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Research Article

On Some Functions Related to e^*-θ-open Sets

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Abstract: In this study, we defined the concept of quasi e^*-θ-closed sets by means of e^*-θ-open sets. Depending on this concept, we introduced approximately $e^*\theta$-open functions and investigated some of its basic properties. Also, we defined and studied contra pre $e^*\theta$-open functions, which are stronger than the approximately $e^*\theta$-open functions. Moreover, we characterized the class of $e^*\theta$-$T_{\frac{1}{2}}$ spaces.

Keywords: $e^*\theta$-open functions, quasi e^*-θ-closed sets, approximately $e^*\theta$-open functions, contra pre $e^*\theta$-open functions, $e^*\theta$-$T_{\frac{1}{2}}$ spaces.

1. Introduction

In 2015, Farhan and Yang [10] introduced a new class of open sets called e^*-θ-open. In the following years, some concepts of open functions in relation to e^*-θ-open sets [10] have been investigated. The notion of $e^*\theta$-open functions is introduced by Ayhan [3] as follows: A function $f : X \to Y$ is said to be $e^*\theta$-open if the image of each open set U of X is e^*-θ-open in Y. In 2018, Ayhan and Özkoç [6] defined a new type of open functions called $e^*\theta$-semiopen functions. Again within the same year, Ayhan and Özkoç [5] defined and studied pre $e^*\theta$-open functions. In 2022, Ayhan [4] introduced and investigated weakly $e^*\theta$-open functions and also obtained some characterizations of its.

Rajesh and Salleh [14] gave the definition of quasi-b-θ-closed sets via b-θ-open sets [13] in their work titled "Some more results on b-θ-open sets". Caldas and Jafari [7] introduced and studied $g\beta\theta$-closed sets through β-θ-openness [12], in 2015.

In this paper, we introduce quasi e^*-θ-closed sets [1] defined with the help of e^*-θ-open sets. Moreover, we define and study approximately $e^*\theta$-open functions and contra pre $e^*\theta$-open functions such that these are weaker than $e^*\theta$-open functions.

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2. Preliminaries

Throughout this paper, X and Y represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. A point $x \in X$ is called to be δ-cluster point [16] of A if $int(cl(U)) \cap A \neq \varnothing$ for every open neighborhood U of x. The set of all δ-cluster points of A is called the δ-closure [16] of A and is denoted by $cl_\delta(A)$. If $A = cl_\delta(A)$, then A is called δ-closed [16] and the complement of a δ-closed set is called δ-open [16]. The set $\{x|(\exists U \in O(X,x))(int(cl(U)) \subseteq A)\}$ is called the δ-interior of A and is denoted by $int_\delta(A)$.

A subset A is called e^*-open [9] $A \subseteq cl(int(cl_\delta(A)))$. The complement of an e^*-open set is called e^*-closed [9]. The intersection of all e^*-closed sets of X containing A is called the e^*-closure [9] of A and is denoted by e^*-$cl(A)$. The union of all e^*-open sets of X containing in A is called the e^*-interior [9] of A and is denoted by e^*-$int(A)$. A subset A is said to be e^*-regular [10] set if it is e^*-open and e^*-closed.

A point x of X is called an e^*-θ-cluster point of A if e^*-$cl(U) \cap A \neq \varnothing$ for every e^*-open set U containing x. The set of all e^*-θ-cluster points of A is called the e^*-θ-closure [10] of A and is denoted by e^*-$cl_\theta(A)$. A subset A is said to be e^*-θ-closed if $A = e^*$-$cl_\theta(A)$. The complement of an e^*-θ-closed set is called an e^*-θ-open [10] set. A point x of X said to be an e^*-θ-interior point [10] of a subset A, denoted by e^*-$int_\theta(A)$, if there exists an e^*-open set U of X containing x such that e^*-$cl(U) \subseteq A$. Also it is noted in [10] that

$$e^*\text{-regular} \Rightarrow e^*\text{-}\theta\text{-open} \Rightarrow e^*\text{-open}.$$

The family of all open (resp. closed, e^*-θ-open, e^*-θ-closed, e^*-open, e^*-closed, e^*-regular) subsets of X is denoted by $O(X)$ (resp. $C(X)$, $e^*\theta O(X)$, $e^*\theta C(X)$, $e^*O(X)$, $e^*C(X)$, $e^*R(X)$). The family of all open (resp. closed, e^*-θ-open, e^*-θ-closed, e^*-open, e^*-closed, e^*-regular) sets of X containing a point x of X is denoted by $O(X,x)$ (resp. $C(X,x)$, $e^*\theta O(X,x)$, $e^*\theta C(X,x)$, $e^*O(X,x)$, $e^*C(X,x)$, $e^*R(X,x)$).

