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Parçacık Sürü Optimizasyon Algoritması Kullanılarak Nakagami Dağılımı için En Çok Olabilirlik Tahmini ve Uygulamaları

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ÖZ

Nakagami dağılımı, radyo sinyallerinin sönümlenmesini modellemek için ortaya çıkmıştır ve çeşitli disiplinlerde yaygın olarak kullanılmaktadır. Bu çalışmada, dağılımın şekil ve ölçek parametrelerini tahmin etmek için en çok olabilirlik (ML) tahmin yöntemi kullanılmıştır. Ancak, bu dağılım için olabilirlik denklemlerinin açık çözümleri bulunmamaktadır. Bu nedenle, bu denklemlerin çözümü için, parçacık sürüsü optimizasyon (PSO), genetik algoritma (GA) ve quasi-newton (QN) algoritmaları olmak üzere üç temel algoritma kullanılmıştır. Bu algoritmaların performanslarının karşılaştırmaları, yan, hata kareler ortalaması (MSE) ve eksiklik (DEF) kriterleri dikkate alınarak, kapsamlı bir Monte-Carlo simülasyon çalışması ile yapılmıştır. Model, kullanışlılığını göstermek amacıyla dört gerçek veri setine uygulanmıştır.

Maximum Likelihood Estimation for the Nakagami Distribution using Particle Swarm Optimization Algorithm with Applications

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ABSTRACT

The Nakagami distribution originated to model the fading of radio signals and is widely used in various disciplines. In this study, the maximum likelihood (ML) estimation method is used to estimate the shape and scale parameters of the distribution. However, there are no explicit solutions to the likelihood equations for this distribution. Therefore, three main algorithms, the particle swarm optimization algorithm (PSO), the genetic algorithm (GA), and the quasi-newton (QN) algorithm, have been used to solve these equations. Comparisons of the performances of these algorithms have been made with a comprehensive Monte-Carlo simulation study, taking into account the bias, mean squared error (MSE), and deficiency (DEF) criteria. The model has been applied to four real data sets in order to demonstrate its usefulness.

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INTRODUCTION

The Nakagami distribution was initially introduced in 1960 to model radio signal fade [1]. This distribution is later used as a model of a wide range of data signals sent by radar objects [2]. It is utilized to simulate a range of data in the communication engineering area in various published research studies in the literature [3, 4]. Additionally, the Nakagami distribution has been effectively applied in a number of different areas, such as hydrology [5, 6], and to model failure times of a variety of electrical components in engineering fields; furthermore, in the healthcare area, to model the time occurrence of tumors and the appearance of lung cancer [7, 8], as well as to model ultrasound data for medical images [9, 10]. Moment estimators for Nakagami distribution were found by [11] and maximum likelihood estimators were found by [12, 13] and demonstrated how the ML-based estimators performed better than the most well-known moment-based estimator after their performance compared to each other. Schwartz et al. (2013) estimated the shape parameter using the improved maximum likelihood estimation method [14]. However, the scale parameter of the Nakagami distribution is estimated by [15] using the Bayesian estimation method. Other Nakagami distribution estimators have been examined and contrasted by using Monte-Carlo simulation [16] and some general characteristics of Nakagami-m distribution are presented by [17]. The core idea of the ML method is to determine the estimate values for the relevant parameters that maximize the likelihood function of the model; however, since nonlinear functions are usually involved, an explicit solution is barely possible, so iterative numerical techniques are therefore required to maximize the likelihood function [18]. The Newton-Raphson algorithm is a popular traditional iterative algorithm for solving equation systems generated by partial derivations of the likelihood function. It's used to find ML estimators numerically, as seen in the literature. However, it only works for functions that can be differentiated at least twice since it employs a gradient-based search technique to identify the optimal parameter values based on the inverse of the hessian matrix. However, every conventional numerical algorithm begins with a point that is chosen at random and continues iteratively to the global optimum, with no guarantee of getting stuck at the local optimum points [19]. The use of well-known meta-heuristic algorithms like particle swarm optimization (PSO) and genetic algorithm (GA) is advised to prevent these problems and ensure that the global optimal solution is reached with greater ease and without reliance on the derivation [20]. However, in large-scale optimization problems, Quasi-Newton (QN), which is another conventional technique, is more efficient than Newton's method since it doesn't require computing second derivatives, which lowers the cost of calculation. So, the key objective of this paper is to determine the most appropriate algorithm from PSO, GA, and QN for estimating the shape (α) and scale (λ) parameters of the Nakagami distribution by a comparison based on the conduct of a comprehensive Monte-Carlo simulation study as well as demonstrate the practicality of this distribution.

