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Optimal Control Problem for Fourth-Order Bianchi Equation in Variable Exponent Sobolev Spaces

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Abstract. This work proposes a necessary and sufficient condition such as Pontryagin's maximum principle for an optimal control problem with distributed parameters, which is described by the fourth-order Bianchi equation involving coefficients in variable exponent Lebesgue spaces. The problem is studied by aid of a novel version of the increment method that essentially uses the concept of the adjoint equation of integral type.

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1. Introduction

As is known, optimality in nature is one of the fundamental facts. The most appropriate choices for certain purposes are of crucial importance. The established models for the determination of such choices in natural science always lead researchers to optimization problems. Optimal control problems are of such optimization problems [3, 7, 11, 12, 16–22, 30–32]. Pontryagin's maximum principle is a fundamental result of the theory of first-order necessary optimality conditions, which is initially proved by R. V. Gamkrelidze in the linear case and V. G. Boltyanskii in the nonlinear case for such control problems described by ordinary differential equations [14, 29]. Under various assumptions, for many optimal control problems described by various differential equations of mathematical physics, a great number of necessary and sufficient conditions of optimality have been proposed by a remarkable number of works in literature.

Bianchi equations are of the partial differential equations which play an essential and key role in mathematical physics [8]. Some investigations for specific fundamental functions, various problems and the classes of such equations have been achieved [15, 23–28]. The works for optimality conditions and optimal control problems involving such equations are relatively less than the ones for the non-Bianchi equations. It is obvious that these works already deal with the classical and conventional spaces such as space of continuous function, Lebesgue and Sobolev spaces (with constant exponent). With the development and advantages of Lebesgue and Sobolev spaces with variable exponents in literature [4, 9, 10, 13], recently in this direction, two successful works on such variable exponent Lebesgue and Sobolev spaces have been observed [5, 6]. It can be clearly seen that the ideas and investigations in that two works may be certainly extended to higher dimensions. But for some reason it has been easily seen that the literature in this direction has been limited to only two and three dimensions. Therefore, inspired by that works, this work by generalization and realizations for four dimension is devoted to contribute on the enrichment of literature.

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2. Preliminaries

Let $\mathbb{R}^4$ be the 4-dimensional Euclidean space of points $x = (x_1, x_2, x_3, x_4)$ equipped with $| x |= (\sum_{i=1}^4 x_i{}^2)^{1/2}$, and $G = G_1 \times G_2 \times G_3 \times G_4 = (x_1^0, h_1) \times (x_2^0, h_2) \times (x_3^0, h_3) \times (x_4^0, h_4)$, $x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$ and let h_i be a fixed real number for $i = 1, 2, 3, 4$.

$\mathcal{P}(G)$ denotes the set of Lebesgue measurable functions such that $p : G \to [1, \infty)$. The function $p \in \mathcal{P}(G)$ is called variable exponent on G. The descriptions

$$\underline{p} = \operatorname*{ess\,inf}_{x\in G} p(x) \quad \text{and} \quad \overline{p} = \operatorname*{ess\,sup}_{x\in G} p(x),$$

and the denotations

$$r_1(x_1) = \lim_{\substack{x_4 \to x_4^0+0\\ x_3 \to x_3^0+0\\ x_2\to x_2^0+0}} p(x_1, x_2, x_3, x_4), \quad r_2(x_2) = \lim_{\substack{x_4 \to x_4^0+0\\ x_3 \to x_3^0+0\\ x_1\to x_1^0+0}} p(x_1, x_2, x_3, x_4),$$

$$r_3(x_3) = \lim_{\substack{x_4 \to x_4^0+0\\ x_2 \to x_2^0+0\\ x_1\to x_1^0+0}} p(x_1, x_2, x_3, x_4), \quad r_4(x_4) = \lim_{\substack{x_3 \to x_3^0+0\\ x_2 \to x_2^0+0\\ x_1\to x_1^0+0}} p(x_1, x_2, x_3, x_4)$$

are considered. Let $q(x)$ be the conjugate variable exponent for $p(x)$ defined by $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Assume $\frac{1}{r_1(x_1)} + \frac{1}{s_1(x_1)} = 1$, $\frac{1}{r_2(x_2)} + \frac{1}{s_2(x_2)} = 1$, $\frac{1}{r_3(x_3)} + \frac{1}{s_3(x_3)} = 1$ and $\frac{1}{r_4(x_4)} + \frac{1}{s_4(x_4)} = 1$, where $x \in G$. Evidently,

$$\operatorname*{ess\,sup}_{x\in G} q(x) = \overline{q} = \frac{\underline{p}}{\underline{p}-1} \quad \text{and} \quad \operatorname*{ess\,inf}_{x\in G} q(x) = \underline{q} = \frac{\overline{p}}{\overline{p}-1}.$$

Definition 2.1 ([10, 13])**.** Let $p \in \mathcal{P}(G)$. By $L_{p(x)}(G)$, we denote the space of Lebesgue measurable functions f on G such that for some $\lambda_0 > 0$

$$\int_G \left(\frac{|f(x)|}{\lambda_0}\right)^{p(x)} dx < \infty.$$

Note that the functional

$$\|f\|_{L_{p(x)}(G)} = \|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \int_G \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\}$$

defines the norm in $L_{p(x)}(G)$ and $L_{p(x)}(G)$ is a Banach function space [10, 13].

Definition 2.2. Let $p \in \mathcal{P}(G)$. By $SW^{(1,1,1,1)}_{p(x)}(G)$, we define the variable exponent Sobolev space of functions with dominating mixed derivatives as

$$SW^{(1,1,1,1)}_{p(x)}(G) := \{u \in L_1^{\text{loc}}(G) : D_1^{i_1} D_2^{i_2} D_3^{i_3} D_4^{i_4} u(x) \in L_{p(x)}(G),\ i_k = 0, 1,\ k = 1, 2, 3, 4\}.$$

Evidently, the norm in $SW^{(1,1,1,1)}_{p(x)}(G)$ is defined as

$$\|u\|_{SW^{(1,1,1,1)}_{p(\cdot)}(G)} = \sum_{i_1,i_2,i_3,i_4=0}^{1} \|D_1^{i_1} D_2^{i_2} D_3^{i_3} D_4^{i_4} u\|_{L_{p(\cdot)}(G)} < \infty.$$

Lemma 2.3. *Let $p \in \mathcal{P}(G)$ and $1 < \underline{p} \le \overline{p} < \infty$. Then, the space $SW^{(1,1,1,1)}_{p(x)}(G)$ is complete.*

Proof. Let$\{u_n\}_{n=1}^\infty$ be a Cauchy sequence in $SW^{(1,1,1,1)}_{p(x)}(G)$. $\{D_1^{i_1} D_2^{i_2} D_3^{i_3} D_4^{i_4} u_n\}$ is a Cauchy sequence in $L_{p(x)}(G)$ for all $0 \le i_1, i_2, i_3, i_4 \le 1$. By the completeness of $L_{p(x)}(G)$ (see [10]), there exists $g_{i_1,i_2,i_3,i_4} \in L_{p(x)}(G)$ such that

$$\|D_1^{i_1} D_2^{i_2} D_3^{i_3} D_4^{i_4} u_n - g_{i_1,i_2,i_3,i_4}\|_{L_{p(x)}(G)} \to 0 \quad \text{as} \quad n \to \infty$$

and for all $0 \le i_1, i_2, i_3, i_4 \le 1$. Applying the Hölder inequality in variable exponent Lebesgue spaces (see [10, 13]), for $\varphi \in C_c^\infty(G)$ we obtain

$$\int_G (D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}u_n(x) - g_{i_1,i_2,i_3,i_4}(x))D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}\varphi(x)\mathrm{d}x$$
$$\leq (\frac{1}{\underline{p}} + \frac{1}{\overline{p}})\|D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}u_n - g_{i_1,i_2,i_3,i_4}\|_{L_{p(x)}(G)}\|D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}\varphi\|_{L_{q(x)}(G)}.$$

Since $\|D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}u_n - g_{i_1,i_2,i_3,i_4}\|_{L_{p(x)}(G)} \to 0$ and $D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}\varphi(x)$ is bounded for any $\varphi \in C_c^\infty(G)$, by the Lebesgue dominated convergence theorem in variable Lebesgue spaces (see [10]), we obtain

$$\lim_{n\to\infty}\int_G u_n(x)D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}\varphi(x)\mathrm{d}x = \int_G g_{i_1,i_2,i_3,i_4}(x)D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}\varphi(x)\mathrm{d}x.$$

Thus, for all $\varphi \in C_c^\infty(G)$, we obtain

$$\begin{aligned}\int_G u(x)D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}\varphi(x)\mathrm{d}x &= \lim_{n\to\infty}\int_G u_n(x)D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}\varphi(x)\mathrm{d}x\\ &=(-1)^{i_1+i_2+i_3+i_4}\lim_{n\to\infty}\int_G D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}u_n(x)\varphi(x)\mathrm{d}x\\ &=(-1)^{i_1+i_2+i_3+i_4}\int_G g_{i_1,i_2,i_3,i_4}(x)\varphi(x)\mathrm{d}x.\end{aligned}$$

This shows that $D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}u$ exists weakly and $g_{i_1,i_2,i_3,i_4} = D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}u$. Thus, $u \in SW_{p(x)}^{(1,1,1,1)}(G)$ and $u_n \to u$ as $n \to \infty$, which completes the proof. □

3. Statement of the Problem

By $L_{(p_1(x_1),p_2(x_2),p_3(x_3),p_4(x_4))}(G)$, we denote the variable Lebesgue space with the mixed norm defined as

$$\|f\|_{L_{(p_1(x_1),p_2(x_2),p_3(x_3),p_4(x_4))}(G)} = \|\|\|\|f\|_{L_{p_1(x_1)}(G_1)}\|_{L_{p_2(x_2)}(G_2)}\|_{L_{p_3(x_3)}(G_3)}\|_{L_{p_4(x_4)}(G_4)} < \infty.$$

Let the controlled object be governed by the Bianchi equation

$$(V_{1,1,1,1}u)(x) \equiv D_1D_2D_3D_4u(x) + \sum_{\substack{i_1,i_2,i_3,i_4=0\\ 0\leq i_1+i_2+i_3+i_4\leq 3}}^{1} a_{i_1,i_2,i_3,i_4}(x)D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}u(x) = \varphi(x,v(x)), \tag{3.1}$$

and the following nonclassical Goursat conditions (see [2])

$$\begin{aligned}
(V_{0,0,0,0}u) &\equiv u(x_1^0,x_2^0,x_3^0,x_4^0) = \varphi_{0,0,0,0},\\
(V_{1,0,0,0}u)(x_1) &\equiv D_1u(x_1,x_2^0,x_3^0,x_4^0) = \varphi_{1,0,0,0}(x_1),\\
(V_{0,1,0,0}u)(x_2) &\equiv D_2u(x_1^0,x_2,x_3^0,x_4^0) = \varphi_{0,1,0,0}(x_2),\\
(V_{0,0,1,0}u)(x_3) &\equiv D_3u(x_1^0,x_2^0,x_3,x_4^0) = \varphi_{0,0,1,0}(x_3),\\
(V_{0,0,0,1}u)(x_4) &\equiv D_4u(x_1^0,x_2^0,x_3^0,x_4) = \varphi_{0,0,0,1}(x_4),\\
(V_{1,1,0,0}u)(x_1,x_2) &\equiv D_1D_2u(x_1,x_2,x_3^0,x_4^0) = \varphi_{1,1,0,0}(x_1,x_2),\\
(V_{0,1,1,0}u)(x_2,x_3) &\equiv D_2D_3u(x_1^0,x_2,x_3,x_4^0) = \varphi_{0,1,1,0}(x_2,x_3),\\
(V_{0,0,1,1}u)(x_3,x_4) &\equiv D_3D_4u(x_1^0,x_2^0,x_3,x_4) = \varphi_{0,0,1,1}(x_3,x_4),\\
(V_{1,0,1,0}u)(x_1,x_3) &\equiv D_1D_3u(x_1,x_2^0,x_3,x_4^0) = \varphi_{1,0,1,0}(x_1,x_3),\\
(V_{0,1,0,1}u)(x_2,x_4) &\equiv D_2D_4u(x_1^0,x_2,x_3^0,x_4) = \varphi_{0,1,0,1}(x_2,x_4),\\
(V_{1,0,0,1}u)(x_1,x_4) &\equiv D_1D_4u(x_1,x_2^0,x_3^0,x_4) = \varphi_{1,0,0,1}(x_1,x_4),\\
(V_{1,1,1,0}u)(x_1,x_2,x_3) &\equiv D_1D_2D_3u(x_1,x_2,x_3,x_4^0) = \varphi_{1,1,1,0}(x_1,x_2,x_3),\\
(V_{0,1,1,1}u)(x_2,x_3,x_4) &\equiv D_2D_3D_4u(x_1^0,x_2,x_3,x_4) = \varphi_{0,1,1,1}(x_2,x_3,x_4),\\
(V_{1,1,0,1}u)(x_1,x_2,x_4) &\equiv D_1D_2D_4u(x_1,x_2,x_3^0,x_4) = \varphi_{1,1,0,1}(x_1,x_2,x_4),\\
(V_{1,0,1,1}u)(x_1,x_3,x_4) &\equiv D_1D_3D_4u(x_1,x_2^0,x_3,x_4) = \varphi_{1,0,1,1}(x_1,x_3,x_4),
\end{aligned} \tag{3.2}$$