We shall use the well-known accepted language almost in the whole of the proofs of the theorems in this article.

Lemma 2.1 *[10, 11] Let X be a topological space and $A, B \subseteq X$. Then the following properties are hold:*

(i) $A \subseteq e^*$-$cl(A) \subseteq e^*$-$cl_\theta(A)$.

(ii) If $A \in e^*\theta O(X)$, *then* e^*-$cl_\theta(A) = e^*$-$cl(A)$.

(iii) If $A \subseteq B$, *then* e^*-$cl_\theta(A) \subseteq e^*$-$cl_\theta(B)$.

(iv) e^*-$cl_\theta(A) \in e^*\theta C(X)$ *and* e^*-$cl_\theta(e^*$-$cl_\theta(A)) = e^*$-$cl_\theta(A)$.

(v) If $A_\alpha \in e^*\theta O(X)$ *for each* $\alpha \in \Lambda$*, then* $\bigcup\{A_\alpha | \alpha \in \Lambda\} \in e^*\theta O(X)$.

(vi) $e^*\text{-}cl_\theta(A) = \bigcap\{F | (A \subseteq F)(F \in e^*\theta C(X))\}$.

(vii) $e^*\text{-}cl_\theta(X \backslash A) = X \backslash e^*\text{-}int_\theta(A)$.

(viii) A *is* e^**-*θ*-open in* X *iff for each* $x \in A$*, there exists* $U \in eR(X,x)$ *such that* $U \subseteq A$.

Definition 2.2 *A function* $f : X \to Y$ *is called* e^**-irresolute [8] if* $f^{-1}[A]$ *is* e^**-*θ*-open in* X *for every* e^**-*θ*-open set* A *of* Y.

3. Quasi e^*-θ-closed Sets

Definition 3.1 *A subset* A *of a space* X *is called quasi* e^**-*θ*-closed [2] (briefly,* $qe^*\theta$*-closed) if* $e^*\text{-}cl_\theta(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is* e^**-*θ*-open in* X*. A subset* A *of a space* X *is said to be quasi* e^**-*θ*-open (briefly,* $qe^*\theta$*-open) if* $X \backslash A$ *is* $qe^*\theta$*-closed. The family of all* $qe^*\theta$*-closed (resp.* $qe^*\theta$*-open) subsets of* X *is denoted by* $qe^*\theta C(X)$ *(resp.* $qe^*\theta O(X)$*).*

Theorem 3.2 *Every* e^**-*θ*-closed set is* $qe^*\theta$*-closed.*

Proof Let $A \in e^*\theta C(X)$, $U \in e^*\theta O(X)$ and $A \subseteq U$.

$$\left.\begin{array}{r} A \in e^*\theta C(X) \\ (U \in e^*\theta O(X))(A \subseteq U) \end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(A) = A \subseteq U.$$

□

Remark 3.3 *This implication is not reversible as shown in the following example.*

Example 3.4 *Let* $X = \{1,2,3\}$*, define a topology* $\tau = \{\varnothing, X, \{1\}, \{2\}, \{1,2\}\}$ *on* X*. It is not difficult to see* $e^*\theta C(X) = 2^X \backslash \{\{3\}\}$ *and the subset* $\{1,2\}$ *is* $qe^*\theta$*-closed but it is not* e^**-*θ*-closed (cf. Example 1 in [10]).*

Lemma 3.5 *A subset* A *of a topological space* X *is* $qe^*\theta$*-open if and only if* $F \subseteq e^*\text{-}int_\theta(A)$ *whenever* F *is* e^**-*θ*-closed in* X *and* $F \subseteq A$.

