö)'. Hence, if $\omega\in$Ƶ\Ç, then $\omega\in$Ƶ∩Ç'∩Ö' or $\omega\in$Ƶ∩Ç'∩Ö. Thus, it is seen that (M,Ƶ)=(S,Ƶ), where Ƶ∩Ç'∩Ö=∅.

In classical theory, symmetric difference operation is commutative, i.e., MΔL= LΔM. However, we have:

4) (ϻ, Ƶ) $\tilde{\Delta}$ (Ʊ,Ç)≠(Ʊ,Ç) $\tilde{\Delta}$(ϻ,Ƶ)

Proof: Let (ϻ, Ƶ) $\tilde{\Delta}$ (Ʊ,Ç)=(ℵ,Ƶ) . Then, $\forall\omega\in$Ƶ;

$$\aleph(\omega)=\begin{cases} \text{ϻ}(\omega), & \omega\ \in\text{Ƶ}\backslash\text{Ç} \\ \text{ϻ}(\omega)\ \Delta\text{Ʊ}(\omega), & \omega\in\text{Ƶ}\cap\text{Ç} \end{cases}$$

Let (Ʊ,Ç) $\tilde{\Delta}$ (ϻ,Ƶ)=(T,Ç) . Then $\forall\omega\in$Ç;

$$\begin{cases} \text{Ʊ}(\omega)\ , & \omega\in\text{Ç}\backslash\text{Ƶ} \end{cases}$$

T(ω)=

$\mho(\omega)\ \Delta ꟽ(\omega)$, $\omega \in Ç\cap Ƶ$

Here, while the PS of the SS of left side is Ƶ; the PS of the SS of right side is Ç. Thus,

$$(ꟽ, Ƶ)\ \tilde{\Delta}(\mho,Ç) \neq (\mho,Ç)\ \tilde{\Delta}(ꟽ,Ƶ).$$

Hence, $\tilde{\Delta}$ is not commutative in $S_E(U)$, where the PSs of the SSs are different. However, it is easy to see that

$$(ꟽ, Ƶ)\tilde{\Delta}(\mho,Ƶ)=(\mho,Ƶ)\ \tilde{\Delta}\ (ꟽ,Ƶ).$$

That is to say, $\tilde{\Delta}$ is commutative, where the PSs of the SSs are the same.

In classical theory, $\emptyset$ is the identity element for the symmetric difference operation, i.e., $M\Delta\emptyset=\emptyset\Delta M=M$. As an analogy, we have:

5) $(ꟽ, Ƶ)\ \tilde{\Delta}\ \emptyset_Ƶ = \emptyset_Ƶ \tilde{\Delta}\ (ꟽ,Ƶ) = (ꟽ, Ƶ)$.

Proof: Let $\emptyset_Ƶ$=(S,Ƶ). Then, $\forall\omega\in Ƶ$; S(ω)= $\emptyset$. Let (ꟽ, Ƶ) $\tilde{\Delta}$(S,Ƶ)=(ℵ,Ƶ), where $\forall\omega\in Ƶ$,

$$\aleph(\omega)=\begin{cases} ꟽ(\omega)\ , & \omega\in Ƶ\backslash Ƶ=\emptyset \\ ꟽ(\omega)\ \Delta\ S(\omega), & \omega\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Hence, $\forall\omega\in Ƶ$; $\aleph(\omega)= ꟽ(\omega)\Delta S(\omega) = ꟽ(\omega)\ \Delta\emptyset= ꟽ(\omega)$. Thus, (ℵ,Ƶ)= (ꟽ, Ƶ) .

Note that, for the SSs whose PS is Ƶ, $\emptyset_Ƶ$ is the identity element for $\tilde{\Delta}$ in $S_Ƶ(U)$.

In classical theory, every element is its own inverse for the symmetric difference operation, i.e., $M\Delta M = \emptyset$. As an analogy, we have:

6) $(ꟽ, Ƶ)\ \tilde{\Delta}\ (ꟽ,Ƶ)= \emptyset_Ƶ$.

Proof: Let (ꟽ, Ƶ) $\tilde{\Delta}$(ꟽ,Ƶ)=(ℵ,Ƶ), where $\forall\omega\in Ƶ$;

$$\aleph(\omega)=\begin{cases} ꟽ(\omega), & \omega\in Ƶ\backslash Ƶ=\emptyset \\ ꟽ(\omega)\ \Delta ꟽ(\omega)\ , & \omega\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Here $\forall\omega\in Ƶ$; $\aleph(\omega)= ꟽ(\omega)\ \Delta ꟽ(\omega)= \emptyset$, thus (ℵ,Ƶ)= $\emptyset_Ƶ$.

This property shows us that every SS is its own inverse for $\tilde{\Delta}$ in $S_Ƶ(U)$ and also $\tilde{\Delta}$ has not idempotency property on $S_E(U)$.

<u>**REMARK 13:**</u> By Theorem 12 (1), (2), (4), (5) and (6), ($S_Ƶ(U)$, $\tilde{\Delta}$) is an abelian group.

7) $(ꟽ, Ƶ)\ \tilde{\Delta}\ \emptyset_E$=(ꟽ,Ƶ).

Proof: Let $\emptyset_E$ =(S,E) . Hence $\forall\omega\in E$; S(ω)=$\emptyset$. Let (ꟽ,Ƶ) $\tilde{\Delta}$ (S,E)=(ℵ,Ƶ) . Thus, $\forall\omega\in Ƶ$,

$$\aleph(\omega)=\begin{cases} ꟽ(\omega), & \omega\in Ƶ\backslash E=\emptyset \\ ꟽ(\omega)\ \Delta S(\omega), & \omega\in Ƶ\cap E=Ƶ \end{cases}$$

Hence, $\forall\omega\in Ƶ$ $\aleph(\omega)= ꟽ(\omega)\ \Delta S(\omega)= ꟽ(\omega)\ \Delta\emptyset= ꟽ(\omega)$, so (ℵ,Ƶ)=(ꟽ,Ƶ).

Note that, for the SSs (no matter what the PS is), $\emptyset_E$ is the right identity element for $\tilde{\Delta}$ in $S_E(U)$.