The remaining sections of this work will be arranged as follows: Section (2) discusses the Nakagami distribution and its basic properties. In Section (3), the ML estimation method for PSO, GA, and QN is covered. An extensive Monte-Carlo simulation study is performed to compare the performance of the parameter estimators in Section (4). Four applications using real-world datasets are carried out in Section (5). In Section (6), the study provides multiple conclusions.

Nakagami Distribution

If we have a random variable, *Y*, that is gamma-distributed with the following probability density function (pdf).

$$f(y;k,\theta) = \frac{1}{\Gamma(k)\theta^k}\, y^{k-1}e^{-\frac{y}{\theta}}, \qquad y > 0, k > 0, \theta > 0 \tag{1}$$

where Γ is the gamma function, k is the shape parameter and θ is the scale parameter, and then $X = \sqrt{Y}$ follows a two-parameter Nakagami distribution, $X \sim Nakagami\ (\alpha,\lambda)$, with shape parameters $\alpha = k$ and scale parameter $\lambda = k\theta$ as in the following probability density function (pdf).

$$f(x;\alpha,\lambda)=\frac{2\alpha^{\alpha}}{\Gamma(\alpha)\lambda^{\alpha}}\,x^{2\alpha-1}e^{\left(-\frac{\alpha x^2}{\lambda}\right)},\qquad x>0,\alpha>0.5,\lambda>0 \tag{2}$$

X's cumulative distribution function (cdf) is:

$$F(x;\alpha,\lambda)=\frac{1}{\Gamma(\alpha)}\,\gamma\left(\alpha,\frac{\alpha}{\lambda}x^2\right). \tag{3}$$

where γ is the regularized (lower) incomplete gamma function. If we have another random variable, *Z*, that is chi-square distributed with an integer-valued parameter equivalent to 2α, then the random variable $U=\sqrt{(\lambda/2\alpha)Z}$ follows Nakagami distribution with parameters α and λ. Furthermore, the Nakagami distribution is related to other distributions, such as if $\alpha=0.5$, it becomes a half-normal distribution, and when $\alpha=1$ it becomes a Rayleigh distribution. The plots of the Nakagami distribution for different values of α and λ are shown in Figure 1. The Nakagami distribution's k^{th} moment general formula is given below.

$$E\left(X^k\right)=\frac{\Gamma\left(\alpha+\frac{k}{2}\right)}{\Gamma(\alpha)}\left(\frac{\lambda}{\alpha}\right)^{\frac{k}{2}} \tag{4}$$

By using formula (4), the mean and variance of the random variable *X* can be calculated as follows:

$$E(X)=\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}} \tag{5}$$

$$Var\,(X)=\lambda-\left[\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}\left(\frac{\lambda}{\alpha}\right)^{2}\right]=\lambda\left[1-\frac{1}{\alpha}\left(\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}\right)^{2}\right] \tag{6}$$