Proof *Necessity.* Let $F \subseteq A$, $F \in e^*\theta C(X)$ and $A \in qe^*\theta O(X)$.

$$\left.\begin{array}{r} A \supseteq F \in e^*\theta C(X) \Rightarrow \backslash A \subseteq \backslash F \in e^*\theta O(X) \\ A \in qe^*\theta O(X) \Rightarrow \backslash A \in qe^*\theta C(X) \end{array}\right\}$$

$$\Rightarrow \backslash e^*\text{-}int_\theta(A) = e^*\text{-}cl_\theta(\backslash A) \subseteq \backslash F$$

$$\Rightarrow F \subseteq e^*\text{-}int_\theta(A).$$

Sufficiency. Let $\setminus F \in e^*\theta O(X)$ and $\setminus A \subseteq \setminus F$.

$$\left.\begin{array}{r}(\setminus F \in e^*\theta O(X))(\setminus A \subseteq \setminus F) \Rightarrow (F \in e^*\theta C(X))(F \subseteq A)\\ \text{Hypothesis}\end{array}\right\}$$

$$\Rightarrow F \subseteq e^*\text{-}int_\theta(A)$$

$$\Rightarrow e^*\text{-}cl_\theta(\setminus A) = \setminus e^*\text{-}int_\theta(A) \subseteq \setminus F$$

Then, $\setminus A \in qe^*\theta C(X)$ and hence $A \in qe^*\theta O(X)$. □

Definition 3.6 *A function $f : X \to Y$ is said to be approximately $e^*\theta$-open (briefly, ap-$e^*\theta$-open) if e^*-$cl_\theta(B) \subseteq f[A]$ whenever $A \in e^*\theta O(X)$, $B \in qe^*\theta C(Y)$ and $B \subseteq f[A]$.*

Definition 3.7 *A function $f : X \to Y$ is said to be:*

(1) *$e^*\theta$-closed [3] (resp. pre $e^*\theta$-closed [5]), if the image of each closed (resp. e^*-θ-closed) set F of X is e^*-θ-closed in Y.*

(2) *$e^*\theta$-open [3] (resp. pre $e^*\theta$-open [5]), if the image of each open (resp. e^*-θ-open) set U of X is e^*-θ-open in Y.*

Theorem 3.8 *Let $f : X \to Y$ be a function. If $f[A]$ is e^*-θ-closed in Y for every $A \in e^*\theta O(X)$, then f is ap-$e^*\theta$-open.*

Proof Let $B \subseteq f[A]$, where $A \in e^*\theta O(X)$ and $B \in qe^*\theta C(Y)$.

$$\left.\begin{array}{r}(A \in e^*\theta O(X))(B \in qe^*\theta C(Y))(B \subseteq f[A])\\ \text{Hypothesis}\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(B) \subseteq e^*\text{-}cl_\theta(f[A]) = f[A]$$

$$\Rightarrow f[A] \in e^*\theta C(Y).$$

□

Theorem 3.9 *Every pre $e^*\theta$-open function is ap-$e^*\theta$-open.*

Proof Let $B \subseteq f[A]$, where $A \in e^*\theta O(X)$ and $B \in qe^*\theta C(Y)$.

$$\left.\begin{array}{r}(A \in e^*\theta O(X))(B \in qe^*\theta C(Y))(B \subseteq f[A])\\ f \text{ is pre } e^*\theta\text{-open}\end{array}\right\} \Rightarrow (f[A] \in e^*\theta O(Y))(B \in qe^*\theta C(Y))(B \subseteq f[A])$$

$$\Rightarrow e^*\text{-}cl_\theta(B) \subseteq f[A].$$ □

Remark 3.10 *This implication is not reversible as shown in the following example.*

Example 3.11 *Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Define the function $f : (X, \tau) \to (X, \tau)$ by $f = \{(a, a), (b, b), (c, b)\}$. It isn't difficult to see $e^*\theta O(X) = 2^X \setminus \{\{a, b\}\}$, $qe^*\theta C(X) = 2^X$*

and hence f is ap-$e^*\theta$-open. However $\{a,c\}$ is e^*-θ-open in X, but $f[\{a,c\}] = \{a,b\}$ is not e^*-θ-open in X. Therefore, f is not pre $e^*\theta$-open.