8) $(\mathrm{M}, Ƶ)\ \tilde{\Delta}\ \emptyset_{\emptyset} = (\mathrm{M}, Ƶ)$.

Proof: Let $\emptyset_{\emptyset}$=(S, ∅). Let $(\mathrm{M}, Ƶ)\ \tilde{\Delta}$(S, ∅)=(ℵ,Ƶ), where ∀ω∈Ƶ,

$$ℵ(\omega)=\begin{cases} \mathrm{M}(\omega), & \omega\in Ƶ\setminus\emptyset=Ƶ \\ \mathrm{M}(\omega)\ \Delta\ \mathrm{S}(\omega), & \omega\in Ƶ\cap\emptyset=\emptyset \end{cases}$$

Hence, ∀ω∈Ƶ; ℵ(ω)= M(ω). Thus, (ℵ,Ƶ)= (M, Ƶ) .

Note that, for the SSs (no matter what the PS is), $\emptyset_{\emptyset}$ is the right identity element for $\tilde{\Delta}$ in $S_E(U)$.

9) $\emptyset_{\emptyset}\tilde{\Delta}\ (\mathrm{M},Ƶ)= \emptyset_{\emptyset}$

Proof: Let (S, ∅)$\tilde{\Delta}$ (M, Ƶ) =(T, ∅). Since, $\emptyset_{\emptyset}$ is the unique SS with empty set, (T, ∅)=$\emptyset_{\emptyset}$. Note that, for the SSs (no matter what the PS is), $\emptyset_{\emptyset}$ is the left absorbing element for $\tilde{\Delta}$ in $S_E(U)$.

In classical theory, MΔU=UΔM=M', where U is the universal set. As an analogy, we have:

10) $(\mathrm{M}, Ƶ)\ \tilde{\Delta}\ U_Ƶ=U_Ƶ\tilde{\Delta}\ (\mathrm{M},Ƶ)=(\mathrm{M},Ƶ)^r$.

Proof: Let $U_Ƶ$ = (T,Ƶ). Then, ∀ω∈Ƶ; T(ω)=U. Let (M,Ƶ) $\tilde{\Delta}$ (T,Ƶ)=(ℵ,Ƶ) , where ∀ω∈Ƶ;

$$ℵ(\omega)=\begin{cases} \mathrm{M}(\omega), & \omega\in Ƶ\setminus Ƶ=\emptyset \\ \mathrm{M}(\omega)\ \Delta \mathrm{T}(\omega), & \omega\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Thus, ∀ω∈Ƶ; ℵ(ω)= M(ω) ΔT(ω)= M(ω) ΔU= M'(ω), hence (ℵ,Ƶ)= $(\mathrm{M},Ƶ)^r$.

11) $(\mathrm{M}, Ƶ)\tilde{\Delta}U_E=(\mathrm{M},Ƶ)^r$

Proof: Let U_E =(T,E). Hence, ∀ω∈E, T(ω)=U. Let (M, Ƶ)$\tilde{\Delta}$ (T, E)=(ℵ,Ƶ) , then ∀ω∈Ƶ ,

$$ℵ(\omega)=\begin{cases} \mathrm{M}(\omega), & \omega\in Ƶ\setminus E=\emptyset \\ \mathrm{M}(\omega)\ \Delta \mathrm{T}(\omega), & \omega\in Ƶ\cap E=Ƶ \end{cases}$$

Hence, ∀ω∈Ƶ, ℵ(ω)= M(ω) ΔT(ω)= M(ω) ΔU= M'(ω), so (ℵ,Ƶ)=$(\mathrm{M},Ƶ)^r$.

In classical theory, MΔM' = M'ΔM=U, where U is the universal set. As an analogy, we have:

12) $(\mathrm{M}, Ƶ)\ \tilde{\Delta}\ (\mathrm{M},Ƶ)^r= (\mathrm{M},Ƶ)^r\ \tilde{\Delta}\ (\mathrm{M},Ƶ)= U_Ƶ$

Proof: Let $(\mathrm{M},Ƶ)^r$=(ℵ,Ƶ). Hence, ∀ω∈Ƶ; ℵ(ω)= M'(ω). Let (M, Ƶ) $\tilde{\Delta}$ (ℵ,Ƶ)=(T,Ƶ), where ∀ω∈Ƶ,

$$\mathrm{T}(\omega)=\begin{cases} \mathrm{M}(\omega), & \omega\in Ƶ\setminus Ƶ=\emptyset \\ \mathrm{M}(\omega)\ \Delta ℵ(\omega), & \omega\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Hence, ∀ω∈Ƶ; T(ω)= M(ω) Δℵ(ω)=M(ω) Δ M'(ω)= U, thus (T,Ƶ)= $U_Ƶ$.

In classical theory, (MΔL)Δ(LΔN)= MΔN. As an analogy, we have:

13) [(M, Ƶ) $\tilde{\Delta}$ (Ʊ,Ş)] $\tilde{\Delta}$[(Ʊ,Ş) $\tilde{\Delta}$ (ℵ,Ƶ)] =(M,Ƶ) $\tilde{\Delta}$(ℵ, Ş).

Proof: Since the PS of the SSs of both hand side is Ƶ, the first condition for the soft equality is satisfied. Now let (M, Ƶ) $\tilde{\Delta}$(Ʊ,Ş)=(ℵ,Ƶ) where ∀ω∈Ƶ;

$$\aleph(\omega)=\begin{cases} \text{M}(\omega), & \omega\in Ƶ\backslash Ş \\ \text{M}(\omega)\ \Delta Ʊ(\omega)\ , & \omega\in Ƶ\cap Ş \end{cases}$$

Let (Ʊ,Ş) $\tilde{\Delta}$(ℵ,Ƶ)=(K,Ş) where $\forall\omega\in$Ƶ;

$$K(\omega)=\begin{cases} Ʊ(\omega), & \omega\in Ş\backslash Ƶ \\ Ʊ(\omega)\ \Delta\aleph\ (\omega)\ , & \omega\in Ş\cap Ƶ \end{cases}$$