where $a_{0,0,0,0}(x) \in L_{p(x)}(G)$, $a_{1,0,0,0}(x) \in L_{(\infty,r_2(x_2),r_3(x_3),r_4(x_4))}(G)$, $a_{0,1,0,0}(x) \in L_{(r_1(x_1),\infty,r_3(x_3),r_4(x_4))}(G)$, $a_{0,0,1,0}(x) \in L_{(r_1(x_1),r_2(x_2),\infty,r_4(x_4))}(G)$, $a_{0,0,0,1}(x) \in L_{(r_1(x_1),r_2(x_2),r_3(x_3),\infty)}(G)$, $a_{1,1,0,0}(x) \in L_{(\infty,\infty,r_3(x_3),r_4(x_4))}(G)$, $a_{0,1,1,0}(x) \in L_{(r_1(x_1),\infty,\infty,r_4(x_4))}(G)$, $a_{0,0,1,1}(x) \in L_{(r_1(x_1),r_2(x_2),\infty,\infty)}(G)$, $a_{1,0,1,0}(x) \in L_{(\infty,r_2(x_2),\infty,r_4(x_4))}(G)$, $a_{0,1,0,1}(x) \in L_{(r_1(x_1),\infty,r_3(x_3),\infty)}(G)$, $a_{1,0,0,1}(x) \in L_{(\infty,r_2(x_2),r_3(x_3),\infty)}(G)$, $a_{1,1,1,0}(x) \in L_{(\infty,\infty,\infty,r_4(x_4))}(G)$, $a_{0,1,1,1}(x) \in L_{(r_1(x_1),\infty,\infty,\infty)}(G)$, $a_{1,1,0,1}(x) \in L_{(\infty,\infty,r_3(x_3),\infty)}(G)$, $a_{1,0,1,1}(x) \in L_{(\infty,r_2(x_2),\infty,\infty)}(G)$, $\varphi_{0,0,0,0} \in \mathbb{R}$, $\varphi_{1,0,0,0}(x_1) \in L_{r_1(x_1)}(G_1)$, $\varphi_{0,1,0,0}(x_2) \in L_{r_2(x_2)}(G_2)$, $\varphi_{0,0,1,0}(x_3) \in L_{r_3(x_3)}(G_3)$, $\varphi_{0,0,0,1}(x_4) \in L_{r_4(x_4)}(G_4)$, $\varphi_{1,1,0,0}(x_1,x_2) \in L_{(r_1(x_1),r_2(x_2))}(G_1 \times G_2)$, $\varphi_{0,1,1,0}(x_2,x_3) \in L_{(r_2(x_2),r_3(x_3))}(G_2 \times G_3)$, $\varphi_{0,0,1,1}(x_3,x_4) \in L_{(r_3(x_3),r_4(x_4))}(G_3 \times G_4)$, $\varphi_{1,0,1,0}(x_1,x_3) \in L_{(r_1(x_1),r_3(x_3))}(G_1 \times G_3)$, $\varphi_{0,1,0,1}(x_2,x_4) \in L_{(r_2(x_2),r_4(x_4))}(G_2 \times G_4)$, $\varphi_{1,0,0,1}(x_1,x_4) \in L_{(r_1(x_1),r_4(x_4))}(G_1 \times G_4)$, $\varphi_{1,1,1,0}(x_1,x_2,x_3) \in L_{(r_1(x_1),r_2(x_2),r_3(x_3))}(G_1 \times G_2 \times G_3)$, $\varphi_{0,1,1,1}(x_2,x_3,x_4) \in L_{(r_2(x_2),r_3(x_3),r_4(x_4))}(G_2 \times G_3 \times G_4)$, $\varphi_{1,1,0,1}(x_1,x_2,x_4) \in L_{(r_1(x_1),r_2(x_2),r_4(x_4))}(G_1 \times G_2 \times G_4)$, $\varphi_{1,0,1,1}(x_1,x_3,x_4) \in L_{(r_1(x_1),r_3(x_3),r_4(x_4))}(G_1 \times G_3 \times G_4)$ and $D_k = \frac{\partial}{\partial x_k}(k = 1,2,3,4)$ is the generalized differential operator in the weak sense. Let $\nu(x) = (\nu_1(x), \nu_2(x), ..., \nu_m(x))$ be a m-dimensional control vector function and $\varphi(x,\nu(x))$ be a given function defined on $G \times \mathbb{R}^m$ and satisfying Carathéodory conditions on $G \times \mathbb{R}^m$:
(1) $\varphi(x,\nu(x))$ is measurable by x in G for all $\nu(x) \in \mathbb{R}^m$,
(2) $\varphi(x,\nu(x))$ is continuous by $\nu(x)$ in $\mathbb{R}^m$ for almost all $x \in G$,
(3) for any $\delta > 0$ there exists $\varphi^0_\delta(x) \in L_{p(x)}(G)$ such that $|\varphi(x,\nu(x))| \le \varphi^0_\delta(x)$ for almost all $x \in G$ and $\|\nu(x)\| = \sum_{i=1}^m |\nu_i(x)| \le \delta$.
Since the coefficients of equation (3.1) are nonsmooth, the solution of problem (3.1)-(3.2) is in the weak sense. Let the vector function $\nu(x)$ be measurable and bounded on G and for almost all $x \in G$ it takes its value from the given set $\Omega \subset \mathbb{R}^m$. Then, the vector function is called an admissible control. The set of all admissible controls is denoted by Ω_∂.

We now consider the following 4D optimal control problem: Find an admissible control $\nu(x)$ from Ω_∂, for which the solution $u \in SW^{(1,1,1,1)}_{p(x)}(G)$ of problem (3.1)-(3.2) minimizes of the multipoint functional

$$\begin{aligned} F(\nu) = \sum_{k=1}^{N} \{ & \alpha_k^{(1,0,0,0)} u(x_1^{(k)},h_2,h_3,h_4) + \alpha_k^{(0,1,0,0)} u(h_1,x_2^{(k)},h_3,h_4) + \alpha_k^{(0,0,1,0)} u(h_1,h_2,x_3^{(k)},h_4) \\ & + \alpha_k^{(0,0,0,1)} u(h_1,h_2,h_3,x_4^{(k)}) + \alpha_k^{(1,1,0,0)} u(x_1^{(k)},x_2^{(k)},h_3,h_4) + \alpha_k^{(0,1,1,0)} u(h_1,x_2^{(k)},x_3^{(k)},h_4) \\ & + \alpha_k^{(0,0,1,1)} u(h_1,h_2,x_3^{(k)},x_4^{(k)}) + \alpha_k^{(1,0,1,0)} u(x_1^{(k)},h_2,x_3^{(k)},h_4) + \alpha_k^{(0,1,0,1)} u(h_1,x_2^{(k)},h_3,x_4^{(k)}) \\ & + \alpha_k^{(1,0,0,1)} u(x_1^{(k)},h_2,h_3,x_4^{(k)}) + \alpha_k^{(1,1,1,0)} u(x_1^{(k)},x_2^{(k)},x_3^{(k)},h_4) + \alpha_k^{(0,1,1,1)} u(h_1,x_2^{(k)},x_3^{(k)},x_4^{(k)}) \\ & + \alpha_k^{(1,1,0,1)} u(x_1^{(k)},x_2^{(k)},h_3,x_4^{(k)}) + \alpha_k^{(1,0,1,1)} u(x_1^{(k)},h_2,x_3^{(k)},x_4^{(k)}) + \alpha_k^{(1,1,1,1)} u(x_1^{(k)},x_2^{(k)},x_3^{(k)},x_4^{(k)}) \} \to \min., \end{aligned} \tag{3.3}$$

where $(x_1^{(k)},x_2^{(k)},x_3^{(k)},x_4^{(k)}) \in \overline{G}$ are the given fixed points, $\alpha_k^{(i_1,i_2,i_3,i_4)}$ are the given real numbers and N is a positive integer, $i_j = 0,1,\ j = 1,2,3,4$ and $1 \le i_1 + i_2 + i_3 + i_4 \le 4$.

4. The Construction of Adjoint Equation for Optimal Control Problem (3.1)-(3.3)

To obtain the necessary and sufficient conditions for optimality, we firstly determine the increment of functional (3.3). Let $\nu(x)$ and $\nu(x) + \Delta\nu(x)$ be different admissible controls and let $u(x)$ and $u(x) + \Delta u(x)$ be solutions of problem (3.1)-(3.2) in $SW^{(1,1,1,1)}_{p(x)}(G)$. Then, the increment of functional (3.3) is of the form

$$\begin{aligned} \Delta F(\nu) = \sum_{k=1}^{N} \{ & \alpha_k^{(1,0,0,0)} \Delta u(x_1^{(k)},h_2,h_3,h_4) + \alpha_k^{(0,1,0,0)} \Delta u(h_1,x_2^{(k)},h_3,h_4) + \alpha_k^{(0,0,1,0)} \Delta u(h_1,h_2,x_3^{(k)},h_4) \\ & + \alpha_k^{(0,0,0,1)} \Delta u(h_1,h_2,h_3,x_4^{(k)}) + \alpha_k^{(1,1,0,0)} \Delta u(x_1^{(k)},x_2^{(k)},h_3,h_4) + \alpha_k^{(0,1,1,0)} \Delta u(h_1,x_2^{(k)},x_3^{(k)},h_4) \\ & + \alpha_k^{(0,0,1,1)} \Delta u(h_1,h_2,x_3^{(k)},x_4^{(k)}) + \alpha_k^{(1,0,1,0)} \Delta u(x_1^{(k)},h_2,x_3^{(k)},h_4) + \alpha_k^{(0,1,0,1)} \Delta u(h_1,x_2^{(k)},h_3,x_4^{(k)}) \\ & + \alpha_k^{(1,0,0,1)} \Delta u(x_1^{(k)},h_2,h_3,x_4^{(k)}) + \alpha_k^{(1,1,1,0)} \Delta u(x_1^{(k)},x_2^{(k)},x_3^{(k)},h_4) + \alpha_k^{(0,1,1,1)} \Delta u(h_1,x_2^{(k)},x_3^{(k)},x_4^{(k)}) \\ & + \alpha_k^{(1,1,0,1)} \Delta u(x_1^{(k)},x_2^{(k)},h_3,x_4^{(k)}) + \alpha_k^{(1,0,1,1)} \Delta u(x_1^{(k)},h_2,x_3^{(k)},x_4^{(k)}) + \alpha_k^{(1,1,1,1)} \Delta u(x_1^{(k)},x_2^{(k)},x_3^{(k)},x_4^{(k)}) \}. \end{aligned} \tag{4.1}$$

Evidently, in this case, $\Delta u \in SW^{(1,1,1,1)}_{p(x)}(G)$ is the solution of the following equation

$$V_{1,1,1,1} \Delta u(x) = \Delta\varphi(x) \tag{4.2}$$

subject to the trivial conditions

$$\begin{gathered} V_{0,0,0,0}\Delta u = 0, (V_{1,0,0,0}\Delta u)(x_1) = 0, (V_{0,1,0,0}\Delta u)(x_2) = 0, \\ (V_{0,0,1,0}\Delta u)(x_3) = 0, (V_{0,0,0,1}\Delta u)(x_4) = 0, (V_{1,1,0,0}\Delta u)(x_1, x_2) = 0, \\ (V_{0,1,1,0}\Delta u)(x_2, x_3) = 0, (V_{0,0,1,1}\Delta u)(x_3, x_4) = 0, (V_{1,0,1,0}\Delta u)(x_1, x_3) = 0, \\ (V_{0,1,0,1}\Delta u)(x_2, x_4) = 0, (V_{1,0,0,1}\Delta u)(x_1, x_4) = 0, (V_{1,1,1,0}\Delta u)(x_1, x_2, x_3) = 0, \\ (V_{0,1,1,1}\Delta u)(x_2, x_3, x_4) = 0, (V_{1,1,0,1}\Delta u)(x_1, x_2, x_4) = 0, (V_{1,0,1,1}\Delta u)(x_1, x_3, x_4) = 0, \end{gathered} \tag{4.3}$$

where $\Delta\varphi(x) = \varphi(x, \nu(x) + \Delta\nu(x)) - \varphi(x, \nu(x))$. Let's denote

$$\begin{aligned} V = (&V_{1,1,1,1}, V_{0,0,0,0}, V_{1,0,0,0}, V_{0,1,0,0}, V_{0,0,1,0}, V_{0,0,0,1}, V_{1,1,0,0}, V_{0,1,1,0}, \\ &V_{0,0,1,1}, V_{1,0,1,0}, V_{0,1,0,1}, V_{1,0,0,1}, V_{1,1,1,0}, V_{0,1,1,1}, V_{1,1,0,1}, V_{1,0,1,1}) \end{aligned}$$

and

$$\begin{aligned} E_{p(x)}(G) \equiv &L_{p(x)}(G) \times \mathbb{R} \times L_{r_1(x_1)}(G_1) \times L_{r_2(x_2)}(G_2) \times L_{r_3(x_3)}(G_3) \times L_{r_4(x_4)}(G_4) \times L_{(r_1(x_1), r_2(x_2))}(G_1 \times G_2) \\ &\times L_{(r_2(x_2), r_3(x_3))}(G_2 \times G_3) \times L_{(r_3(x_3), r_4(x_4))}(G_3 \times G_4) \times L_{(r_1(x_1), r_3(x_3))}(G_1 \times G_3) \\ &\times L_{(r_2(x_2), r_4(x_4))}(G_2 \times G_4) \times L_{(r_1(x_1), r_4(x_4))}(G_1 \times G_4) \times L_{(r_1(x_1), r_2(x_2), r_3(x_3))}(G_1 \times G_2 \times G_3) \\ &\times L_{(r_2(x_2), r_3(x_3), r_4(x_4))}(G_2 \times G_3 \times G_4) \times L_{(r_1(x_1), r_2(x_2), r_4(x_4))}(G_1 \times G_2 \times G_4) \times L_{(r_1(x_1), r_3(x_3), r_4(x_4))}(G_1 \times G_3 \times G_4). \end{aligned}$$