Theorem 3.12 *Let $f : X \to Y$ be a function. If the e^*-θ-open and e^*-θ-closed sets of Y coincide, then f is ap-$e^*\theta$-open if and only if $f[W] \in e^*\theta C(Y)$ for every e^*-θ-open subset W of X.*

Proof *Necessity.* Let A be an arbitrary subset of Y such that $A \subseteq U$, where $U \in e^*\theta O(Y)$ and let $W \in e^*\theta O(X)$.

$$\left.\begin{array}{r}(A \subseteq U)(U \in e^*\theta O(Y)) \\ e^*\theta O(Y) = e^*\theta C(Y)\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(A) \subseteq e^*\text{-}cl_\theta(U) = U$$

Therefore all subset of Y are $qe^*\theta$-closed and hence all are $qe^*\theta$-open.

$$\left.\begin{array}{r}(W \in e^*\theta O(X))(Y \supseteq f[W] \in qe^*\theta O(Y))(f[W] \subseteq f[W]) \\ f \text{ is ap-}e^*\theta\text{-open}\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(f[W]) \subseteq f[W]$$

$$\Rightarrow f[W] \in e^*\theta C(Y).$$

Sufficiency. It is obvious from Theorem 3.8. □

Corollary 3.13 *Let $f : X \to Y$ be a function. If the e^*-θ-open and e^*-θ-closed sets of Y coincide, then f is ap-$e^*\theta$-open if and only if f is pre $e^*\theta$-open.*

Definition 3.14 *A function $f : X \to Y$ is said to be contra pre $e^*\theta$-open (resp. contra pre $e^*\theta$-closed) if the image of each e^*-θ-open (resp. e^*-θ-closed) set U of X is e^*-θ-closed (resp. e^*-θ-open) in Y.*

Theorem 3.15 *Every contra pre $e^*\theta$-open function is ap-$e^*\theta$-open.*

Proof Let $B \subseteq f[A]$, where $A \in e^*\theta O(X)$ and $B \in qe^*\theta C(Y)$.

$$\left.\begin{array}{r}(A \in e^*\theta O(X))(B \in qe^*\theta C(Y))(B \subseteq f[A]) \\ f \text{ is contra pre } e^*\theta\text{-open}\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(B) \subseteq e^*\text{-}cl_\theta(f[A]) = f[A].$$

□

Remark 3.16 *This implication is not reversible as shown in the following example.*

Example 3.17 *Consider the same topology in Example 3.11. Define the identity function $f : (X,\tau) \to (X,\tau)$. Then, f is ap-$e^*\theta$-open. However $\{c\}$ is e^*-θ-open in X, but $f[\{c\}] = \{c\}$ is not e^*-θ-closed in X. Therefore, f is not contra pre $e^*\theta$-open.*

Remark 3.18 *The following examples show that contra pre $e^*\theta$-openness and pre $e^*\theta$-openness are independent notions.*

Example 3.19 *Define the same function on the topology in Example 3.11. Since the image of every e^*-θ-open set of X is e^*-θ-closed in X, then f is contra pre $e^*\theta$-open. However, f is not pre $e^*\theta$-open.*

Example 3.20 *Consider the same topology in Example 3.11. Define the identity function $f : (X,\tau) \to (X,\tau)$. Since the image of every e^*-θ-open set of X is e^*-θ-open in X, then f is pre $e^*\theta$-open. However $\{c\}$ is e^*-θ-open in X, but $f[\{c\}] = \{c\}$ is not e^*-θ-closed in X. Therefore, f is not contra pre $e^*\theta$-open.*

Remark 3.21 *From Definitions 3.6, 3.7, 3.14, we have the relation among ap-$e^*\theta$-open functions, contra pre $e^*\theta$-open functions and other well-known functions in topological spaces. The converses of the below implications are not true in general, as shown in the previous examples.*

$$\begin{array}{ccccc} & & \textit{pre } e^*\theta\textit{-open function} & & \\ & \not\nearrow\ \not\swarrow & & \searrow\ \not\nwarrow & \\ e^*\theta\textit{-open function} & & \not\downarrow\ \not\uparrow & & \textit{ap-}e^*\theta\textit{-open function} \\ & \not\searrow\ \not\nwarrow & & \nearrow\ \not\swarrow & \\ & & \textit{contra pre } e^*\theta\textit{-open function} & & \end{array}$$