Let (ℵ,Ƶ)$\tilde{\Delta}$ (K,Ş)=(S,Ƶ), where for $\forall\omega\in$Ƶ;

$$S(\omega)=\begin{cases} \aleph(\omega), & \omega\in Ƶ\backslash Ş \\ \aleph(\omega)\ \Delta K(\omega)\ , & \omega\in Ƶ\cap Ş \end{cases}$$

Hence,

$$S(\omega)=\begin{cases} \text{M}(\omega), & \omega\in(Ƶ\backslash Ş)\backslash Ş=Ƶ\backslash Ş \\ \text{M}(\omega)\ \Delta Ʊ(\omega)\ , & \omega\in(Ƶ\cap Ş)\backslash Ş=\emptyset \\ \text{M}(\omega)\ \Delta\ Ʊ(\omega), & \omega\in(Ƶ\backslash Ş)\cap(Ş\backslash Ƶ)=\emptyset \\ \text{M}(\omega)\ \Delta\ (Ʊ(\omega)\ \Delta\aleph\ (\omega)), & \omega\in(Ƶ\backslash Ş)\cap(Ş\cap Ƶ)=\emptyset \\ (\text{M}(\omega)\ \Delta Ʊ(\omega))\ \Delta Ʊ(\omega), & \omega\in(Ƶ\cap Ş)\cap(Ş\backslash Ƶ)=\emptyset \\ [\text{M}(\omega)\ \Delta Ʊ(\omega)]\ \Delta\ [Ʊ(\omega)\ \Delta\aleph\ (\omega)], & \omega\in(Ƶ\cap Ş)\cap Ş=Ƶ\cap Ş \end{cases}$$

Thus,

$$S(\omega)=\begin{cases} \text{M}(\omega), & \omega\in Ƶ\backslash Ş \\ [\text{M}(\omega)\ \Delta Ʊ(\omega)]\ \Delta\ [Ʊ(\omega)\ \Delta\aleph\ (\omega)], & \omega\in Ƶ\cap Ş \end{cases}$$

Therefore,

$$S(\omega)=\begin{cases} \text{M}(\omega), & \omega\in Ƶ\backslash Ş \\ \text{M}(\omega)\ \Delta\aleph\ (\omega), & \omega\in Ƶ\cap Ş \end{cases}$$

Hence, (S,Ƶ)= (M, Ƶ) $\tilde{\Delta}$ (ℵ,Ş).

In classical theory, M’ΔL’=MΔL. Now, we have the folllowing:

14) $(\text{M},Ƶ)^r\ \tilde{\Delta}(Ʊ,Ƶ)^r=(\text{M},Ƶ)\ \tilde{\Delta}(Ʊ,Ƶ)$

Proof: Let $(\text{M},Ƶ)^r\ \tilde{\Delta}(Ʊ,Ƶ)^r=(\aleph,Ƶ)$. Then, $\forall\omega\in$Ƶ,

$$\aleph(\omega)=\begin{cases} \text{M}'(\omega), & \omega\in Ƶ\backslash Ƶ=\emptyset \\ \ldots \end{cases}$$

$\mathrm{M}'(\omega)\ \Delta\mho'(\omega),\quad \omega\in Ƶ\cap Ƶ=Ƶ$

Since $\forall\omega\in Ƶ$, $\aleph(\omega)=\mathrm{M}'(\omega)\ \Delta\mho'(\omega)=\mathrm{M}(\omega)\ \Delta\mho(\omega)$. Thus, $(\aleph,Ƶ)=(\mathrm{M},Ƶ)\ \tilde{\Delta}(\mho,Ƶ)$.

In classical theory, for all M, $\emptyset\subseteq$ M. As an analogy, we have:

15) $\emptyset_Ƶ\ \tilde{\subseteq}(\mathrm{M},Ƶ)\ \tilde{\Delta}(\mho,Ç)$ and $\emptyset_Ç\ \tilde{\subseteq}(\mho,Ç)\ \tilde{\Delta}(\mathrm{M},Ƶ)$.

In classical theory, for all M, M $\subseteq$ U. As an analogy, we have:

16) $(\mathrm{M},Ƶ)\ \tilde{\Delta}(\mho,Ç)\ \tilde{\subseteq}\ U_Ƶ$ and $(\mho,Ç)\ \tilde{\Delta}(\mathrm{M},Ƶ)\ \tilde{\subseteq}\ U_Ç$.

In classical theory, MΔL = MΔN$\Rightarrow$L=N (Cancellation Law). As an analogy, we have:

17) $(\mathrm{M},Ƶ)\tilde{\Delta}(\mho,Ş)=(\mathrm{M},Ƶ)\ \tilde{\Delta}(\aleph,Ş)\ \Rightarrow(\mho,Ƶ\cap Ş)=(\aleph,Ƶ\cap Ş)$.

Proof: Let $(\mathrm{M},Ƶ)\tilde{\Delta}(\mho,Ş)=(\aleph,Ƶ)$. Then, $\forall\omega\in Ƶ$,

$$\aleph(\omega)=\begin{cases}\mathrm{M}(\omega), & \omega\in Ƶ\backslash Ş\\ \mathrm{M}(\omega)\ \Delta\mho(\omega), & \omega\in Ƶ\cap Ş\end{cases}$$

Let, $(\mathrm{M},Ƶ)\ \tilde{\Delta}(\aleph,Ş)=(T,Ƶ)$, where $\forall\omega\in Ƶ$,

$$T(\omega)=\begin{cases}\mathrm{M}(\omega), & \omega\in Ƶ\backslash Ş\\ \mathrm{M}(\omega)\ \Delta\aleph(\omega), & \omega\in Ƶ\cap Ş\end{cases}$$

Since, $(\aleph,Ƶ)=(T,Ƶ)$, then for all $\omega\in Ƶ\cap Ş$; $\mathrm{M}(\omega)\ \Delta\mho(\omega)=\mathrm{M}(\omega)\ \Delta\aleph(\omega)$, thus $\mho(\omega)=\aleph(\omega)$ for all $\omega\in Ƶ\cap Ş$. Hence, $(\mho,Ƶ\cap Ş)=(\aleph,Ƶ\cap Ş)$. Here note that $(\mathrm{M},Ƶ)\tilde{\Delta}(\mho,Ş)=(\mathrm{M},Ƶ)\ \tilde{\Delta}(\aleph,Ş)$ does not imply that $(\mho,Ƶ)=(\aleph,Ş)$.