Let $B(G)$ denote the set of variable exponents $p(x)$ such that V is bounded from $SW^{(1,1,1,1)}_{p(x)}(G)$ to $E_{p(x)}(G)$. Then, operator $V : SW^{(1,1,1,1)}_{p(x)}(G) \to E_{p(x)}(G)$ generated by problem (3.1)-(3.2) is bounded by the considered assumptions. The integral representation of any function $u \in SW^{(1,1,1,1)}_{p(x)}(G)$ can be given as [1]

$$\begin{aligned} u(x) = &u(x_1^0, x_2^0, x_3^0, x_4^0) + \int_{x_1^0}^{x_1} u_{\alpha_1}(\alpha_1, x_2^0, x_3^0, x_4^0)\mathrm{d}\alpha_1 + \int_{x_2^0}^{x_2} u_{\alpha_2}(x_1^0, \alpha_2, x_3^0, x_4^0)\mathrm{d}\alpha_2 + \int_{x_3^0}^{x_3} u_{\alpha_3}(x_1^0, x_2^0, \alpha_3, x_4^0)\mathrm{d}\alpha_3 \\ &+ \int_{x_4^0}^{x_4} u_{\alpha_4}(x_1^0, x_2^0, x_3^0, \alpha_4)\mathrm{d}\alpha_4 + \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2}(\alpha_1, \alpha_2, x_3^0, x_4^0)\mathrm{d}\alpha_2\mathrm{d}\alpha_1 + \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{\alpha_2\alpha_3}(x_1^0, \alpha_2, \alpha_3, x_4^0)\mathrm{d}\alpha_3\mathrm{d}\alpha_2 \\ &+ \int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{\alpha_3\alpha_4}(x_1^0, x_2^0, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3 + \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3} u_{\alpha_1\alpha_3}(\alpha_1, x_2^0, \alpha_3, x_4^0)\mathrm{d}\alpha_3\mathrm{d}\alpha_1 \\ &+ \int_{x_2^0}^{x_2}\int_{x_4^0}^{x_4} u_{\alpha_2\alpha_4}(x_1^0, \alpha_2, x_3^0, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2 + \int_{x_1^0}^{x_1}\int_{x_4^0}^{x_4} u_{\alpha_1\alpha_4}(\alpha_1, x_2^0, x_3^0, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_1 \\ &+ \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{\alpha_1\alpha_2\alpha_3}(\alpha_1, \alpha_2, \alpha_3, x_4^0)\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1 + \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{\alpha_2\alpha_3\alpha_4}(x_1^0, \alpha_2, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_2 \\ &+ \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_4^0}^{x_4} u_{\alpha_1\alpha_2\alpha_4}(\alpha_1, \alpha_2, x_3^0, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2\mathrm{d}\alpha_1 + \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{\alpha_1\alpha_3\alpha_4}(\alpha_1, x_2^0, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_1 \\ &+ \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{\alpha_1\alpha_2\alpha_3\alpha_4}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1. \end{aligned} \tag{4.4}$$

It is clear that, the weak derivatives are of form

$$\begin{aligned} D_1u(x) = &u_{x_1}(x_1, x_2^0, x_3^0, x_4^0) + \int_{x_2^0}^{x_2} u_{x_1\alpha_2}(x_1, \alpha_2, x_3^0, x_4^0)\mathrm{d}\alpha_2 + \int_{x_3^0}^{x_3} u_{x_1\alpha_3}(x_1, x_2^0, \alpha_3, x_4^0)\mathrm{d}\alpha_3 + \int_{x_4^0}^{x_4} u_{x_1\alpha_4}(x_1, x_2^0, x_3^0, \alpha_4)\mathrm{d}\alpha_4 \\ &+ \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{x_1\alpha_2\alpha_3}(x_1, \alpha_2, \alpha_3, x_4^0)\mathrm{d}\alpha_3\mathrm{d}\alpha_2 + \int_{x_2^0}^{x_2}\int_{x_4^0}^{x_4} u_{x_1\alpha_2\alpha_4}(x_1, \alpha_2, x_3^0, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2 \\ &+ \int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{x_1\alpha_3\alpha_4}(x_1, x_2^0, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3 + \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{x_1\alpha_2\alpha_3\alpha_4}(x_1, \alpha_2, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_2, \end{aligned}$$

$$D_2u(x) = u_{x_2}(x_1^0, x_2, x_3^0, x_4^0) + \int_{x_1^0}^{x_1} u_{\alpha_1 x_2}(\alpha_1, x_2, x_3^0, x_4^0)\mathrm{d}\alpha_1 + \int_{x_3^0}^{x_3} u_{x_2\alpha_3}(x_1^0, x_2, \alpha_3, x_4^0)\mathrm{d}\alpha_3 + \int_{x_4^0}^{x_4} u_{x_2\alpha_4}(x_1^0, x_2, x_3^0, \alpha_4)\mathrm{d}\alpha_4$$
$$+ \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3} u_{\alpha_1 x_2\alpha_3}(\alpha_1, x_2, \alpha_3, x_4^0)\mathrm{d}\alpha_3\mathrm{d}\alpha_1 + \int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{x_2\alpha_3\alpha_4}(x_1^0, x_2, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3$$
$$+ \int_{x_1^0}^{x_1}\int_{x_4^0}^{x_4} u_{\alpha_1 x_2\alpha_4}(\alpha_1, x_2, x_3^0, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_1 + \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{\alpha_1 x_2\alpha_3\alpha_4}(\alpha_1, x_2, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_1,$$

$$D_3u(x) = u_{x_3}(x_1^0, x_2^0, x_3, x_4^0) + \int_{x_2^0}^{x_2} u_{\alpha_2 x_3}(x_1^0, \alpha_2, x_3, x_4^0)\mathrm{d}\alpha_2 + \int_{x_4^0}^{x_4} u_{x_3\alpha_4}(x_1^0, x_2^0, x_3, \alpha_4)\mathrm{d}\alpha_4 + \int_{x_1^0}^{x_1} u_{\alpha_1 x_3}(\alpha_1, x_2^0, x_3, x_4^0)\mathrm{d}\alpha_1$$
$$+ \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2 x_3}(\alpha_1, \alpha_2, x_3, x_4^0)\mathrm{d}\alpha_2\mathrm{d}\alpha_1 + \int_{x_2^0}^{x_2}\int_{x_4^0}^{x_4} u_{\alpha_2 x_3\alpha_4}(x_1^0, \alpha_2, x_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2$$
$$+ \int_{x_1^0}^{x_1}\int_{x_4^0}^{x_4} u_{\alpha_1 x_3\alpha_4}(\alpha_1, x_2^0, x_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_1 + \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_4^0}^{x_4} u_{\alpha_1\alpha_2 x_3\alpha_4}(\alpha_1, \alpha_2, x_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2\mathrm{d}\alpha_1,$$

$$D_4u(x) = u_{x_4}(x_1^0, x_2^0, x_3^0, x_4) + \int_{x_3^0}^{x_3} u_{\alpha_3 x_4}(x_1^0, x_2^0, \alpha_3, x_4)\mathrm{d}\alpha_3 + \int_{x_2^0}^{x_2} u_{\alpha_2 x_4}(x_1^0, \alpha_2, x_3^0, x_4)\mathrm{d}\alpha_2 + \int_{x_1^0}^{x_1} u_{\alpha_1 x_4}(\alpha_1, x_2^0, x_3^0, x_4)\mathrm{d}\alpha_1$$
$$+ \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{\alpha_2\alpha_3 x_4}(x_1^0, \alpha_2, \alpha_3, x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_2 + \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2 x_4}(\alpha_1, \alpha_2, x_3^0, x_4)\mathrm{d}\alpha_2\mathrm{d}\alpha_1$$
$$+ \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3} u_{\alpha_1\alpha_3 x_4}(\alpha_1, x_2^0, \alpha_3, x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_1 + \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{\alpha_1\alpha_2\alpha_3 x_4}(\alpha_1, \alpha_2, \alpha_3, x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1,$$

$$D_1D_2u(x) = u_{x_1x_2}(x_1, x_2, x_3^0, x_4^0) + \int_{x_3^0}^{x_3} u_{x_1x_2\alpha_3}(x_1, x_2, \alpha_3, x_4^0)\mathrm{d}\alpha_3 + \int_{x_4^0}^{x_4} u_{x_1x_2\alpha_4}(x_1, x_2, x_3^0, \alpha_4)\mathrm{d}\alpha_4$$
$$+ \int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{x_1x_2\alpha_3\alpha_4}(x_1, x_2, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3,$$

$$D_2D_3u(x) = u_{x_2x_3}(x_1^0, x_2, x_3, x_4^0) + \int_{x_1^0}^{x_1} u_{\alpha_1x_2x_3}(\alpha_1, x_2, x_3, x_4^0)\mathrm{d}\alpha_1 + \int_{x_4^0}^{x_4} u_{x_2x_3\alpha_4}(x_1^0, x_2, x_3, \alpha_4)\mathrm{d}\alpha_4$$
$$+ \int_{x_1^0}^{x_1}\int_{x_4^0}^{x_4} u_{\alpha_1x_2x_3\alpha_4}(\alpha_1, x_2, x_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_1,$$

$$D_3D_4u(x) = u_{x_3x_4}(x_1^0, x_2^0, x_3, x_4) + \int_{x_2^0}^{x_2} u_{\alpha_2x_3x_4}(x_1^0, \alpha_2, x_3, x_4)\mathrm{d}\alpha_2 + \int_{x_1^0}^{x_1} u_{\alpha_1x_3x_4}(\alpha_1, x_2^0, x_3, x_4)\mathrm{d}\alpha_1$$
$$+ \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2x_3x_4}(\alpha_1, \alpha_2, x_3, x_4)\mathrm{d}\alpha_2\mathrm{d}\alpha_1,$$

$$D_1D_3u(x) = u_{x_1x_3}(x_1, x_2^0, x_3, x_4^0) + \int_{x_2^0}^{x_2} u_{x_1\alpha_2x_3}(x_1, \alpha_2, x_3, x_4^0)\mathrm{d}\alpha_2 + \int_{x_4^0}^{x_4} u_{x_1x_3\alpha_4}(x_1, x_2^0, x_3, \alpha_4)\mathrm{d}\alpha_4$$
$$+ \int_{x_2^0}^{x_2}\int_{x_4^0}^{x_4} u_{x_1\alpha_2x_3\alpha_4}(x_1, \alpha_2, x_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2,$$

$$D_2D_4u(x) = u_{x_2x_4}(x_1^0, x_2, x_3^0, x_4) + \int_{x_3^0}^{x_3} u_{x_2\alpha_3x_4}(x_1^0, x_2, \alpha_3, x_4)\mathrm{d}\alpha_3 + \int_{x_1^0}^{x_1} u_{\alpha_1x_2x_4}(\alpha_1, x_2, x_3^0, x_4)\mathrm{d}\alpha_1$$
$$+ \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3} u_{\alpha_1x_2\alpha_3x_4}(\alpha_1, x_2, \alpha_3, x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_1,$$

$$D_1D_4u(x) = u_{x_1x_4}(x_1, x_2^0, x_3^0, x_4) + \int_{x_2^0}^{x_2} u_{x_1\alpha_2x_4}(x_1, \alpha_2, x_3^0, x_4)\mathrm{d}\alpha_2 + \int_{x_3^0}^{x_3} u_{x_1\alpha_3x_4}(x_1, x_2^0, \alpha_3, x_4)\mathrm{d}\alpha_3$$
$$+ \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{x_1\alpha_2\alpha_3x_4}(x_1, \alpha_2, \alpha_3, x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_2,$$

$$D_1D_2D_3u(x) = u_{x_1x_2x_3}(x_1, x_2, x_3, x_4^0) + \int_{x_4^0}^{x_4} u_{x_1x_2x_3\alpha_4}(x_1, x_2, x_3, \alpha_4)\mathrm{d}\alpha_4,$$
$$D_2D_3D_4u(x) = u_{x_2x_3x_4}(x_1^0, x_2, x_3, x_4) + \int_{x_1^0}^{x_1} u_{\alpha_1x_2x_3x_4}(\alpha_1, x_2, x_3, x_4)\mathrm{d}\alpha_1,$$
$$D_1D_2D_4u(x) = u_{x_1x_2x_4}(x_1, x_2, x_3^0, x_4) + \int_{x_3^0}^{x_3} u_{x_1x_2\alpha_3x_4}(x_1, x_2, \alpha_3, x_4)\mathrm{d}\alpha_3,$$
$$D_1D_3D_4u(x) = u_{x_1x_3x_4}(x_1, x_2^0, x_3, x_4) + \int_{x_2^0}^{x_2} u_{x_1\alpha_2x_3x_4}(x_1, \alpha_2, x_3, x_4)\mathrm{d}\alpha_2.$$