Theorem 3.22 *If $f : X \to Y$ is e^*-irresolute and ap-$e^*\theta$-open surjection, then $f^{-1}[B]$ is $qe^*\theta$-open in X whenever B is $qe^*\theta$-open subset of Y.*

Proof Let $B \in qe^*\theta O(Y)$. Suppose that $A \subseteq f^{-1}[B]$, where $A \in e^*\theta C(X)$.

$$\left.\begin{array}{r} (A \in e^*\theta C(X) \Rightarrow \setminus A \in e^*\theta O(X))(B \in qe^*\theta O(Y) \Rightarrow \setminus B \in qe^*\theta C(Y)) \\ A \subseteq f^{-1}[B] \Rightarrow f^{-1}[\setminus B] \subseteq \setminus A \Rightarrow f[f^{-1}[\setminus B]] \overset{f \text{ is surj.}}{=} \setminus B \subseteq f[\setminus A] \\ f \text{ is ap-}e^*\theta\text{-open} \end{array}\right\}$$

$$\Rightarrow \setminus e^*\text{-}int_\theta(B) = e^*\text{-}cl_\theta(\setminus B) \subseteq f[\setminus A]$$

$$\Rightarrow \left.\begin{array}{r} \setminus f^{-1}[e^*\text{-}int_\theta(B)] \subseteq \setminus A \Rightarrow A \subseteq f^{-1}[e^*\text{-}int_\theta(B)] \\ f \text{ is } e^*\text{-irresolute} \end{array}\right\} \Rightarrow f^{-1}[e^*\text{-}int_\theta(B)] \in e^*\theta O(X)$$

$$\Rightarrow A \subseteq f^{-1}[e^*\text{-}int_\theta(B)] = e^*\text{-}int_\theta(f^{-1}[e^*\text{-}int_\theta(B)]) \subseteq e^*\text{-}int_\theta(f^{-1}[B]).$$

This implies that by Lemma 3.5, $f^{-1}[B]$ is $qe^*\theta$-open in X. □

Definition 3.23 *A function $f : X \to Y$ is called quasi $e^*\theta$-irresolute (briefly, $qe^*\theta$-irresolute) if $f^{-1}[A]$ is $qe^*\theta$-closed in X for every $qe^*\theta$-closed set A of Y.*

Theorem 3.24 *Let $f : X \to Y$, $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$. Then:*

(i) $g \circ f$ is ap-$e^*\theta$-open if f is pre $e^*\theta$-open and g is ap-$e^*\theta$-open.

(ii) $g \circ f$ is ap-$e^*\theta$-open if f is ap-$e^*\theta$-open and g is bijective pre $e^*\theta$-closed and $qe^*\theta$-irresolute.

Proof (i): Let $A \in e^*\theta O(X)$ and $B \in qe^*\theta C(Z)$, where $B \subseteq (gof)[A]$.

$$\left.\begin{array}{r}(A \in e^*\theta O(X))(B \in qe^*\theta C(Z))(B \subseteq (gof)[A] = g[f[A]]) \\ f \text{ is pre } e^*\theta\text{-open}\end{array}\right\} \Rightarrow \left.\begin{array}{r} f[A] \in e^*\theta O(Y) \\ g \text{ is ap-}e^*\theta\text{-open}\end{array}\right\}$$

$$\Rightarrow e^*\text{-}cl_\theta(B) \subseteq g[f[A]] = (gof)[A].$$

This implies that $g \circ f$ is ap-$e^*\theta$-open.