In classical theory, MΔL $\subseteq$ M $\cup$L. As an analogy, we have:

18) $(\mathrm{M},Ƶ)\ \tilde{\Delta}(\mho,Ş)\ \tilde{\subseteq}\ (\mathrm{M},Ƶ)\ \tilde{\cup}\ (\mho,Ş)$.

Proof: Since the PS of the SSs of both hand side is Ƶ, the first condition for the soft subset is satisfied. Let $(\mathrm{M},Ƶ)\ \tilde{\Delta}(\mho,Ş)=(\aleph,Ƶ)$, where $\forall\omega\in Ƶ$,

$$\aleph(\omega)=\begin{cases}\mathrm{M}(\omega), & \omega\in Ƶ\backslash Ş\\ \mathrm{M}(\omega)\ \Delta\mho(\omega), & \omega\in Ƶ\cap Ş\end{cases}$$

Now let $(\mathrm{M},Ƶ)\ \tilde{\cup}(\mho,Ş)=(T,Ƶ)$, where $\forall\omega\in Ƶ$,

$$T(\omega)=\begin{cases}\mathrm{M}(\omega), & \omega\in Ƶ\backslash Ş\\ \mathrm{M}(\omega)\ \cup\mho(\omega), & \omega\in Ƶ\cap Ş\end{cases}$$

Since for all $\omega\in Ƶ\backslash Ş$, $\mathrm{M}(\omega)\subseteq\mathrm{M}(\omega)$ and $\forall\omega\in Ƶ\cap Ş$, $\mathrm{M}(\omega)\ \Delta\mho(\omega)\subseteq\mathrm{M}(\omega)\cup\mho(\omega)$, thus for all $\forall\omega\in Ƶ$, $\aleph(\omega)\subseteq T(\omega)$. Hence, $(\aleph,Ƶ)\ \tilde{\subseteq}\ (T,Ƶ)$.

In set theory, MΔL = $\emptyset$ $\Leftrightarrow$ M=L. As an analogy, we have (19) and (20) as below:

19) $(\mathrm{M},Ƶ)\ \tilde{\Delta}(\mho,Ƶ)=\emptyset_Ƶ\ \Leftrightarrow(\mathrm{M},Ƶ)=(\mho,Ƶ)$

Proof: Necessity: Let $(\text{Ɱ}, Ƶ)\ \tilde{\Delta}(Ʊ, Ƶ) = (T,Ƶ)$. Hence, $\forall\omega \in Ƶ$,

$$T(\omega)= \begin{cases} Ɱ(\omega), & \omega \in Ƶ \setminus Ƶ = \emptyset \\ Ɱ(\omega)\ \Delta Ʊ(\omega), & \omega \in Ƶ \cap Ƶ = Ƶ \end{cases}$$

Since $(T,Ƶ)= \emptyset_Ƶ$, $\forall\omega\in Ƶ$, $T(\omega)= \emptyset$. Thus, $\forall\omega\in Ƶ$, $Ɱ(\omega)\ \Delta Ʊ(\omega)= \emptyset$. Hence, $\forall\omega \in Ƶ$, $Ɱ(\omega)= Ʊ(\omega)$.

So, $(Ɱ, Ƶ) = (Ʊ, Ƶ)$

Sufficiency: Let $(Ɱ, Ƶ) = (Ʊ, Ƶ)$. Thus, $\forall\omega \in Ƶ$, $Ɱ(\omega)= Ʊ(\omega)$. Then, $(Ɱ, Ƶ)\ \tilde{\Delta}(Ʊ, Ƶ)=\emptyset_Ƶ$.

20) $(Ɱ, Ƶ)\ \tilde{\Delta}(Ʊ, Ş)= \emptyset_Ƶ \Leftrightarrow (Ɱ, Ƶ \setminus Ş) = \emptyset_{Ƶ\setminus Ş}$ and $(Ɱ, Ƶ \cap Ş)=(Ʊ, Ƶ \cap Ş)$.

Proof: Necessity: Let $(Ɱ, Ƶ)\ \tilde{\Delta}(Ʊ, Ş) = (T,Ƶ)$. Hence, $\forall\omega\in Ƶ$,

$$T(\omega)= \begin{cases} Ɱ(\omega), & \omega \in Ƶ \setminus Ş \\ Ɱ(\omega)\ \Delta Ʊ(\omega), & \omega \in Ƶ \cap Ş \end{cases}$$

Since $(T,Ƶ)= \emptyset_Ƶ$, $\forall\omega\in Ƶ$, $T(\omega)= \emptyset$. Thus, $\forall\omega\in Ƶ\setminus Ş$, $Ɱ(\omega) = \emptyset$ and $\forall\omega\in Ƶ\cap Ş$, $Ɱ(\omega)\ \Delta Ʊ(\omega)= \emptyset$. Hence, $\forall\omega \in Ƶ \cap Ş$, $Ɱ(\omega)=Ʊ(\omega)$. Therefore, $(Ɱ, Ƶ\setminus Ş) = \emptyset_{Ƶ\setminus Ş}$ and $(Ɱ, Ƶ \cap Ş)=(Ʊ, Ƶ \cap Ş)$. This completes the proof of necessity condition.

Sufficiency: Let $(Ɱ, Ƶ)\ \tilde{\Delta}(Ʊ, Ş) = (T,Ƶ)$. Hence, $\forall\omega\in Ƶ$,

$$T(\omega)= \begin{cases} Ɱ(\omega), & \omega \in Ƶ \setminus Ş \\ Ɱ(\omega)\ \Delta Ʊ(\omega), & \omega \in Ƶ \cap Ş \end{cases}$$

Assume that $(Ɱ, Ƶ\setminus Ş) = \emptyset_{Ƶ\setminus Ş}$ and $(Ɱ, Ƶ \cap Ş)=(Ʊ, Ƶ \cap Ş)$. Thus,

$$T(\omega)= \begin{cases} \emptyset, & \omega \in Ƶ \setminus Ş \\ \emptyset, & \omega \in Ƶ \cap Ş \end{cases}$$

Thus, $(T,Ƶ)= \emptyset_Ƶ$. This completes the proof.