Now, we show that operator V has an adjoint operator $V^\star = (w_{1,1,1,1}, w_{0,0,0,0}, w_{1,0,0,0}, w_{0,1,0,0}, w_{0,0,1,0}, w_{0,0,0,1}, w_{1,1,0,0}, w_{0,1,1,0}, w_{0,0,1,1}, w_{1,0,1,0}, w_{0,1,0,1}, w_{1,0,0,1}, w_{1,1,1,0}, w_{0,1,1,1}, w_{1,1,0,1}, w_{1,0,1,1})$, which boundedly acts on spaces

$$\begin{aligned} E_{q(x)}(G) \equiv &L_{q(x)}(G) \times \mathbb{R} \times L_{s_1(x_1)}(G_1) \times L_{s_2(x_2)}(G_2) \times L_{s_3(x_3)}(G_3) \times L_{s_4(x_4)}(G_4) \times L_{(s_1(x_1),s_2(x_2))}(G_1 \times G_2) \\ &\times L_{(s_2(x_2),s_3(x_3))}(G_2 \times G_3) \times L_{(s_3(x_3),s_4(x_4))}(G_3 \times G_4) \times L_{(s_1(x_1),s_3(x_3))}(G_1 \times G_3) \times L_{(s_2(x_2),s_4(x_4))}(G_2 \times G_4) \\ &\times L_{(s_1(x_1),s_4(x_4))}(G_1 \times G_4) \times L_{(s_1(x_1),s_2(x_2),s_3(x_3))}(G_1 \times G_2 \times G_3) \times L_{(s_2(x_2),s_3(x_3),s_4(x_4))}(G_2 \times G_3 \times G_4) \\ &\times L_{(s_1(x_1),s_2(x_2),s_4(x_4))}(G_1 \times G_2 \times G_4) \times L_{(s_1(x_1),s_3(x_3),s_4(x_4))}(G_1 \times G_3 \times G_4) \end{aligned}$$

and satisfies (4.2) and (4.3). Let $f = (f_{1,1,1,1}(x), f_{0,0,0,0}, f_{1,0,0,0}(x_1), f_{0,1,0,0}(x_2), f_{0,0,1,0}(x_3), f_{0,0,0,1}(x_4), f_{1,1,0,0}(x_1, x_2), f_{0,1,1,0}(x_2, x_3), f_{0,0,1,1}(x_3, x_4), f_{1,0,1,0}(x_1, x_3), f_{0,1,0,1}(x_2, x_4), f_{1,0,0,1}(x_1, x_4), f_{1,1,1,0}(x_1, x_2, x_3), f_{0,1,1,1}(x_2, x_3, x_4), f_{1,1,0,1}(x_1, x_2, x_4), f_{1,0,1,1}(x_1, x_3, x_4)) \in E_{q(x)}(G)$ be any linear bounded functional on $E_{p(x)}(G), u \in SW^{(1,1,1,1)}_{p(x)}(G)$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Then, by the general form of linear functional in $E_{p(x)}(G)$, we have

$$\begin{aligned} f(Vu) = &\int_G f_{1,1,1,1}(x)(V_{1,1,1,1}u)(x)\mathrm{d}x + f_{0,0,0,0}V_{0,0,0,0}u + \int_{G_1} f_{1,0,0,0}(x_1)(V_{1,0,0,0}u)(x_1)\mathrm{d}x_1 \\ &+ \int_{G_2} f_{0,1,0,0}(x_2)(V_{0,1,0,0}u)(x_2)\mathrm{d}x_2 + \int_{G_3} f_{0,0,1,0}(x_3)(V_{0,0,1,0}u)(x_3)\mathrm{d}x_3 + \int_{G_4} f_{0,0,0,1}(x_4)(V_{0,0,0,1}u)(x_4)\mathrm{d}x_4 \\ &+ \int_{G_1}\int_{G_2} f_{1,1,0,0}(x_1, x_2)(V_{1,1,0,0}u)(x_1, x_2)\mathrm{d}x_2\mathrm{d}x_1 + \int_{G_2}\int_{G_3} f_{0,1,1,0}(x_2, x_3)(V_{0,1,1,0}u)(x_2, x_3)\mathrm{d}x_3\mathrm{d}x_2 \\ &+ \int_{G_3}\int_{G_4} f_{0,0,1,1}(x_3, x_4)(V_{0,0,1,1}u)(x_3, x_4)\mathrm{d}x_4\mathrm{d}x_3 + \int_{G_1}\int_{G_3} f_{1,0,1,0}(x_1, x_3)(V_{1,0,1,0}u)(x_1, x_3)\mathrm{d}x_3\mathrm{d}x_1 \\ &+ \int_{G_2}\int_{G_4} f_{0,1,0,1}(x_2, x_4)(V_{0,1,0,1}u)(x_2, x_4)\mathrm{d}x_4\mathrm{d}x_2 + \int_{G_1}\int_{G_4} f_{1,0,0,1}(x_1, x_4)(V_{1,0,0,1}u)(x_1, x_4)\mathrm{d}x_4\mathrm{d}x_1 \\ &+ \int_{G_1}\int_{G_2}\int_{G_3} f_{1,1,1,0}(x_1, x_2, x_3)(V_{1,1,1,0}u)(x_1, x_2, x_3)\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 \\ &+ \int_{G_2}\int_{G_3}\int_{G_4} f_{0,1,1,1}(x_2, x_3, x_4)(V_{0,1,1,1}u)(x_2, x_3, x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_2 \\ &+ \int_{G_1}\int_{G_2}\int_{G_4} f_{1,1,0,1}(x_1, x_2, x_4)(V_{1,1,0,1}u)(x_1, x_2, x_4)\mathrm{d}x_4\mathrm{d}x_2\mathrm{d}x_1 \\ &+ \int_{G_1}\int_{G_3}\int_{G_4} f_{1,0,1,1}(x_1, x_3, x_4)(V_{1,0,1,1}u)(x_1, x_3, x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_1. \end{aligned}$$

By (3.2), we obtain

$$
\begin{aligned}
f(Vu) = &\int_G f_{1,1,1,1}(x)\left[D_1D_2D_3D_4u(x) + \sum_{\substack{i_1,i_2,i_3,i_4=0\\0\le i_1+i_2+i_3+i_4\le 3}}^{1} a_{i_1,i_2,i_3,i_4}(x)D_1^{i_1}D_2^{i_2}D_3^{i_3}D_4^{i_4}u(x)\right]\mathrm{d}x\\
&+ f_{0,0,0,0}u(x_1^0,x_2^0,x_3^0,x_4^0) + \int_{G_1} f_{1,0,0,0}(x_1)D_1u(x_1,x_2^0,x_3^0,x_4^0)\mathrm{d}x_1 + \int_{G_2} f_{0,1,0,0}(x_2)D_2u(x_1^0,x_2,x_3^0,x_4^0)\mathrm{d}x_2\\
&+ \int_{G_3} f_{0,0,1,0}(x_3)D_3u(x_1^0,x_2^0,x_3,x_4^0)\mathrm{d}x_3 + \int_{G_4} f_{0,0,0,1}(x_4)D_4u(x_1^0,x_2^0,x_3^0,x_4)\mathrm{d}x_4\\
&+ \int_{G_1}\int_{G_2} f_{1,1,0,0}(x_1,x_2)D_1D_2u(x_1,x_2,x_3^0,x_4^0)\mathrm{d}x_2\mathrm{d}x_1 + \int_{G_2}\int_{G_3} f_{0,1,1,0}(x_2,x_3)D_2D_3u(x_1^0,x_2,x_3,x_4^0)\mathrm{d}x_3\mathrm{d}x_2\\
&+ \int_{G_3}\int_{G_4} f_{0,0,1,1}(x_3,x_4)D_3D_4u(x_1^0,x_2^0,x_3,x_4)\mathrm{d}x_4\mathrm{d}x_3 + \int_{G_1}\int_{G_3} f_{1,0,1,0}(x_1,x_3)D_1D_3u(x_1,x_2^0,x_3,x_4^0)\mathrm{d}x_3\mathrm{d}x_1\\
&+ \int_{G_2}\int_{G_4} f_{0,1,0,1}(x_2,x_4)D_2D_4u(x_1^0,x_2,x_3^0,x_4)\mathrm{d}x_4\mathrm{d}x_2 + \int_{G_1}\int_{G_4} f_{1,0,0,1}(x_1,x_4)D_1D_4u(x_1,x_2^0,x_3^0,x_4)\mathrm{d}x_4\mathrm{d}x_1\\
&+ \int_{G_1}\int_{G_2}\int_{G_3} f_{1,1,1,0}(x_1,x_2,x_3)D_1D_2D_3u(x_1,x_2,x_3,x_4^0)\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1\\
&+ \int_{G_2}\int_{G_3}\int_{G_4} f_{0,1,1,1}(x_2,x_3,x_4)D_2D_3D_4u(x_1^0,x_2,x_3,x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_2\\
&+ \int_{G_1}\int_{G_2}\int_{G_4} f_{1,1,0,1}(x_1,x_2,x_4)D_1D_2D_4u(x_1,x_2,x_3^0,x_4)\mathrm{d}x_4\mathrm{d}x_2\mathrm{d}x_1\\
&+ \int_{G_1}\int_{G_3}\int_{G_4} f_{1,0,1,1}(x_1,x_3,x_4)D_1D_3D_4u(x_1,x_2^0,x_3,x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_1.
\end{aligned}
\tag{4.5}
$$

By substituting forms of the weak derivatives and (4.4) into (4.5), we obtain

$$
\begin{aligned}
f(Vu) = \int_G f_{1,1,1,1}(x)\Big\{&D_1D_2D_3D_4u(x) + a_{1,1,1,0}(x)\left[u_{x_1x_2x_3}(x_1,x_2,x_3,x_4^0) + \int_{x_4^0}^{x_4} u_{x_1x_2x_3\alpha_4}(x_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\right]\\
&+ a_{0,1,1,1}(x)\left[u_{x_2x_3x_4}(x_1^0,x_2,x_3,x_4) + \int_{x_1^0}^{x_1} u_{\alpha_1x_2x_3x_4}(\alpha_1,x_2,x_3,x_4)\mathrm{d}\alpha_1\right]\\
&+ a_{1,1,0,1}(x)\left[u_{x_1x_2x_4}(x_1,x_2,x_3^0,x_4) + \int_{x_3^0}^{x_3} u_{x_1x_2\alpha_3x_4}(x_1,x_2,\alpha_3,x_4)\mathrm{d}\alpha_3\right]\\
&+ a_{1,0,1,1}(x)\left[u_{x_1x_3x_4}(x_1,x_2^0,x_3,x_4) + \int_{x_2^0}^{x_2} u_{x_1\alpha_2x_3x_4}(x_1,\alpha_2,x_3,x_4)\mathrm{d}\alpha_2\right]\\
&+ a_{1,1,0,0}(x)\Big[u_{x_1x_2}(x_1,x_2,x_3^0,x_4^0) + \int_{x_3^0}^{x_3} u_{x_1x_2\alpha_3}(x_1,x_2,\alpha_3,x_4^0)\mathrm{d}\alpha_3 + \int_{x_4^0}^{x_4} u_{x_1x_2\alpha_4}(x_1,x_2,x_3^0,\alpha_4)\mathrm{d}\alpha_4\\
&+ \int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{x_1x_2\alpha_3\alpha_4}(x_1,x_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\Big]\\
&+ a_{0,1,1,0}(x)\Big[u_{x_2x_3}(x_1^0,x_2,x_3,x_4^0) + \int_{x_1^0}^{x_1} u_{\alpha_1x_2x_3}(\alpha_1,x_2,x_3,x_4^0)\mathrm{d}\alpha_1 + \int_{x_4^0}^{x_4} u_{x_2x_3\alpha_4}(x_1^0,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\\
&+ \int_{x_1^0}^{x_1}\int_{x_4^0}^{x_4} u_{\alpha_1x_2x_3\alpha_4}(\alpha_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_1\Big]
\end{aligned}
$$