(ii): Let $A \in e^*\theta O(X)$ and $B \in qe^*\theta C(Z)$, where $B \subseteq (gof)[A]$.

$$\left.\begin{array}{r}(A \in e^*\theta O(X))(B \in qe^*\theta C(Z))(B \subseteq (gof)[A] = g[f[A]]) \\ g \text{ is } qe^*\theta\text{-irresolute}\end{array}\right\}$$

$$\Rightarrow (A \in e^*\theta O(X))(g^{-1}[B] \in qe^*\theta C(Y))(g^{-1}[B] \subseteq g^{-1}[g[f[A]]] \overset{\text{g is bijective}}{=} f[A]) \left.\begin{array}{r} \\ f \text{ is ap-}e^*\theta\text{-open}\end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{r} e^*\text{-}cl_\theta(g^{-1}[B]) \subseteq f[A] \\ g \text{ is pre } e^*\theta\text{-closed}\end{array}\right\}$$

$$\Rightarrow e^*\text{-}cl_\theta(B) \subseteq e^*\text{-}cl_\theta(g[g^{-1}[B]]) \subseteq g[e^*\text{-}cl_\theta(g^{-1}[B])] \subseteq g[f[A]] = (gof)[A].$$

This implies that $g \circ f$ is ap-$e^*\theta$-open. □

Theorem 3.25 *Let $f : X \to Y$, $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$. Then:*

(i) $g \circ f$ is contra pre $e^*\theta$-open if f is pre $e^*\theta$-open and g is contra pre $e^*\theta$-open.

(ii) $g \circ f$ is contra pre $e^*\theta$-open if f is contra pre $e^*\theta$-open and g is pre $e^*\theta$-closed.

Proof (i): Let $U \in e^*\theta O(X)$.

$$\left.\begin{array}{r} U \in e^*\theta O(X) \\ f \text{ is pre } e^*\theta\text{-open}\end{array}\right\} \Rightarrow \left.\begin{array}{r} f[U] \in e^*\theta O(Y) \\ g \text{ is contra pre } e^*\theta\text{-open}\end{array}\right\} \Rightarrow g[f[U]] = (gof)[U] \in e^*\theta C(Z).$$

This implies that $g \circ f$ is contra pre $e^*\theta$-open.

(ii): Let $U \in e^*\theta O(X)$.

$$\left.\begin{array}{r} U \in e^*\theta O(X) \\ f \text{ is contra pre } e^*\theta\text{-open}\end{array}\right\} \Rightarrow \left.\begin{array}{r} f[U] \in e^*\theta C(Y) \\ g \text{ is pre } e^*\theta\text{-closed}\end{array}\right\} \Rightarrow g[f[U]] = (gof)[U] \in e^*\theta C(Z).$$

This implies that $g \circ f$ is contra pre $e^*\theta$-open. □

Theorem 3.26 *Let $f : X \to Y$, $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$ is contra pre $e^*\theta$-open. Then:*

(i) If f is an e^*-irresolute surjection, then g is contra pre $e^*\theta$-open.

(ii) If g is an e^*-irresolute injection, then f is contra pre $e^*\theta$-open.

Proof (i): Let $U \in e^*\theta O(Y)$.

$$\left.\begin{array}{r} U \in e^*\theta O(Y) \\ f \text{ is } e^*\text{-irresolute} \end{array}\right\} \Rightarrow \left.\begin{array}{r} f^{-1}[U] \in e^*\theta O(X) \\ g \circ f \text{ is contra pre } e^*\theta\text{-open} \end{array}\right\}$$

$$\Rightarrow (gof)[f^{-1}[U]] = g[f[f^{-1}[U]]] \overset{f \text{ is surj.}}{=} g[U] \in e^*\theta C(Z).$$

This implies that g is contra pre $e^*\theta$-open.

(ii): Let $U \in e^*\theta O(X)$.

$$\left.\begin{array}{r} U \in e^*\theta O(X) \\ g \circ f \text{ is contra pre } e^*\theta\text{-open} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (gof)[U] = g[f[U]] \in e^*\theta C(Z) \\ g \text{ is } e^*\text{-irresolute} \end{array}\right\}$$

$$\Rightarrow g^{-1}[g[f[U]]] \overset{g \text{ is inj.}}{=} f[U] \in e^*\theta C(Y).$$