In classical theory, $M \Delta L = M \cup L \Leftrightarrow M \cap L=\emptyset$. As an analogy, we have (21) and (22).

21) $(Ɱ, Ƶ)\ \tilde{\Delta}(Ʊ,Ƶ) =(Ɱ, Ƶ)\ \tilde{\cup}\ (Ʊ, Ƶ) \Leftrightarrow (Ɱ, Ƶ)\ \tilde{\cap}\ (Ʊ, Ƶ)= \emptyset_Ƶ$

Proof: Let $(Ɱ, Ƶ)\ \tilde{\Delta}(Ʊ,Ƶ)=(\aleph,Ƶ)$ and $(Ɱ, Ƶ)\ \tilde{\cup}(Ʊ,Ƶ)=(T,Ƶ)$. Then,

$$\aleph(\omega)= \begin{cases} Ɱ(\omega), & \omega \in Ƶ \setminus Ƶ = \emptyset \\ Ɱ(\omega)\ \Delta Ʊ(\omega), & \omega \in Ƶ \cap Ƶ = Ƶ \end{cases}$$

and

$$T(\omega)= \begin{cases} Ɱ(\omega), & \omega \in Ƶ \setminus Ƶ = \emptyset \\ \ldots \end{cases}$$

$ꟽ(\omega) \cup Ʊ(\omega), \quad \omega \in Ƶ \cap Ƶ = Ƶ$

Since (ℵ,Ƶ)=(T,Ƶ), then $\forall\omega\in Ƶ$, $ℵ(\omega)=ꟽ(\omega)\Delta Ʊ(\omega)=ꟽ(\omega)\cup Ʊ(\omega)=T(\omega)$. Thus, $\forall\omega\in Ƶ$, $ꟽ(\omega)\cap Ʊ(\omega)=\emptyset$. Hence,(ꟽ, Ƶ) $\tilde{\cap}$ (Ʊ, Ƶ)= $\emptyset_{Ƶ}$.

22) (ꟽ, Ƶ) $\tilde{\Delta}$(Ʊ,Ş) =(ꟽ, Ƶ) $\tilde{\cup}$ (Ʊ, Ş) ⇔ (ꟽ, Ƶ) $\cap_R$ (Ʊ, Ş)= $\emptyset_{Ƶ\cap Ş}$.

Proof: Let (ꟽ, Ƶ) $\tilde{\Delta}$(Ʊ,Ş)=(ℵ,Ƶ) and (ꟽ, Ƶ) $\tilde{\cup}$(Ʊ,Ş)=(T,Ƶ). Then,

$$ℵ(\omega)=\begin{cases} ꟽ(\omega), & \omega\in Ƶ\setminus Ş \\ ꟽ(\omega)\,\Delta Ʊ(\omega), & \omega\in Ƶ\cap Ş \end{cases}$$

and

$$T(\omega)=\begin{cases} ꟽ(\omega), & \omega\in Ƶ\setminus Ş \\ ꟽ(\omega)\cup Ʊ(\omega), & \omega\in Ƶ\cap Ş \end{cases}$$

Since (ℵ,Ƶ)=(T,Ƶ), then $\forall\omega\in Ƶ\cap Ş$, $ꟽ(\omega)\Delta Ʊ(\omega)=ꟽ(\omega)\cup Ʊ(\omega)$. Thus, $\forall\omega\in Ƶ\cap Ş$, $ꟽ(\omega)\cap Ʊ(\omega)=\emptyset$. Hence, (ꟽ, Ƶ) $\cap_R$ (Ʊ, Ş)= $\emptyset_{Ƶ\cap Ş}$.

In classical theory, M⊆ L ⟹ MΔL= L\M. As an analogy, we have (23) and (24):

23) (ꟽ, Ƶ) $\tilde{\subseteq}$ (Ʊ, Ƶ) ⟹(ꟽ, Ƶ) $\tilde{\Delta}$(Ʊ,Ƶ)= (Ʊ,Ƶ) $\tilde{\setminus}$ (ꟽ,Ƶ) .

Proof: Let (ꟽ, Ƶ) $\tilde{\subseteq}$ (Ʊ, Ƶ). Then, $\forall\omega\in Ƶ$, $ꟽ(\omega) \subseteq Ʊ(\omega)$ and let (ꟽ, Ƶ) $\tilde{\Delta}$(Ʊ, , Ƶ)=(ℵ,Ƶ). Then, $\forall\omega\in Ƶ$,

$$ℵ(\omega)=\begin{cases} ꟽ(\omega), & \omega\in Ƶ\setminus Ƶ=\emptyset \\ ꟽ(\omega)\,\Delta Ʊ(\omega), & \omega\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Since $\forall\omega\in Ƶ$, $ꟽ(\omega) \subseteq Ʊ(\omega)$, and $ℵ(\omega)= ꟽ(\omega)\,\Delta Ʊ(\omega)=Ʊ(\omega)\setminus ꟽ(\omega)$. Thus, (ℵ,Ƶ)=(Ʊ,Ƶ) $\tilde{\setminus}$ (ꟽ,Ƶ).

24) (ꟽ, Ƶ) $\tilde{\subseteq}$ (Ʊ, Ş) ⟹(ꟽ, Ƶ) $\tilde{\Delta}$(Ʊ,Ş) $\tilde{\subseteq}$ (Ʊ,Ş) $\tilde{\setminus}$ (ꟽ,Ƶ) .