$$+ a_{0,0,1,1}(x)\Bigg[u_{x_3x_4}(x_1^0,x_2^0,x_3,x_4) + \int_{x_2^0}^{x_2} u_{\alpha_2x_3x_4}(x_1^0,\alpha_2,x_3,x_4)\mathrm{d}\alpha_2 + \int_{x_1^0}^{x_1} u_{\alpha_1x_3x_4}(\alpha_1,x_2^0,x_3,x_4)\mathrm{d}\alpha_1$$
$$+ \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2x_3x_4}(\alpha_1,\alpha_2,x_3,x_4)\mathrm{d}\alpha_2\mathrm{d}\alpha_1\Bigg]$$
$$+ a_{1,0,1,0}(x)\Bigg[u_{x_1x_3}(x_1,x_2^0,x_3,x_4^0) + \int_{x_2^0}^{x_2} u_{x_1\alpha_2x_3}(x_1,\alpha_2,x_3,x_4^0)\mathrm{d}\alpha_2 + \int_{x_4^0}^{x_4} u_{x_1x_3\alpha_4}(x_1,x_2^0,x_3,\alpha_4)\mathrm{d}\alpha_4$$
$$+ \int_{x_2^0}^{x_2}\int_{x_4^0}^{x_4} u_{x_1\alpha_2x_3\alpha_4}(x_1,\alpha_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2\Bigg]$$
$$+ a_{0,1,0,1}(x)\Bigg[u_{x_2x_4}(x_1^0,x_2,x_3^0,x_4) + \int_{x_3^0}^{x_3} u_{x_2\alpha_3x_4}(x_1^0,x_2,\alpha_3,x_4)\mathrm{d}\alpha_3 + \int_{x_1^0}^{x_1} u_{\alpha_1x_2x_4}(\alpha_1,x_2,x_3^0,x_4)\mathrm{d}\alpha_1$$
$$+ \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3} u_{\alpha_1x_2\alpha_3x_4}(\alpha_1,x_2,\alpha_3,x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_1\Bigg]$$
$$+ a_{1,0,0,1}(x)\Bigg[u_{x_1x_4}(x_1,x_2^0,x_3^0,x_4) + \int_{x_2^0}^{x_2} u_{x_1\alpha_2x_4}(x_1,\alpha_2,x_3^0,x_4)\mathrm{d}\alpha_2 + \int_{x_3^0}^{x_3} u_{x_1\alpha_3x_4}(x_1,x_2^0,\alpha_3,x_4)\mathrm{d}\alpha_3$$
$$+ \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{x_1\alpha_2\alpha_3x_4}(x_1,\alpha_2,\alpha_3,x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_2\Bigg]$$
$$+ a_{1,0,0,0}(x)\Bigg[u_{x_1}(x_1,x_2^0,x_3^0,x_4^0) + \int_{x_2^0}^{x_2} u_{x_1\alpha_2}(x_1,\alpha_2,x_3^0,x_4^0)\mathrm{d}\alpha_2 + \int_{x_3^0}^{x_3} u_{x_1\alpha_3}(x_1,x_2^0,\alpha_3,x_4^0)\mathrm{d}\alpha_3$$
$$+ \int_{x_4^0}^{x_4} u_{x_1\alpha_4}(x_1,x_2^0,x_3^0,\alpha_4)\mathrm{d}\alpha_4 + \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{x_1\alpha_2\alpha_3}(x_1,\alpha_2,\alpha_3,x_4^0)\mathrm{d}\alpha_3\mathrm{d}\alpha_2$$
$$+ \int_{x_2^0}^{x_2}\int_{x_4^0}^{x_4} u_{x_1\alpha_2\alpha_4}(x_1,\alpha_2,x_3^0,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2 + \int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{x_1\alpha_3\alpha_4}(x_1,x_2^0,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3$$
$$+ \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{x_1\alpha_2\alpha_3\alpha_4}(x_1,\alpha_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_2\Bigg]$$
$$+ a_{0,1,0,0}(x)\Bigg[u_{x_2}(x_1^0,x_2,x_3^0,x_4^0) + \int_{x_1^0}^{x_1} u_{\alpha_1x_2}(\alpha_1,x_2,x_3^0,x_4^0)\mathrm{d}\alpha_1 + \int_{x_3^0}^{x_3} u_{x_2\alpha_3}(x_1^0,x_2,\alpha_3,x_4^0)\mathrm{d}\alpha_3$$
$$+ \int_{x_4^0}^{x_4} u_{x_2\alpha_4}(x_1^0,x_2,x_3^0,\alpha_4)\mathrm{d}\alpha_4 + \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3} u_{\alpha_1x_2\alpha_3}(\alpha_1,x_2,\alpha_3,x_4^0)\mathrm{d}\alpha_3\mathrm{d}\alpha_1$$
$$+ \int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{x_2\alpha_3\alpha_4}(x_1^0,x_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3 + \int_{x_1^0}^{x_1}\int_{x_4^0}^{x_4} u_{\alpha_1x_2\alpha_4}(\alpha_1,x_2,x_3^0,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_1$$
$$+ \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{\alpha_1x_2\alpha_3\alpha_4}(\alpha_1,x_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_1\Bigg]$$
$$+ a_{0,0,1,0}(x)\Bigg[u_{x_3}(x_1^0,x_2^0,x_3,x_4^0) + \int_{x_2^0}^{x_2} u_{\alpha_2x_3}(x_1^0,\alpha_2,x_3,x_4^0)\mathrm{d}\alpha_2 + \int_{x_4^0}^{x_4} u_{x_3\alpha_4}(x_1^0,x_2^0,x_3,\alpha_4)\mathrm{d}\alpha_4$$
$$+ \int_{x_1^0}^{x_1} u_{\alpha_1x_3}(\alpha_1,x_2^0,x_3,x_4^0)\mathrm{d}\alpha_1 + \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2x_3}(\alpha_1,\alpha_2,x_3,x_4^0)\mathrm{d}\alpha_2\mathrm{d}\alpha_1$$
$$+ \int_{x_2^0}^{x_2}\int_{x_4^0}^{x_4} u_{\alpha_2x_3\alpha_4}(x_1^0,\alpha_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2 + \int_{x_1^0}^{x_1}\int_{x_4^0}^{x_4} u_{\alpha_1x_3\alpha_4}(\alpha_1,x_2^0,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_1$$
$$+ \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_4^0}^{x_4} u_{\alpha_1\alpha_2x_3\alpha_4}(\alpha_1,\alpha_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2\mathrm{d}\alpha_1\Bigg]$$
$$+ a_{0,0,0,1}(x)\Bigg[u_{x_4}(x_1^0,x_2^0,x_3^0,x_4) + \int_{x_3^0}^{x_3} u_{\alpha_3x_4}(x_1^0,x_2^0,\alpha_3,x_4)\mathrm{d}\alpha_3 + \int_{x_2^0}^{x_2} u_{\alpha_2x_4}(x_1^0,\alpha_2,x_3^0,x_4)\mathrm{d}\alpha_2$$

$$+\int_{x_1^0}^{x_1} u_{\alpha_1 x_4}(\alpha_1, x_2^0, x_3^0, x_4)\mathrm{d}\alpha_1 + \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{\alpha_2\alpha_3 x_4}(x_1^0, \alpha_2, \alpha_3, x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_2$$

$$+\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2 x_4}(\alpha_1, \alpha_2, x_3^0, x_4)\mathrm{d}\alpha_2\mathrm{d}\alpha_1 + \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3} u_{\alpha_1\alpha_3 x_4}(\alpha_1, x_2^0, \alpha_3, x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_1$$

$$\left.+\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{\alpha_1\alpha_2\alpha_3 x_4}(\alpha_1, \alpha_2, \alpha_3, x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1\right]$$

$$+ a_{0,0,0,0}(x)\left[u(x_1^0, x_2^0, x_3^0, x_4^0) + \int_{x_1^0}^{x_1} u_{\alpha_1}(\alpha_1, x_2^0, x_3^0, x_4^0)\mathrm{d}\alpha_1 + \int_{x_2^0}^{x_2} u_{\alpha_2}(x_1^0, \alpha_2, x_3^0, x_4^0)\mathrm{d}\alpha_2\right.$$

$$+\int_{x_3^0}^{x_3} u_{\alpha_3}(x_1^0, x_2^0, \alpha_3, x_4^0)\mathrm{d}\alpha_3 + \int_{x_4^0}^{x_4} u_{\alpha_4}(x_1^0, x_2^0, x_3^0, \alpha_4)\mathrm{d}\alpha_4$$

$$+\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2}(\alpha_1, \alpha_2, x_3^0, x_4^0)\mathrm{d}\alpha_2\mathrm{d}\alpha_1 + \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{\alpha_2\alpha_3}(x_1^0, \alpha_2, \alpha_3, x_4^0)\mathrm{d}\alpha_3\mathrm{d}\alpha_2$$

$$+\int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{\alpha_3\alpha_4}(x_1^0, x_2^0, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3 + \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3} u_{\alpha_1\alpha_3}(\alpha_1, x_2^0, \alpha_3, x_4^0)\mathrm{d}\alpha_3\mathrm{d}\alpha_1$$

$$+\int_{x_2^0}^{x_2}\int_{x_4^0}^{x_4} u_{\alpha_2\alpha_4}(x_1^0, \alpha_2, x_3^0, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2 + \int_{x_1^0}^{x_1}\int_{x_4^0}^{x_4} u_{\alpha_1\alpha_4}(\alpha_1, x_2^0, x_3^0, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_1$$

$$+\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{\alpha_1\alpha_2\alpha_3}(\alpha_1, \alpha_2, \alpha_3, x_4^0)\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1 + \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{\alpha_2\alpha_3\alpha_4}(x_1^0, \alpha_2, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_2$$

$$+\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_4^0}^{x_4} u_{\alpha_1\alpha_2\alpha_4}(\alpha_1, \alpha_2, x_3^0, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2\mathrm{d}\alpha_1 + \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{\alpha_1\alpha_3\alpha_4}(\alpha_1, x_2^0, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_1$$

$$\left.\left.+\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\int_{x_4^0}^{x_4} u_{\alpha_1\alpha_2\alpha_3\alpha_4}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1\right]\right\}\mathrm{d}x$$

$$+ f_{0,0,0,0}u(x_1^0, x_2^0, x_3^0, x_4^0) + \int_{G_1} f_{1,0,0,0}(x_1)D_1u(x_1, x_2^0, x_3^0, x_4^0)\mathrm{d}x_1 + \int_{G_2} f_{0,1,0,0}(x_2)D_2u(x_1^0, x_2, x_3^0, x_4^0)\mathrm{d}x_2$$

$$+\int_{G_3} f_{0,0,1,0}(x_3)D_3u(x_1^0, x_2^0, x_3, x_4^0)\mathrm{d}x_3 + \int_{G_4} f_{0,0,0,1}(x_4)D_4u(x_1^0, x_2^0, x_3^0, x_4)\mathrm{d}x_4$$

$$+\int_{G_1}\int_{G_2} f_{1,1,0,0}(x_1, x_2)D_1D_2u(x_1, x_2, x_3^0, x_4^0)\mathrm{d}x_2\mathrm{d}x_1 + \int_{G_2}\int_{G_3} f_{0,1,1,0}(x_2, x_3)D_2D_3u(x_1^0, x_2, x_3, x_4^0)\mathrm{d}x_3\mathrm{d}x_2$$

$$+\int_{G_3}\int_{G_4} f_{0,0,1,1}(x_3, x_4)D_3D_4u(x_1^0, x_2^0, x_3, x_4)\mathrm{d}x_4\mathrm{d}x_3 + \int_{G_1}\int_{G_3} f_{1,0,1,0}(x_1, x_3)D_1D_3u(x_1, x_2^0, x_3, x_4^0)\mathrm{d}x_3\mathrm{d}x_1$$

$$+\int_{G_2}\int_{G_4} f_{0,1,0,1}(x_2, x_4)D_2D_4u(x_1^0, x_2, x_3^0, x_4)\mathrm{d}x_4\mathrm{d}x_2 + \int_{G_1}\int_{G_4} f_{1,0,0,1}(x_1, x_4)D_1D_4u(x_1, x_2^0, x_3^0, x_4)\mathrm{d}x_4\mathrm{d}x_1$$

$$+\int_{G_1}\int_{G_2}\int_{G_3} f_{1,1,1,0}(x_1, x_2, x_3)D_1D_2D_3u(x_1, x_2, x_3, x_4^0)\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1$$

$$+\int_{G_2}\int_{G_3}\int_{G_4} f_{0,1,1,1}(x_2, x_3, x_4)D_2D_3D_4u(x_1^0, x_2, x_3, x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_2$$

$$+\int_{G_1}\int_{G_2}\int_{G_4} f_{1,1,0,1}(x_1, x_2, x_4)D_1D_2D_4u(x_1, x_2, x_3^0, x_4)\mathrm{d}x_4\mathrm{d}x_2\mathrm{d}x_1$$

$$+\int_{G_1}\int_{G_3}\int_{G_4} f_{1,0,1,1}(x_1, x_3, x_4)D_1D_3D_4u(x_1, x_2^0, x_3, x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_1.$$