This implies that f is contra pre $e^*\theta$-open. □

Definition 3.27 *Let X and Y be two topological spaces. A function $f : X \to Y$ has an $e^*\theta$-closed graph if its $G(f) = \{(x, f(x)) | x \in X\}$ is e^*-θ-closed in the product space $X \times Y$.*

Definition 3.28 *The product space $X = X_1 \times \ldots \times X_n$ has property $P_{e^*\theta}$ [5] if A_i is an e^*-θ-open set in a topological spaces X_i for $i = 1, 2, \ldots, n$, then $A_1 \times \ldots \times A_n$ is also e^*-θ-open in the product space $X = X_1 \times \ldots \times X_n$.*

Theorem 3.29 *If $f : X \to Y$ is a contra pre $e^*\theta$-open function with $e^*\theta$-closed fibers which has the property $P_{e^*\theta}$, then f has an $e^*\theta$-closed graph.*

Proof Let $(x, y) \notin G(f)$.

$$\left.\begin{array}{r} (x,y) \notin G(f) \Rightarrow (x,y) \in X \times Y \setminus G(f) \Rightarrow x \in \setminus f^{-1}[\{y\}] \\ f^{-1}[\{y\}] \text{ is } e^*\theta\text{-closed} \end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{r} (\exists E \in e^*\theta O(X, x))(E \subseteq \setminus f^{-1}[\{y\}]) \\ f \text{ is contra pre } e^*\theta\text{-open} \end{array}\right\} \Rightarrow A := \setminus f[E] \in e^*\theta O(Y, y)$$

$$\Rightarrow \left.\begin{array}{r} (x,y) \in E \times A \subseteq X \times Y \setminus G(f) \\ X \times Y \text{ has the property } P_{e^*\theta} \end{array}\right\} \Rightarrow E \times A \in e^*\theta O(X \times Y)$$

$$\Rightarrow X \times Y \backslash G(f) \in e^*\theta O(X \times Y)$$

$$\Rightarrow G(f) \in e^*\theta C(X \times Y).$$

□

4. Characterizations of $e^*\theta$-$T_{\frac{1}{2}}$ Spaces

Definition 4.1 *A topological space X is said to be $e^*\theta$-$T_{\frac{1}{2}}$ [1] if every $qe^*\theta$-closed set is $e^*\theta$-closed.*

Lemma 4.2 *Let X be a topological space and $A \subseteq X$. If $A \in qe^*\theta C(X)$, then $F \nsubseteq e^*$-$cl_\theta(A)\backslash A$ where $\varnothing \neq F \in e^*\theta C(X)$.*

Proof Let $A \in qe^*\theta C(X)$. Suppose that $F \subseteq e^*$-$cl_\theta(A)\backslash A$, where $\varnothing \neq F \in e^*\theta C(X)$.

$$\left.\begin{array}{r}(\varnothing \neq F \in e^*\theta C(X))(F \subseteq e^*\text{-}cl_\theta(A)\backslash A) \Rightarrow (\backslash F \in e^*\theta O(X))(A \subseteq \backslash F) \\ A \in qe^*\theta C(X)\end{array}\right\}$$

$$\left.\begin{array}{r}\Rightarrow e^*\text{-}cl_\theta(A) \subseteq \backslash F \Rightarrow F \subseteq \backslash e^*\text{-}cl_\theta(A) \\ F \subseteq e^*\text{-}cl_\theta(A)\backslash A \Rightarrow F \subseteq e^*\text{-}cl_\theta(A)\end{array}\right\} \Rightarrow F \subseteq (\backslash e^*\text{-}cl_\theta(A)) \cap e^*\text{-}cl_\theta(A) \Rightarrow F = \varnothing.$$

This is a contradiction and hence e^*-$cl_\theta(A)\backslash A$ does not contain any non-empty e^*-θ-closed set.

□

Theorem 4.3 *For a topological space X, the following statements are equivalent:*

(i) X is $e^*\theta$-$T_{\frac{1}{2}}$,

(ii) For each $x \in X$, $\{x\}$ is e^*-θ-closed or e^*-θ-open.