Proof: Let (ꟽ, Ƶ) $\tilde{\subseteq}$ (Ʊ, Ş). Then, Ƶ⊆Ş, and so the first condition for the soft subset is satisfied. Moreover, since (ꟽ, Ƶ) $\tilde{\subseteq}$ (Ʊ, Ş), $\forall\omega\in Ƶ$, $ꟽ(\omega) \subseteq Ʊ(\omega)$. Let (ꟽ, Ƶ) $\tilde{\Delta}$(Ʊ, Ş)=(ℵ,Ƶ). Then, $\forall\omega\in Ƶ$,

$$ℵ(\omega)=\begin{cases} ꟽ(\omega), & \omega\in Ƶ\setminus Ş=\emptyset \\ ꟽ(\omega)\,\Delta Ʊ(\omega), & \omega\in Ƶ\cap Ş=Ƶ \end{cases}$$

Let (Ʊ,Ş) $\tilde{\setminus}$ (ꟽ, Ƶ)=(T,Ş). Then, $\forall\omega\in Ş$,

$$T(\omega)=\begin{cases} ꟽ(\omega), & \omega\in Ş\setminus Ƶ \\ Ʊ(\omega)\setminus ꟽ(\omega), & \omega\in Ş\cap Ƶ=Z \end{cases}$$

Since $\forall\omega\in Ƶ$, $ꟽ(\omega) \subseteq Ʊ(\omega)$, thus $ꟽ(\omega)\,\Delta Ʊ(\omega)=Ʊ(\omega)\setminus ꟽ(\omega)$. Therefore, (ℵ,Ƶ) $\tilde{\subseteq}$ (T,Ş).

In classical theory, M Δ(M ∩ L)=M\L. As an analogy, we have:

25) (ʍ,Ƶ) $\tilde{\Delta}$[(ɰ, Ƶ) $\tilde{\cap}$(Ʊ,Ƶ)]=(ʍ, Ƶ) $\tilde{\setminus}$(Ʊ,Ƶ).

Proof: Let (ɰ, Ƶ) $\tilde{\cap}$(Ʊ,Ƶ)=(ℵ,Ƶ). Then, ∀ω∈Ƶ,

ℵ(ω)= { ʍ(ω), ω∈Ƶ\Ƶ=∅ ; ʍ(ω) ∩Ʊ(ω), ω∈Ƶ∩Ƶ=Ƶ

Let, (ʍ,Ƶ) $\tilde{\Delta}$(ℵ,Ƶ)=(T,Ƶ), where , ∀ω∈Ƶ,

T(ω) { ʍ(ω), ω∈Ƶ\Ƶ=∅ ; ʍ(ω) Δℵ(ω), ω∈Ƶ∩Ƶ=Ƶ

Hence,

T(ω)= { ʍ(ω), ω∈Ƶ\Ƶ=∅ ; ʍ(ω) Δ[ʍ(ω) ∩Ʊ(ω)], ω∈Ƶ∩Ƶ=Ƶ

So,

T(ω)= { ʍ(ω), ω∈Ƶ\Ƶ=∅ ; ʍ(ω) \Ʊ(ω), ω∈Ƶ∩Ƶ=Ƶ

Thus, (T,Ƶ)=(ʍ, Ƶ) $\tilde{\setminus}$(Ʊ,Ƶ).

In classical theory, M∪ L = (MΔL) ∪(M∩L). As an analogy, we have:

26) (ɰ, Ƶ) $\tilde{\cup}$(Ʊ,Ş)=[(ʍ,Ƶ) $\tilde{\Delta}$(Ʊ, Ş)] $\tilde{\cup}$ [(ɰ, Ƶ) $\tilde{\cap}$(Ʊ,Ş)].

Proof: Since the PS of the SSs of both hand side is Ƶ, the first condition for the soft equality is satisfied. First let's consider right side. Let (ɰ, Ƶ) $\tilde{\cup}$(Ʊ,Ş)=(ℵ,Ƶ). Then, ∀ω∈Ƶ,

ℵ(ω)= { ʍ(ω), ω∈Ƶ\Ş ; ʍ(ω) ∪Ʊ(ω), ω∈Ƶ∩Ş

Now let's consider left side. Let (ɰ, Ƶ)$\tilde{\Delta}$(Ʊ,Ş)=(K,Ƶ). Then, ∀ω∈Ƶ,

K(ω)= { ʍ(ω), ω∈Ƶ\Ş ; ʍ(ω) ΔƱ(ω), ω∈Ƶ∩Ş

Let, (ʍ,Ƶ) $\tilde{\cap}$(Ʊ,Ş)=(T,Ƶ), where ∀ω∈Ƶ,

{ ʍ(ω), ω∈Ƶ\Ş

$$T(\omega)= \quad \text{ᴍ}(\omega) \cap \text{Ʊ}(\omega), \quad \omega\in \text{Ƶ}\cap\text{Ş}$$

Now, let (K,Ƶ) $\tilde{\cup}$(T,Ƶ)=(S,Ƶ), where ∀ω∈Ƶ,

$$S(\omega)=\begin{cases} K(\omega), & \omega\in \text{Ƶ}\setminus\text{Ƶ}=\emptyset \\ K(\omega)\cup T(\omega), & \omega\in \text{Ƶ}\cap\text{Ƶ}=\text{Ƶ} \end{cases}$$

Thus,

$$S(\omega)=\begin{cases} \text{ᴍ}(\omega)\cup \text{ᴍ}(\omega) & \omega\in(\text{Ƶ}\setminus\text{Ş})\cap(\text{Ƶ}\setminus\text{Ş})=\text{Ƶ}\setminus\text{Ş} \\ \text{ᴍ}(\omega)\cup[\text{ᴍ}(\omega)\cap\text{Ʊ}(\omega)], & \omega\in(\text{Ƶ}\setminus\text{Ş})\setminus\cap(\text{Ƶ}\cap\text{Ş})=\emptyset \\ [\text{ᴍ}(\omega)\Delta\text{Ʊ}(\omega)]\cup\text{ᴍ}(\omega), & \omega\in(\text{Ƶ}\cap\text{Ş})\cap(\text{Ƶ}\setminus\text{Ş})=\emptyset \\ [\text{ᴍ}(\omega)\Delta\text{Ʊ}(\omega)]\cup[\text{ᴍ}(\omega)\cap\text{Ʊ}(\omega)], & \omega\in(\text{Ƶ}\cap\text{Ş})\cap(\text{Ƶ}\cap\text{Ş})=\text{Ƶ}\cap\text{Ş} \end{cases}$$

Thus,

$$S(\omega)=\begin{cases} \text{ᴍ}(\omega), & \omega\in\text{Ƶ}\setminus\text{Ş} \\ \text{ᴍ}(\omega)\cup\text{Ʊ}(\omega), & \omega\in\text{Ƶ}\cap\text{Ş} \end{cases}$$

Thus, (ℵ,Ƶ)=(S,Ƶ). This completes the proof.