We denote

$$w_{0,0,0,0}f \equiv \int_G f_{1,1,1,1}(x)a_{0,0,0,0}(x)\mathrm{d}x + f_{0,0,0,0},$$

$$
\begin{aligned}
(w_{1,0,0,0}f)(x_1) \equiv & \int_{x_1}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3^0}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(\alpha_1,x_2,x_3,x_4)a_{0,0,0,0}(\alpha_1,x_2,x_3,x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}\alpha_1 \\
& + \int_{x_2^0}^{h_2}\int_{x_3^0}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x)a_{1,0,0,0}(x)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_2 + f_{1,0,0,0}(x_1), \\
(w_{0,1,0,0}f)(x_2) \equiv & \int_{x_1^0}^{h_1}\int_{x_2}^{h_2}\int_{x_3^0}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x_1,\alpha_2,x_3,x_4)a_{0,0,0,0}(x_1,\alpha_2,x_3,x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}\alpha_2\mathrm{d}x_1 \\
& + \int_{x_1^0}^{h_1}\int_{x_3^0}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x)a_{0,1,0,0}(x)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_1 + f_{0,1,0,0}(x_2), \\
(w_{0,0,1,0}f)(x_3) \equiv & \int_{x_1^0}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x_1,x_2,\alpha_3,x_4)a_{0,0,0,0}(x_1,x_2,\alpha_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_3\mathrm{d}x_2\mathrm{d}x_1 \\
& + \int_{x_1^0}^{h_1}\int_{x_2^0}^{h_2}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x)a_{0,0,1,0}(x)\mathrm{d}x_4\mathrm{d}x_2\mathrm{d}x_1 + f_{0,0,1,0}(x_3), \\
(w_{0,0,0,1}f)(x_4) \equiv & \int_{x_1^0}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3^0}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,x_2,x_3,\alpha_4)a_{0,0,0,0}(x_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 \\
& + \int_{x_1^0}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3^0}^{h_3} f_{1,1,1,1}(x)a_{0,0,0,1}(x)\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 + f_{0,0,0,1}(x_4), \\
(w_{1,1,0,0}f)(x_1,x_2) \equiv & \int_{x_1}^{h_1}\int_{x_2}^{h_2}\int_{x_3^0}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(\alpha_1,\alpha_2,x_3,x_4)a_{0,0,0,0}(\alpha_1,\alpha_2,x_3,x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1 \\
& + \int_{x_2}^{h_2}\int_{x_3^0}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x_1,\alpha_2,x_3,x_4)a_{1,0,0,0}(x_1,\alpha_2,x_3,x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}\alpha_2 \\
& + \int_{x_1}^{h_1}\int_{x_3^0}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(\alpha_1,x_2,x_3,x_4)a_{0,1,0,0}(\alpha_1,x_2,x_3,x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}\alpha_1 \\
& + \int_{x_3^0}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x)a_{1,1,0,0}(x)\mathrm{d}x_4\mathrm{d}x_3 + f_{1,1,0,0}(x_1,x_2), \\
(w_{0,1,1,0}f)(x_2,x_3) \equiv & \int_{x_1^0}^{h_1}\int_{x_2}^{h_2}\int_{x_3}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x_1,\alpha_2,\alpha_3,x_4)a_{0,0,0,0}(x_1,\alpha_2,\alpha_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}x_1 \\
& + \int_{x_1^0}^{h_1}\int_{x_3}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x_1,x_2,\alpha_3,x_4)a_{0,1,0,0}(x_1,x_2,\alpha_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_3\mathrm{d}x_1 \\
& + \int_{x_1^0}^{h_1}\int_{x_2}^{h_2}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x_1,\alpha_2,x_3,x_4)a_{0,0,1,0}(x_1,\alpha_2,x_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_2\mathrm{d}x_1 \\
& + \int_{x_1^0}^{h_1}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x)a_{0,1,1,0}(x)\mathrm{d}x_4\mathrm{d}x_1 + f_{0,1,0,0}(x_2,x_3), \\
(w_{0,0,1,1}f)(x_3,x_4) \equiv & \int_{x_1^0}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,x_2,\alpha_3,\alpha_4)a_{0,0,0,0}(x_1,x_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2\mathrm{d}x_2\mathrm{d}x_1 \\
& + \int_{x_1^0}^{h_1}\int_{x_2^0}^{h_2}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,x_2,x_3,\alpha_4)a_{0,0,1,0}(x_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_2\mathrm{d}x_1 \\
& + \int_{x_1^0}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3}^{h_3} f_{1,1,1,1}(x_1,x_2,\alpha_3,x_4)a_{0,0,0,1}(x_1,x_2,\alpha_3,x_4)\mathrm{d}\alpha_3\mathrm{d}x_2\mathrm{d}x_1 \\
& + \int_{x_1^0}^{h_1}\int_{x_2^0}^{h_2} f_{1,1,1,1}(x)a_{0,0,1,1}(x)\mathrm{d}x_2\mathrm{d}x_1 + f_{0,0,1,1}(x_3,x_4), \\
(w_{1,0,1,0}f)(x_1,x_3) \equiv & \int_{x_1}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(\alpha_1,x_2,\alpha_3,x_4)a_{0,0,0,0}(\alpha_1,x_2,\alpha_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_3\mathrm{d}x_2\mathrm{d}\alpha_1 \\
& + \int_{x_2^0}^{h_2}\int_{x_3}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x_1,x_2,\alpha_3,x_4)a_{1,0,0,0}(x_1,x_2,\alpha_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_3\mathrm{d}x_2
\end{aligned}
$$

$$+\int_{x_1}^{h_1}\int_{x_2^0}^{h_2}\int_{x_4^0}^{h_4} f_{1,1,1,1}(\alpha_1,x_2,x_3,x_4)a_{0,0,1,0}(\alpha_1,x_2,x_3,x_4)\mathrm{d}x_4\mathrm{d}x_2\mathrm{d}\alpha_1$$
$$+\int_{x_2^0}^{h_2}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x)a_{1,0,1,0}(x)\mathrm{d}x_4\mathrm{d}x_2 + f_{1,0,1,0}(x_1,x_3),$$

$$(w_{0,1,0,1}f)(x_2,x_4) \equiv \int_{x_1^0}^{h_1}\int_{x_2}^{h_2}\int_{x_3^0}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,\alpha_2,x_3,\alpha_4)a_{0,0,0,0}(x_1,\alpha_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_3\mathrm{d}\alpha_2\mathrm{d}x_1$$
$$+\int_{x_1^0}^{h_1}\int_{x_3^0}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,x_2,x_3,\alpha_4)a_{0,1,0,0}(x_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_3\mathrm{d}x_1$$
$$+\int_{x_1^0}^{h_1}\int_{x_2}^{h_2}\int_{x_3^0}^{h_3} f_{1,1,1,1}(x_1,\alpha_2,x_3,x_4)a_{0,0,0,1}(x_1,\alpha_2,x_3,x_4)\mathrm{d}x_3\mathrm{d}\alpha_2\mathrm{d}x_1$$
$$+\int_{x_1^0}^{h_1}\int_{x_3^0}^{h_3} f_{1,1,1,1}(x)a_{0,1,0,1}(x)\mathrm{d}x_3\mathrm{d}x_1 + f_{0,1,0,1}(x_2,x_4),$$

$$(w_{1,0,0,1}f)(x_1,x_4) \equiv \int_{x_1}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3^0}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(\alpha_1,x_2,x_3,\alpha_4)a_{0,0,0,0}(\alpha_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}\alpha_1$$
$$+\int_{x_2^0}^{h_2}\int_{x_3^0}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,x_2,x_3,\alpha_4)a_{1,0,0,0}(x_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_3\mathrm{d}x_2$$
$$+\int_{x_1}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3^0}^{h_3} f_{1,1,1,1}(\alpha_1,x_2,x_3,x_4)a_{0,0,0,1}(\alpha_1,x_2,x_3,x_4)\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}\alpha_1$$
$$+\int_{x_2^0}^{h_2}\int_{x_3^0}^{h_3} f_{1,1,1,1}(x)a_{1,0,0,1}(x)\mathrm{d}x_3\mathrm{d}x_2 + f_{1,0,0,1}(x_1,x_4),$$

$$(w_{1,1,1,0}f)(x_1,x_2,x_3) \equiv \int_{x_1}^{h_1}\int_{x_2}^{h_2}\int_{x_3}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(\alpha_1,\alpha_2,\alpha_3,x_4)a_{0,0,0,0}(\alpha_1,\alpha_2,\alpha_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1$$
$$+\int_{x_1}^{h_1}\int_{x_2}^{h_2}\int_{x_4^0}^{h_4} f_{1,1,1,1}(\alpha_1,\alpha_2,x_3,x_4)a_{0,0,1,0}(\alpha_1,\alpha_2,x_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_2\mathrm{d}\alpha_1$$
$$+\int_{x_1}^{h_1}\int_{x_3}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(\alpha_1,x_2,\alpha_3,x_4)a_{0,1,0,0}(\alpha_1,x_2,\alpha_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_3\mathrm{d}\alpha_1$$
$$+\int_{x_2}^{h_2}\int_{x_3}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x_1,\alpha_2,\alpha_3,x_4)a_{1,0,0,0}(x_1,\alpha_2,\alpha_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_3\mathrm{d}\alpha_2$$
$$+\int_{x_2}^{h_2}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x_1,\alpha_2,x_3,x_4)a_{1,0,1,0}(x_1,\alpha_2,x_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_2$$
$$+\int_{x_1}^{h_1}\int_{x_4^0}^{h_4} f_{1,1,1,1}(\alpha_1,x_2,x_3,x_4)a_{0,1,1,0}(\alpha_1,x_2,x_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_1$$
$$+\int_{x_3}^{h_3}\int_{x_4^0}^{h_4} f_{1,1,1,1}(x_1,x_2,\alpha_3,x_4)a_{1,1,0,0}(x_1,x_2,\alpha_3,x_4)\mathrm{d}x_4\mathrm{d}\alpha_3$$
$$+\int_{x_4^0}^{h_4} f_{1,1,1,1}(x)a_{1,1,1,0}(x)\mathrm{d}x_4 + f_{1,1,1,0}(x_1,x_2,x_3),$$

$$(w_{0,1,1,1}f)(x_2,x_3,x_4) \equiv \int_{x_1^0}^{h_1}\int_{x_2}^{h_2}\int_{x_3}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,\alpha_2,\alpha_3,\alpha_4)a_{0,0,0,0}(x_1,\alpha_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}x_1$$
$$+\int_{x_1^0}^{h_1}\int_{x_2}^{h_2}\int_{x_3}^{h_3} f_{1,1,1,1}(x_1,\alpha_2,\alpha_3,x_4)a_{0,0,0,1}(x_1,\alpha_2,\alpha_3,x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}x_1$$