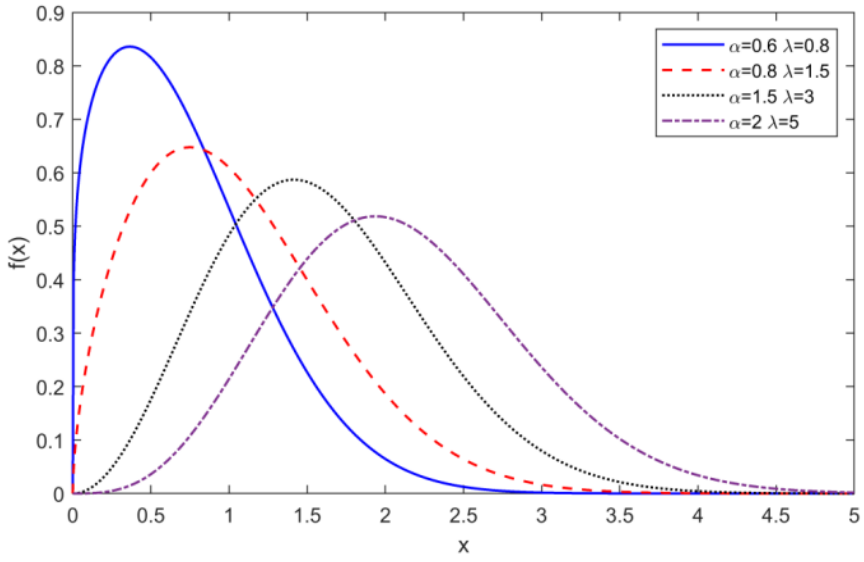

Figure 1. *Nakagami pdf for different values of α and λ*

Obtaining the values that maximize the likelihood function to its maximum is the basis for this method; commonly, the likelihood function's logarithm is applied to simplify the calculations; this is considered the study model. To estimate the unknown parameters for the Nakagami distribution in this study, the log-likelihood (log *L*) function is presented below.

$$\log L(\alpha,\lambda) = nlog(2) + n\alpha log(\alpha) - nlog\Gamma(\alpha) - n\alpha log(\lambda) + (2\alpha - 1)\sum_{i=1}^{n} log(x_i) - \frac{\alpha}{\lambda}\sum_{i=1}^{n} {x_i}^2 \quad (7)$$

To estimate the likelihood parameters of the log L function for the Nakagami distribution, the partial derivatives corresponding to the parameters under consideration are determined and set to zero. The likelihood equations are given as follows:

$$\frac{\partial\, log\, L(\alpha,\lambda)}{\partial\alpha} = \frac{n^2}{\alpha} - n\frac{\partial log\Gamma(\alpha)}{\partial\alpha} - nlog(\lambda) + 2\sum_{i=1}^{n} log(x_i) - \frac{1}{\lambda}\sum_{i=1}^{n} {x_i}^2 = 0 \quad (8)$$

and

$$\frac{\partial\, log\, L(\alpha,\lambda)}{\partial\lambda} = \frac{n\alpha}{\lambda} + \frac{\alpha}{\lambda^2}\sum_{i=1}^{n} {x_i}^2 = 0 \Rightarrow \hat{\lambda} = \frac{\sum_{i=1}^{n} {x_i}^2}{n} \quad (9)$$

It is unlikely to find an explicit solution to the likelihood equation for estimating the shape parameter α since it contains a nonlinear function, as illustrated by the equation (8). Iterative numerical techniques are therefore necessary to get ML estimations for parameter α. The next subsections provide a brief introduction to the highly efficient algorithms PSO, GA, and QN that are used in this study as numerical techniques for estimating the likelihood estimators for the Nakagami distribution.

Particle Swarm Optimization (PSO)

PSO is a population-based heuristic algorithm that was initially proposed in 1995 by Kennedy and Eberhart [21]. It is based on a simulation of the swarming behavior of birds. With this algorithm, which has a wide range of applications in the literature [22, 23], every solution is referred to as a particle and any group of solutions is referred to as a population. The PSO method's fundamental concept can be stated as a procedure that involves continually traveling a swarm of particles in a defined search space with respect to a set of formulas until finally obtaining the ideal answer [24]. Considering there are n dimensions in the search space, Every particle i in the swarm has a position and velocity vectors $X_i = (x_{i1}, x_{i2}, \dots, x_{in})$ and, $V_i = (v_{i1}, v_{i2}, \dots, v_{in})$, $i = 1,2,\dots,m$ respectively, where m is the amount of particles. In PSO, the initial solutions are represented by the population of particles' initial positions randomly generated during the first iteration of their search over a multidimensional search space. The fitness value for the main study's model, represented by the log L function, is used to evaluate each particle's position, and the best position that's equivalent to the fitness function's highest value for each particle is then determined and called the "personal best" (*pbest*). After that, the best position along with all the particles according to the best fitness value of all particles is found and called the "global best" (*gbest*). Then each particle's position is updated by the following equation:

$$V_i^{k+1} = \omega V_i^k + c_1 R_{rand_1}\left(pbest_i^k - X_i^k\right) + c_2 R_{rand_2}\left(gbest^k - X_i^k\right) \quad (10)$$