In classical theory, intersection distributes over symmetric difference from both left and right side, that is, M∩(LΔN) =(M∩L)Δ(M∩N) and (MΔL) ∩N=(M∩N)Δ(L∩N) for all M,L,N. As an analogy, we have the following two properties:

27) (ᴍ,Ƶ) $\cap_R$[(Ʊ,Ş) $\tilde{\Delta}$ (ℵ,C)]=[(ᴍ,Ƶ) $\cap_R$(Ʊ,Ş)]$\tilde{\Delta}$ [(ᴍ,Ƶ) $\cap_R$ (ℵ,C)]

Proof: Let's first consider the left side. Let (Ʊ,Ş)$\tilde{\Delta}$ (ℵ,C)=(M,Ş), where ∀ω ∈Ş;

$$M(\omega)=\begin{cases} \text{Ʊ}(\omega), & \omega\in\text{Ş}\setminus C \\ \text{Ʊ}(\omega)\Delta\aleph(\omega), & \omega\in\text{Ş}\cap C \end{cases}$$

Assume that (ᴍ,Ƶ) $\cap_R$(M,Ş) =(N,Ƶ∩Ş) , where ∀ω∈ Ƶ∩Ş; N(ω)=ᴍ(ω)∩M(ω). Hence,

$$N(\omega)=\begin{cases} \text{ᴍ}(\omega)\cap\text{Ʊ}(\omega), & \omega\in \text{Ƶ}\cap(\text{Ş}\setminus C) \\ \text{ᴍ}(\omega)\cap[\text{Ʊ}(\omega)\Delta\aleph(\omega)], & \omega\in \text{Ƶ}\cap(\text{Ş}\cap C) \end{cases}$$

Now let's consider the right side: [(ᴍ,Ƶ) $\cap_R$(Ʊ,Ş)] $\tilde{\Delta}$[(ᴍ,Ƶ) $\cap_R$ (ℵ,C)] . Let (ᴍ,Ƶ) $\cap_R$ (Ʊ,Ş) =(K,Ƶ∩Ş), where ∀ω∈ Ƶ∩Ş, K(ω)=ᴍ(ω)∩Ʊ(ω). Let (ᴍ,Ƶ) $\cap_R$(ℵ,C)=(T,Ƶ∩C), where ∀ω∈Ƶ∩C; T(ω)=ᴍ(ω)∩ℵ(ω). Thus, (K,Ƶ∩Ş)$\tilde{\Delta}$ (T,Ƶ∩C)=(L,Ƶ∩Ş), where ∀ω∈ Ƶ∩Ş;

$$L(\omega)=\begin{cases} K(\omega), & \omega\in(\text{Ƶ}\cap\text{Ş})\setminus(\text{Ƶ}\cap C) \\ K(\omega)\Delta T(\omega), & \omega\in(\text{Ƶ}\cap\text{Ş})\cap(\text{Ƶ}\cap C) \end{cases}$$

Thus,

$$L(\omega)=\begin{cases} \text{ᴍ}(\omega)\cap\text{Ʊ}(\omega), & \omega\in (\text{Ƶ}\cap\text{Ş})\setminus(\text{Ƶ}\cap C)=\text{Ƶ}\cap(\text{Ş}\setminus C)\\ [\text{ᴍ}(\omega)\cap\text{Ʊ}(\omega)]\Delta[\text{ᴍ}(\omega)\cap\aleph(\omega)], & \omega\in (\text{Ƶ}\cap\text{Ş})\cap(\text{Ƶ}\cap C)=\text{Ƶ}\cap(\text{Ş}\cap C)\end{cases}$$

Hence, (N,Ƶ∩Ş)=(L,Ƶ∩Ş). Here note that if Ƶ∩Ş=∅, then the left hand side is equal to $\emptyset_\emptyset$ and the right hand side is $\emptyset_\emptyset\tilde{\Delta}[(\text{ᴍ},\text{Ƶ})\cap_R(\aleph,C)]=\emptyset_\emptyset$, too.

28) $[(\text{ᴍ},\text{Ƶ})\,\tilde{\Delta}\,(\text{Ʊ},\text{Ş})]\cap_R(\aleph,C)=[(\text{ᴍ},\text{Ƶ})\cap_R(\aleph,C)]\,\tilde{\Delta}\,[(\text{Ʊ},\text{Ş})\cap_R(\aleph,C)]$

Proof: Let's consider first the left side. Let $(\text{ᴍ},\text{Ƶ})\,\tilde{\Delta}\,(\text{Ʊ},\text{Ş})=(M,\text{Ƶ})$, where $\forall\omega\in\text{Ƶ}$;

$$M(\omega)=\begin{cases}\text{ᴍ}(\omega), & \omega\in\text{Ƶ}\setminus\text{Ş}\\ \text{ᴍ}(\omega)\,\Delta\text{Ʊ}(\omega), & \omega\in\text{Ƶ}\cap\text{Ş}\end{cases}$$

Now, let $(M,\text{Ƶ})\cap_R(\aleph,C)=(W,\text{Ƶ}\cap C)$, where $\forall\omega\in\text{Ƶ}\cap C$; $W(\omega)=M(\omega)\cap\aleph(\omega)$. Thus,

$$W(\omega)=\begin{cases}\text{ᴍ}(\omega)\cap\aleph(\omega), & \omega\in(\text{Ƶ}\setminus\text{Ş})\cap C\\ [\text{ᴍ}(\omega)\,\Delta\text{Ʊ}(\omega)]\cap\aleph(\omega), & \omega\in(\text{Ƶ}\cap\text{Ş})\cap C\end{cases}$$