$$+\int_{x_1^0}^{h_1}\int_{x_2}^{h_2}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,\alpha_2,x_3,\alpha_4)a_{0,0,1,0}(x_1,\alpha_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2\mathrm{d}x_1$$
$$+\int_{x_1^0}^{h_1}\int_{x_3}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,x_2,\alpha_3,\alpha_4)a_{0,1,0,0}(x_1,x_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}x_1$$
$$+\int_{x_1^0}^{h_1}\int_{x_3}^{h_3} f_{1,1,1,1}(x_1,x_2,\alpha_3,x_4)a_{0,1,0,1}(x_1,x_2,\alpha_3,x_4)\mathrm{d}\alpha_3\mathrm{d}x_1$$
$$+\int_{x_1^0}^{h_1}\int_{x_2}^{h_2} f_{1,1,1,1}(x_1,\alpha_2,x_3,x_4)a_{0,0,1,1}(x_1,\alpha_2,x_3,x_4)\mathrm{d}\alpha_2\mathrm{d}x_1$$
$$+\int_{x_1^0}^{h_1}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,x_2,x_3,\alpha_4)a_{0,1,1,0}(x_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_1$$
$$+\int_{x_1^0}^{h_1} f_{1,1,1,1}(x)a_{0,1,1,1}(x)\mathrm{d}x_1 + f_{0,1,1,1}(x_2,x_3,x_4),$$
$$(w_{1,1,0,1}f)(x_1,x_2,x_4) \equiv \int_{x_1}^{h_1}\int_{x_2}^{h_2}\int_{x_3^0}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(\alpha_1,\alpha_2,x_3,\alpha_4)a_{0,0,0,0}(\alpha_1,\alpha_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1$$
$$+\int_{x_1}^{h_1}\int_{x_2}^{h_2}\int_{x_3^0}^{h_3} f_{1,1,1,1}(\alpha_1,\alpha_2,x_3,x_4)a_{0,0,0,1}(\alpha_1,\alpha_2,x_3,x_4)\mathrm{d}x_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1$$
$$+\int_{x_1}^{h_1}\int_{x_3^0}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(\alpha_1,x_2,x_3,\alpha_4)a_{0,1,0,0}(\alpha_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_3\mathrm{d}\alpha_1$$
$$+\int_{x_2}^{h_2}\int_{x_3^0}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,\alpha_2,x_3,\alpha_4)a_{1,0,0,0}(x_1,\alpha_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_3\mathrm{d}\alpha_2$$
$$+\int_{x_2}^{h_2}\int_{x_3^0}^{h_3} f_{1,1,1,1}(x_1,\alpha_2,x_3,x_4)a_{1,0,0,1}(x_1,\alpha_2,x_3,x_4)\mathrm{d}x_3\mathrm{d}\alpha_2$$
$$+\int_{x_1}^{h_1}\int_{x_3^0}^{h_3} f_{1,1,1,1}(\alpha_1,x_2,x_3,x_4)a_{0,1,0,1}(\alpha_1,x_2,x_3,x_4)\mathrm{d}x_3\mathrm{d}\alpha_1$$
$$+\int_{x_3^0}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,x_2,x_3,\alpha_4)a_{1,1,0,0}(x_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_3$$
$$+\int_{x_3^0}^{h_3} f_{1,1,1,1}(x)a_{1,1,0,1}(x)\mathrm{d}x_3 + f_{1,1,0,1}(x_1,x_2,x_4),$$
$$(w_{1,0,1,1}f)(x_1,x_3,x_4) \equiv \int_{x_1}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(\alpha_1,x_2,\alpha_3,\alpha_4)a_{0,0,0,0}(\alpha_1,x_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}x_2\mathrm{d}\alpha_1$$
$$+\int_{x_1}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3}^{h_3} f_{1,1,1,1}(\alpha_1,x_2,\alpha_3,x_4)a_{0,0,0,1}(\alpha_1,x_2,\alpha_3,x_4)\mathrm{d}\alpha_3\mathrm{d}x_2\mathrm{d}\alpha_1$$
$$+\int_{x_1}^{h_1}\int_{x_2^0}^{h_2}\int_{x_4}^{h_4} f_{1,1,1,1}(\alpha_1,x_2,x_3,\alpha_4)a_{0,0,1,0}(\alpha_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_2\mathrm{d}\alpha_1$$
$$+\int_{x_2^0}^{h_2}\int_{x_3}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,x_2,\alpha_3,\alpha_4)a_{1,0,0,0}(x_1,x_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}x_2$$

$$+\int_{x_2^0}^{h_2}\int_{x_3}^{h_3} f_{1,1,1,1}(x_1,x_2,\alpha_3,x_4)a_{1,0,0,1}(x_1,x_2,\alpha_3,x_4)\mathrm{d}\alpha_3\mathrm{d}x_2$$
$$+\int_{x_1}^{h_1}\int_{x_2^0}^{h_2} f_{1,1,1,1}(\alpha_1,x_2,x_3,x_4)a_{0,0,1,1}(\alpha_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}\alpha_1$$
$$+\int_{x_2^0}^{h_2}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,x_2,x_3,\alpha_4)a_{1,0,1,0}(x_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}x_2$$
$$+\int_{x_2^0}^{h_2} f_{1,1,1,1}(x)a_{1,0,1,1}(x)\mathrm{d}x_2+f_{1,0,1,1}(x_1,x_3,x_4)$$

and

$$(w_{1,1,1,1}f)(x)\equiv\int_{x_1}^{h_1}\int_{x_2}^{h_2}\int_{x_3}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)a_{0,0,0,0}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1$$
$$+\int_{x_1}^{h_1}\int_{x_2}^{h_2}\int_{x_3}^{h_3} f_{1,1,1,1}(\alpha_1,\alpha_2,\alpha_3,x_4)a_{0,0,0,1}(\alpha_1,\alpha_2,\alpha_3,x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1$$
$$+\int_{x_1}^{h_1}\int_{x_2}^{h_2}\int_{x_4}^{h_4} f_{1,1,1,1}(\alpha_1,\alpha_2,x_3,\alpha_4)a_{0,0,1,0}(\alpha_1,\alpha_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2\mathrm{d}\alpha_1$$
$$+\int_{x_1}^{h_1}\int_{x_3}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(\alpha_1,x_2,\alpha_3,\alpha_4)a_{0,1,0,0}(\alpha_1,x_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_1$$
$$+\int_{x_2}^{h_2}\int_{x_3}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,\alpha_2,\alpha_3,\alpha_4)a_{1,0,0,0}(x_1,\alpha_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3\mathrm{d}\alpha_2$$
$$+\int_{x_2}^{h_2}\int_{x_3}^{h_3} f_{1,1,1,1}(x_1,\alpha_2,\alpha_3,x_4)a_{1,0,0,1}(x_1,\alpha_2,\alpha_3,x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_2$$
$$+\int_{x_1}^{h_1}\int_{x_3}^{h_3} f_{1,1,1,1}(\alpha_1,x_2,\alpha_3,x_4)a_{0,1,0,1}(\alpha_1,x_2,\alpha_3,x_4)\mathrm{d}\alpha_3\mathrm{d}\alpha_1$$
$$+\int_{x_2}^{h_2}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,\alpha_2,x_3,\alpha_4)a_{1,0,1,0}(x_1,\alpha_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_2$$
$$+\int_{x_1}^{h_1}\int_{x_2}^{h_2} f_{1,1,1,1}(\alpha_1,\alpha_2,x_3,x_4)a_{0,0,1,1}(\alpha_1,\alpha_2,x_3,x_4)\mathrm{d}\alpha_2\mathrm{d}\alpha_1$$
$$+\int_{x_1}^{h_1}\int_{x_4}^{h_4} f_{1,1,1,1}(\alpha_1,x_2,x_3,\alpha_4)a_{0,1,1,0}(\alpha_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_1$$
$$+\int_{x_3}^{h_3}\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,x_2,\alpha_3,\alpha_4)a_{1,1,0,0}(x_1,x_2,\alpha_3,\alpha_4)\mathrm{d}\alpha_4\mathrm{d}\alpha_3$$
$$+\int_{x_2}^{h_2} f_{1,1,1,1}(x_1,\alpha_2,x_3,x_4)a_{1,0,1,1}(x_1,\alpha_2,x_3,x_4)\mathrm{d}\alpha_2$$
$$+\int_{x_3}^{h_3} f_{1,1,1,1}(x_1,x_2,\alpha_3,x_4)a_{1,1,0,1}(x_1,x_2,\alpha_3,x_4)\mathrm{d}\alpha_3$$
$$+\int_{x_1}^{h_1} f_{1,1,1,1}(\alpha_1,x_2,x_3,x_4)a_{0,1,1,1}(\alpha_1,x_2,x_3,x_4)\mathrm{d}\alpha_1$$
$$+\int_{x_4}^{h_4} f_{1,1,1,1}(x_1,x_2,x_3,\alpha_4)a_{1,1,1,0}(x_1,x_2,x_3,\alpha_4)\mathrm{d}\alpha_4+f_{1,1,1,1}(x).$$

Consequently, we have

$$
\begin{aligned}
f(Vu) =& u(x_1^0, x_2^0, x_3^0, x_4^0) w_{0,0,0,0} f + \int_{x_1^0}^{h_1} (w_{1,0,0,0} f)(x_1) D_1 u(x_1, x_2^0, x_3^0, x_4^0) \mathrm{d}x_1 \\
&+ \int_{x_2^0}^{h_2} (w_{0,1,0,0} f)(x_2) D_2 u(x_1^0, x_2, x_3^0, x_4^0) \mathrm{d}x_2 + \int_{x_3^0}^{h_3} (w_{0,0,1,0} f)(x_3) D_3 u(x_1^0, x_2^0, x_3, x_4^0) \mathrm{d}x_3 \\
&+ \int_{x_4^0}^{h_4} (w_{0,0,0,1} f)(x_4) D_4 u(x_1^0, x_2^0, x_3^0, x_4) \mathrm{d}x_4 + \int_{x_1^0}^{h_1} \int_{x_2^0}^{h_2} (w_{1,1,0,0} f)(x_1, x_2) D_1 D_2 u(x_1, x_2, x_3^0, x_4^0) \mathrm{d}x_2 \mathrm{d}x_1 \\
&+ \int_{x_2^0}^{h_2} \int_{x_3^0}^{h_3} (w_{0,1,1,0} f)(x_2, x_3) D_2 D_3 u(x_1^0, x_2, x_3, x_4^0) \mathrm{d}x_3 \mathrm{d}x_2 \\
&+ \int_{x_3^0}^{h_3} \int_{x_4^0}^{h_4} (w_{0,0,1,1} f)(x_3, x_4) D_3 D_4 u(x_1^0, x_2^0, x_3, x_4) \mathrm{d}x_4 \mathrm{d}x_3 \\
&+ \int_{x_1^0}^{h_1} \int_{x_3^0}^{h_3} (w_{1,0,1,0} f)(x_1, x_3) D_1 D_3 u(x_1, x_2^0, x_3, x_4^0) \mathrm{d}x_3 \mathrm{d}x_1 \\
&+ \int_{x_2^0}^{h_2} \int_{x_4^0}^{h_4} (w_{0,1,0,1} f)(x_2, x_4) D_2 D_4 u(x_1^0, x_2, x_3^0, x_4) \mathrm{d}x_4 \mathrm{d}x_2 \\
&+ \int_{x_1^0}^{h_1} \int_{x_4^0}^{h_4} (w_{1,0,0,1} f)(x_1, x_4) D_1 D_4 u(x_1, x_2^0, x_3^0, x_4) \mathrm{d}x_4 \mathrm{d}x_1 \\
&+ \int_{x_1^0}^{h_1} \int_{x_2^0}^{h_2} \int_{x_3^0}^{h_3} (w_{1,1,1,0} f)(x_1, x_2, x_3) D_1 D_2 D_3 u(x_1, x_2, x_3, x_4^0) \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \\
&+ \int_{x_2^0}^{h_2} \int_{x_3^0}^{h_3} \int_{x_4^0}^{h_4} (w_{0,1,1,1} f)(x_2, x_3, x_4) D_2 D_3 D_4 u(x_1^0, x_2, x_3, x_4) \mathrm{d}x_4 \mathrm{d}x_3 \mathrm{d}x_2 \\
&+ \int_{x_1^0}^{h_1} \int_{x_2^0}^{h_2} \int_{x_4^0}^{h_4} (w_{1,1,0,1} f)(x_1, x_2, x_4) D_1 D_2 D_4 u(x_1, x_2, x_3^0, x_4) \mathrm{d}x_4 \mathrm{d}x_2 \mathrm{d}x_1 \\
&+ \int_{x_1^0}^{h_1} \int_{x_3^0}^{h_3} \int_{x_4^0}^{h_4} (w_{1,0,1,1} f)(x_1, x_3, x_4) D_1 D_3 D_4 u(x_1, x_2^0, x_3, x_4) \mathrm{d}x_4 \mathrm{d}x_3 \mathrm{d}x_1 \\
&+ \int_G (w_{1,1,1,1} f)(x) D_1 D_2 D_3 D_4 u(x) \mathrm{d}x \equiv (V^\star f)(u).
\end{aligned}
\tag{4.6}
$$

Thus, $(V^\star f)(u)$ is a finite sum of the Hardy-type operators. It is known that, if variable exponent $p(x)$ satisfies Dini-Lipschitz condition, then Hardy-type operators are bounded on variable Lebesgue spaces $L_{q(x)}(G)$ [10,13]. Eventually, we have proved the following lemma:

Lemma 4.1. *Let $p \in B(G) \bigcap \mathcal{P}(G)$ and $1 < \underline{p} \le \overline{p} < \infty$. Then, the operator*

$$V : SW_{p(x)}^{(1,1,1,1)}(G) \to E_{p(x)}(G)$$

has an adjoint operator $V^\star$, which acts boundedly on the spaces $E_{q(x)}(G)$.