Proof $(i) \Rightarrow (ii)$: Suppose that for any $x \in X$, $\{x\} \notin e^*\theta C(X)$.

$$\left.\begin{array}{r}\{x\} \notin e^*\theta C(X) \Rightarrow X\backslash\{x\} \in e^*\theta O(X) \\ X\backslash\{x\} \subseteq X \in e^*\theta O(X)\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(X\backslash\{x\}) \subseteq X$$

$$\left.\begin{array}{r}\Rightarrow X\backslash\{x\} \in qe^*\theta C(X) \\ X \text{ is } e^*\theta\text{-}T_{\frac{1}{2}}\end{array}\right\} \Rightarrow X\backslash\{x\} \in e^*\theta C(X).$$

Thus $X\backslash\{x\} \in e^*\theta C(X)$ or equivalently $\{x\} \in e^*\theta O(X)$.

$(ii) \Rightarrow (i)$: Let $A \in qe^*\theta C(X)$ and $x \in e^*$-$cl_\theta(A)$.

Case I. If $\{x\} \in e^*\theta C(X)$:

$$A \in qe^*\theta C(X) \overset{\text{Lemma 4.2}}{\Rightarrow} (\{x\} \in e^*\theta C(X))(\{x\} \nsubseteq e^*\text{-}cl_\theta(A)\backslash A) \Rightarrow x \in A.$$

Case II. If $\{x\} \in e^*\theta O(X)$:

$$\{x\} \in e^*\text{-}cl_\theta(A) \Rightarrow (\{x\} \in e^*\theta O(X,x))(\{x\} \cap A \neq \varnothing) \Rightarrow x \in A.$$

As can be seen, in both cases $x \in A$. Thus $e^*\text{-}cl_\theta(A) \subseteq A$. Since there is always $A \subseteq e^*\text{-}cl_\theta(A)$, A is $e^*\text{-}\theta$-closed. □

Theorem 4.4 *For a topological space Y, the following statements are equivalent:*

(i) *Y is $e^*\theta\text{-}T_{\frac{1}{2}}$,*

(ii) *For every space X, every map $f : X \to Y$ is ap-$e^*\theta$-open.*

Proof $(i) \Rightarrow (ii)$: Let $B \in qe^*\theta C(Y)$ and let $B \subseteq f[A]$, where $A \in e^*\theta O(X)$.

$$\left.\begin{array}{r}(A \in e^*\theta O(X))(B \in qe^*\theta C(Y))(B \subseteq f[A]) \\ Y \text{ is } e^*\theta\text{-}T_{\frac{1}{2}}\end{array}\right\} \Rightarrow (A \in e^*\theta O(X))(B \in e^*\theta C(Y))(B \subseteq f[A])$$
$$\Rightarrow e^*\text{-}cl_\theta(B) = B \subseteq f[A]$$

Then, f is ap-$e^*\theta$-open.

$(ii) \Rightarrow (i)$: Let $B \in qe^*\theta C(Y)$. Suppose that $B \subseteq f[B]$, where $B \in e^*\theta O(X)$.

$$\left.\begin{array}{r}(B \in qe^*\theta C(Y))(B \in e^*\theta O(X))(B \subseteq f[B]) \\ f \text{ is ap-}e^*\theta\text{-open}\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(B) \subseteq f[B] = B \Rightarrow B \in e^*\theta C(Y).$$

Then, Y is $e^*\theta\text{-}T_{\frac{1}{2}}$. □

Theorem 4.5 *[2] For a topological space X, the following statements are equivalent:*

(i) *X is $e^*\theta\text{-}T_{\frac{1}{2}}$,*

(ii) *X is $e^*\theta\text{-}T_1$.*

5. Conclusion

Various forms of closed sets have been worked on by many topologist in recent years. This paper is concerned with the concept of quasi $e^*\text{-}\theta$-closed sets and which are defined by utilizing the notion of $e^*\text{-}\theta$-open set. Also, we defined approximately $e^*\theta$-open functions via quasi $e^*\text{-}\theta$-closed sets and $e^*\theta$-open sets. We demonstrated that newly defined these functions are weaker than $e^*\theta$-open functions, pre $e^*\theta$-open functions and contra pre $e^*\theta$-open functions (cf. Remark 3.21). We believe that this study will help researchers to support further studies on continuous functions.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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