Now let's consider the right side: $[(\text{ᴍ},\text{Ƶ})\cap_R(\aleph,C)]\,\tilde{\Delta}[(\text{Ʊ},\text{Ş})\cap_R(\aleph,C)]$. Let $(\text{ᴍ},\text{Ƶ})\cap(\aleph,C)=(K,\text{Ƶ}\cap C)$, where $\forall\omega\in\text{Ƶ}\cap C$, $K(\omega)=\text{ᴍ}(\omega)\cap\aleph(\omega)$. Let $(\text{Ʊ},\text{Ş})\cap_R(\aleph,C)=(T,\text{Ş}\cap C)$, where $\forall\omega\in\text{Ş}\cap C$; $T(\omega)=\text{Ʊ}(\omega)\cap\aleph(\omega)$. Thus, $(K,\text{Ƶ}\cap C)\,\tilde{\Delta}(T,\text{Ş}\cap C)=(R,\text{Ƶ}\cap C)$, where $\forall\omega\in\text{Ƶ}\cap C$;

$$R(\omega)=\begin{cases}K(\omega), & \omega\in(\text{Ƶ}\cap C)\setminus(\text{Ş}\cap C)\\ K(\omega)\,\Delta T(\omega), & \omega\in(\text{Ƶ}\cap C)\cap(\text{Ş}\cap C)\end{cases}$$

Thus,

$$Q(\omega)=\begin{cases}\text{ᴍ}(\omega)\cap\aleph(\omega), & \omega\in(\text{Ƶ}\cap C)\setminus(\text{Ş}\cap C)=(\text{Ƶ}\setminus\text{Ş})\cap C\\ [\text{ᴍ}(\omega)\cap\aleph(\omega)]\,\Delta\,[\text{Ʊ}(\omega)\cap\aleph(\omega)], & \omega\in(\text{Ƶ}\cap C)\cap(\text{Ş}\cap C)=(\text{Ƶ}\cap\text{Ş})\cap C\end{cases}$$

Hence (W,Ƶ∩C)= (Q,Ƶ∩C), Here note that if Ƶ∩C=∅, then right hand side is $\emptyset_\emptyset\tilde{\Delta}\,[(\text{Ʊ},\text{Ş})\cap_R(\aleph,C)]=\emptyset_\emptyset$, and the left hand side is $\emptyset_\emptyset$, too.

<u>**REMARK 14:**</u> In Remark13, we show that $(S_A(U),\tilde{\Delta})$ is an abelian group with identity $\emptyset_A$ and every element is its own inverse. Hence, we can deduce that $(S_A(U),\tilde{\Delta})$ is a semigroup. Moreover, in [3,5,17], it was proved that $(S_A(U),\cap_R)$ is a commutative monoid with identity U_A. Hence, we can deduce that $(S_A(U),\cap_R)$ is a semigroup. Moreover, by Theorem 12. (27) and (28), $\cap_R$ distributes over $\tilde{\Delta}$ from both sides. Therefore, $(S_A(U),\tilde{\Delta},\cap_R)$ is a semiring. Further, by Theorem 12 (4) $(F,A)\tilde{\Delta}(G,A)=(G,A)\tilde{\Delta}(F,A)$. That is to say, $\tilde{\Delta}$ is commutative in $S_A(U)$ and $(F,A)\tilde{\Delta}\emptyset_A=\emptyset_A\tilde{\Delta}(F,A)=(F,A)$ and $(F,A)\cap_R\emptyset_A=\emptyset_A\cap_R(F,A)=\emptyset_A$. That is to say, $\emptyset_A$ is the zero element of $(S_A(U),\tilde{\Delta},\cap_R)$. Therefore, $(S_A(U),\tilde{\Delta},\cap_R)$ is a hemiring. Besides, since $(F,A)\cap_R U_A=U_A\cap_R(F,A)=(F,A)$ and $(F,A)\cap_R(G,A)=(G,A)\cap_R(F,A)$ (see [3,5,17]) , $(S_A(U),\tilde{\Delta},\cap_R)$ is a commutative hemiring with identity U_A.

Also, since $(S_A(U),\tilde{\Delta})$ is an abelian group by Remark 13, $(S_A(U),\cap_R)$ is a semigroup by [3,5,17] and $\cap_R$ distributes over $\tilde{\Delta}$ from both side by Theorem 12. (27) and (28), we can also deduce that $(S_A(U),\tilde{\Delta},\cap_R)$ is a ring. Also, since $(F,A) \cap_R (G,A) = (G,A) \cap_R (F,A)$ and $(F,A) \cap_R U_A = U_A \cap_R (F,A) = (F,A)$, (see [3,5,17]), $(S_A(U),\tilde{\Delta},\cap_R)$ is a commutative ring with identity U_A. Moreover, $(F,A)^2 = (F,A) \cap_R (F,A) = (F,A)$ for all $(F,A) \in S_A(U)$. Thus, $(S_A(U),\tilde{\Delta},\cap_R)$ is a Boolean ring and $(F,A)\tilde{\Delta}(F,A)= \emptyset_A$ and $(F,A)\tilde{\Delta}G,A)=(G,A)\tilde{\Delta}(F,A)$ is satisfied naturally as a result of being Boolean ring.

CONCLUSIONS

To treat uncertain objects, the soft set and soft operations are powerful parametric tools. In order to consider problems containing parametric data, creating new soft operations and deriving their algebraic properties and implementations will offer new perspectives. In this regard, this research represents a novel form of soft set operation, which we call soft binary piecewise symmetric difference operation. The basic algebraic properties of the operations are examined. By examining the distribution rules, we determine the connections between this new soft set operation and restricted intersection operation. Additionally, we demonstrate that the set of all the soft sets with a fixed parameter set together with the soft binary piecewise symmetric difference operation and the restricted intersection operation is a commutative hemiring with identity and also Boolean ring. New varieties of soft set operations could be developed in upcoming studies. Additionally, as the soft set operation is a potent mathematical tool for the identification of uncertain objects, researchers may propose some novel encryption or decision-making techniques as a result of this study. The operation outlined in this study can also be used to revisit studies on soft algebraic structures in terms of their algebraic properties.

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