Additionally, we need the following lemma:

Lemma 4.2. *Let $f \in E_{q(x)}(G)$. Then, the increment of the functional (3.3) has the integral form*

$$\Delta F(v) = -\int_G f_{1,1,1,1}(x) \Delta\varphi(x) \mathrm{d}x.$$

Proof. Now, instead of $u(x)$ in (4.6), we substitute the solution of the problem (4.2)-(4.3), i.e., replace the function $u(x)$ by $\Delta u(x)$. Then, the equality

$$f(V\Delta u) = \int_G f_{1,1,1,1}(x) \Delta\varphi(x) \mathrm{d}x = \int_G (w_{1,1,1,1} f)(x) D_1 D_2 D_3 D_4 \Delta u(x) \mathrm{d}x \equiv (V^\star f)(\Delta u)$$

holds for all $f \in E_{q(x)}(G)$. In other words,

$$-\int_G f_{1,1,1,1}(x)\Delta\varphi(x)\mathrm{d}x + \int_G (w_{1,1,1,1}f)(x)D_1D_2D_3D_4\Delta u(x)\mathrm{d}x = 0. \tag{4.7}$$

Thus, the function $\Delta u(x)$ as an element of $SW^{(1,1,1,1)}_{p(x)}(G)$ satisfies condition (4.3). By using the integral representation (4.4), we obtain

$$\begin{aligned}
&\alpha_k^{(1,0,0,0)}\Delta u(x_1^{(k)}, h_2, h_3, h_4) + \alpha_k^{(0,1,0,0)}\Delta u(h_1, x_2^{(k)}, h_3, h_4) + \alpha_k^{(0,0,1,0)}\Delta u(h_1, h_2, x_3^{(k)}, h_4)\\
&+ \alpha_k^{(0,0,0,1)}\Delta u(h_1, h_2, h_3, x_4^{(k)}) + \alpha_k^{(1,1,0,0)}\Delta u(x_1^{(k)}, x_2^{(k)}, h_3, h_4) + \alpha_k^{(0,1,1,0)}\Delta u(h_1, x_2^{(k)}, x_3^{(k)}, h_4)\\
&+ \alpha_k^{(0,0,1,1)}\Delta u(h_1, h_2, x_3^{(k)}, x_4^{(k)}) + \alpha_k^{(1,0,1,0)}\Delta u(x_1^{(k)}, h_2, x_3^{(k)}, h_4) + \alpha_k^{(0,1,0,1)}\Delta u(h_1, x_2^{(k)}, h_3, x_4^{(k)})\\
&+ \alpha_k^{(1,0,0,1)}\Delta u(x_1^{(k)}, h_2, h_3, x_4^{(k)}) + \alpha_k^{(1,1,1,0)}\Delta u(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, h_4) + \alpha_k^{(0,1,1,1)}\Delta u(h_1, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})\\
&+ \alpha_k^{(1,1,0,1)}\Delta u(x_1^{(k)}, x_2^{(k)}, h_3, x_4^{(k)}) + \alpha_k^{(1,0,1,1)}\Delta u(x_1^{(k)}, h_2, x_3^{(k)}, x_4^{(k)}) + \alpha_k^{(1,1,1,1)}\Delta u(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})\\
&= \int_G B_k(x)D_1D_2D_3D_4\Delta u(x)\mathrm{d}x,
\end{aligned}$$

where

$$\begin{aligned}
B_k(x) =&\alpha_k^{(1,0,0,0)}\theta(x_1^{(k)} - x_1) + \alpha_k^{(0,1,0,0)}\theta(x_2^{(k)} - x_2) + \alpha_k^{(0,0,1,0)}\theta(x_3^{(k)} - x_3) + \alpha_k^{(0,0,0,1)}\theta(x_4^{(k)} - x_4)\\
&+ \alpha_k^{(1,1,0,0)}\theta(x_1^{(k)} - x_1)\theta(x_2^{(k)} - x_2) + \alpha_k^{(0,1,1,0)}\theta(x_2^{(k)} - x_2)\theta(x_3^{(k)} - x_3) + \alpha_k^{(0,0,1,1)}\theta(x_3^{(k)} - x_3)\theta(x_4^{(k)} - x_4)\\
&+ \alpha_k^{(1,0,1,0)}\theta(x_1^{(k)} - x_1)\theta(x_3^{(k)} - x_3) + \alpha_k^{(0,1,0,1)}\theta(x_2^{(k)} - x_2)\theta(x_4^{(k)} - x_4) + \alpha_k^{(1,0,0,1)}\theta(x_1^{(k)} - x_1)\theta(x_4^{(k)} - x_4)\\
&+ \alpha_k^{(1,1,1,0)}\theta(x_1^{(k)} - x_1)\theta(x_2^{(k)} - x_2)\theta(x_3^{(k)} - x_3) + \alpha_k^{(0,1,1,1)}\theta(x_2^{(k)} - x_2)\theta(x_3^{(k)} - x_3)\theta(x_4^{(k)} - x_4)\\
&+ \alpha_k^{(1,1,0,1)}\theta(x_1^{(k)} - x_1)\theta(x_2^{(k)} - x_2)\theta(x_4^{(k)} - x_4) + \alpha_k^{(1,0,1,1)}\theta(x_1^{(k)} - x_1)\theta(x_3^{(k)} - x_3)\theta(x_4^{(k)} - x_4)\\
&+ \alpha_k^{(1,1,1,1)}\theta(x_1^{(k)} - x_1)\theta(x_2^{(k)} - x_2)\theta(x_3^{(k)} - x_3)\theta(x_4^{(k)} - x_4)
\end{aligned}$$

and $\theta(t) = \begin{cases} 1, & \text{if} \quad t > 0 \\ 0, & \text{if} \quad t \leq 0 \end{cases}$ is Heaviside function. Thus, increment (4.1) of functional (3.3) can be represented by

$$\Delta F(v) = \int_G \sum_{k=1}^{N} B_k(x)D_1D_2D_3D_4\Delta u(x)\mathrm{d}x$$

or

$$\Delta F(v) = \int_G B(x)D_1D_2D_3D_4\Delta u(x)\mathrm{d}x, \tag{4.8}$$

where $B(x) = \sum_{k=1}^{N} B_k(x)$. According to (4.7), increment (4.8) can be represented by the form

$$\Delta F(v) = \int_G [B(x) + (w_{1,1,1,1}f)(x)]D_1D_2D_3D_4\Delta u(x)\mathrm{d}x - \int_G f_{1,1,1,1}(x)\Delta\varphi(x)\mathrm{d}x. \tag{4.9}$$

Since $w_{1,1,1,1}$ depends only on $f_{1,1,1,1}$, equality (4.9) holds for all $f_{1,1,1,1} \in L_{q(x)}(G)$. For integro-differential expression (4.9), we consider the equation

$$(w_{1,1,1,1}f)(x) + B(x) = 0, \quad x \in G, \tag{4.10}$$

that is said to be an adjoint equation for optimal control problem (3.1)-(3.3). As the function $f_{1,1,1,1}(x)$ we consider the solution of equation (4.10) in $L_{q(x)}(G)$. Then, increment (4.9) has the integral form

$$\Delta F(v) = -\int_G f_{1,1,1,1}(x)\Delta\varphi(x)\mathrm{d}x.$$

This completes the proof. □

5. The Main Result

Now, for a fixed $(\tau_1, \tau_2, \tau_3, \tau_4) \in G$, we consider the following needle variation of the admissible control $\nu(x)$

$$\Delta\nu_\varepsilon(x) = \begin{cases} \widehat{\nu} - \nu(x) & , \quad \text{if} \quad x \in G_\varepsilon \\ 0 & , \quad \text{if} \quad x \in G \setminus G_\varepsilon, \end{cases}$$

where $\widehat{\nu} \in \Omega_\partial$, ε is a sufficiently small parameter that is positive and $G_\varepsilon = (\tau_1 - \frac{\varepsilon}{2}, \tau_1 + \frac{\varepsilon}{2}) \times (\tau_2 - \frac{\varepsilon}{2}, \tau_2 + \frac{\varepsilon}{2}) \times (\tau_3 - \frac{\varepsilon}{2}, \tau_3 + \frac{\varepsilon}{2}) \times (\tau_4 - \frac{\varepsilon}{2}, \tau_4 + \frac{\varepsilon}{2}) \subset G$. A control $\nu_\varepsilon(x)$ defined by $\nu_\varepsilon(x) = \nu(x) + \Delta\nu_\varepsilon(x)$ is an admissible control for all sufficiently small positive ε and all $\widehat{\nu} \in \Omega_\partial$ called a needle perturbation given by control $\nu(x)$ where $(\tau_1, \tau_2, \tau_3, \tau_4) \in G$ is a fixed point. Clearly,

$$F(\nu_\varepsilon) - F(\nu) = -\int_{G_\varepsilon} f_{1,1,1,1}(x)[\varphi(x, \nu(x) + \Delta\nu_\varepsilon(x)) - \varphi(x, \nu(x))]\mathrm{d}x = -\int_{G_\varepsilon} f_{1,1,1,1}(x)[\varphi(x, \widehat{\nu}(x)) - \varphi(x, \nu(x))]\mathrm{d}x. \quad (5.1)$$

Since the optimal control problem is linear, that theorem follows from (5.1):

Theorem 5.1. *Let $f_{1,1,1,1} \in L_{q(x)}(G)$ be a solution of adjoint equation (4.10). Then, for the optimality of the admissible control $\nu(x)$, it is necessary and sufficient that for almost all $x \in G$ Pontryagin's maximum condition*

$$\max_{\widehat{\nu} \in \Omega_\partial} H(x, f_{1,1,1,1}(x), \widehat{\nu}) = H(x, f_{1,1,1,1}(x), \nu)$$

be satisfied, where $H(x, f_{1,1,1,1}(x), \nu) = f_{1,1,1,1}(x)\varphi(x, \nu)$ is Hamilton-Pontryagin function.

Proof. Assume that a control $\nu(x_1, x_2, x_3, x_4) \in \Omega_\partial$ gives the minimum value of functional (3.3). Hence, by (5.1), we obtain

$$-\int_{G_\varepsilon} [H(x_1, x_2, x_3, x_4, f_{1,1,1,1}(x_1, x_2, x_3, x_4), \widehat{\nu}) - H(x_1, x_2, x_3, x_4, f_{1,1,1,1}(x_1, x_2, x_3, x_4), \nu(x_1, x_2, x_3, x_4)]\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 \geq 0. \quad (5.2)$$

Dividing the both sides of (5.2) by ε^4 and later taking the limit as ε approaches $+0$, for almost all $(\tau_1, \tau_2, \tau_3, \tau_4) \in G$ and using analogue of Lebesgue differentiation theorem in $L_{p(x)}$ (see [10]) for all $\nu \in \Omega_\partial$, we have

$$H(\tau_1, \tau_2, \tau_3, \tau_4, f_{1,1,1,1}(\tau_1, \tau_2, \tau_3, \tau_4), \nu(\tau_1, \tau_2, \tau_3, \tau_4)) - H(\tau_1, \tau_2, \tau_3, \tau_4, f_{1,1,1,1}(\tau_1, \tau_2, \tau_3, \tau_4), \widehat{\nu}) \geq 0. \quad (5.3)$$

Thus, for optimal control of $\nu(x_1, x_2, x_3, x_4) \in \Omega_\partial$, it is necessary to satisfy condition (5.3). Moreover, the equality

$$\Delta F(\nu) = -\int_G \Delta H(x_1, x_2, x_3, x_4, f_{1,1,1,1}(x_1, x_2, x_3, x_4), \nu(x_1, x_2, x_3, x_4))\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1$$

shows that this condition is also sufficient for optimal control $\nu(x_1, x_2, x_3, x_4)$, where

$$\Delta H(x_1, x_2, x_3, x_4, f_{1,1,1,1}, \nu) = H(x_1, x_2, x_3, x_4, f_{1,1,1,1}, \nu + \Delta\nu) - H(x_1, x_2, x_3, x_4, f_{1,1,1,1}, \nu).$$

This completes the proof. □

Example 5.2. Equation (3.1) generalizes the four-dimensional analogue of vibrating string equation and the four-dimensional telegraph equation. Essentially, if we take $a_{0,0,0,0}(x) = -k$, k = constant ≥ 0 and $a_{i_1,i_2,i_3,i_4}(x) = 0$, $0 < i_1 + i_2 + i_3 + i_4 \leq 3$ in Bianchi equation (3.1), we obtain

$$D_1D_2D_3D_4u(x) - ku(x) = \varphi(x, \nu(x)). \quad (5.4)$$

It is known that equation (5.4) is a controlled process described by the four-dimensional telegraph equation. This equation arises in the four-dimensional mathematical modeling of filtering and telecommunication. Assume that k is zero. Then, adjoint equation (4.10) for four-dimensional optimal control problem (3.1)-(3.3) is of a simpler form as follows:

$$f_{1,1,1,1}(x) + B(x) = 0, \quad x \in G.$$

6. Conclusions

A necessary and sufficient condition is proposed for an optimal control problem with distributed parameters, which is described by the fourth-order Bianchi equation involving coefficients in variable exponent Lebesgue spaces. This condition can be seen as a novel approach to Pontryagin's maximum principle for the considered problem throughout the work. The novelty arises from the facts that the problem is studied with the help of such a version of the increment method that essentially uses the concept of the adjoint equation of integral type, and also that the problem is studied in variable exponent case instead of conventional constant exponent case of Lebesgue and Sobolev spaces.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this manuscript.

Authors Contribution Statement

The author has read and agreed to the manuscript.

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