and each particle's velocity is updated by the following equation

$$x_i^{k+1} = x_i^k + V_i^{k+1}\,. \quad (11)$$

Where V_i^k velocity of particle i at iteration k, ω inertia weight parameter, c_1, c_2 acceleration coefficients, R_{rand_1}, R_{rand_2} random numbers uniformly distributed between 0 and 1, X_i^k position of particle i at iteration k, $pbest_i^k$ best position of particle i until iteration k, $gbest^k$ best position of the group until iteration k. Until convergence is achieved, this process will continue.

Genetic Algorithm (GA)

The GA is a powerful technique for finding highly precise, approximate solutions to optimization problems that is based on the law of natural selection and Darwinian evolution. John Holland initially introduced

it [25], and David Goldberg [26] thoroughly investigated it. Every solution is a chromosome, and every set of solutions (chromosomes) is a population. Initializing the GA algorithm involves creating a population at random once the fitness function, search space, and initial GA fixed parameters have been determined. Each chromosome in the population has its fitness value evaluated; the highest-scoring chromosomes are chosen as the best and the lowest-scoring chromosomes are replaced with new, randomly-generated individuals. A specific amount of elite chromosomes is chosen and transferred unchanged to the next generation, and the two chromosomes with the highest score are considered the parents of the other individuals. The processes of crossover and mutation are then carried out to produce a new, more desirable offspring. The mutation phase is critical to avoiding becoming stuck in the local optimum, which indicates exploration. This technique will be repeated until convergence is reached.

Quasi-Newton (QN)

The Quasi-Newton (QN) is a well-known type of conventional algorithm for unconstrained nonlinear numerical optimization problems that uses a quadratic approximation to find the global minimum of the objective function. It was invented by William Davidon [27], and it is regarded as a modified algorithm version of the Newton method with enhanced computing efficiency, which saves time and costs while computing partial derivatives in each iteration [28]. Like any other conventional algorithm, it starts with a carefully selected initial point, and then the gradient of the objective function, which is the same as the log L function in this study, is calculated to identify the search direction. The step size for each iteration is calculated by continuously updating the inverse Hessian matrix approximation and determining if the further approximated point is optimal or not. This process is continuing until the convergence requirements are satisfied. For more details, see [29-34].

Monte-Carlo Simulations Study

This section contains the simulated results of the Monte Carlo simulation study for a variety of sample sizes. Using iterative techniques such as PSO, GA, and QN, the performance of the ML estimators of the model parameters is compared. Matlab R2021a software is used for performing all the calculations for the simulation study. All the simulations are based on 2000 executions for Monte Carlo runs. The shape α and scale λ parameters are considered to be 0.6 and 0.8, respectively, for a number of sample sizes n that are taken to be 10, 20, 30, 40, and 60 in a search space ranging from 0 to 20. The simulations' resulting estimates for the shape and scale parameters are denoted, respectively, by $\hat{\alpha}$ and $\hat{\lambda}$. The mathematical formulas (12–16) below are used to simulate the mean, bias, mean square error (MSE), and deficiency (Def) values, which are used for comparing and evaluating the performance of the estimators.

$$Mean\left(\hat{\theta}\right) = \frac{\sum_i^n \hat{\theta}_i}{n} \tag{12}$$

$$Bias\left(\hat{\theta}\right) = E\left(\hat{\theta}\right) - \theta \tag{13}$$

$$Var\left(\hat{\theta}\right) = \frac{1}{n-1}\sum_{i=1}^{n}\left(\hat{\theta}_i - Mean\,\hat{\theta}\right)^2 \tag{14}$$

$$MSE\left(\hat{\theta}\right) = Var\left(\hat{\theta}\right) + \left(Bias\left(\hat{\theta}\right)\right)^2 \tag{15}$$

$$Def\left(\hat{\alpha}, \hat{\lambda}\right) = MSE(\hat{\alpha}) + MSE\left(\hat{\lambda}\right) \tag{16}$$

where $\theta = (\alpha, \lambda) \in R \times R^+$. The resulting simulated values of mean, bias, MSE, and Def for $\hat{\alpha}$ and $\hat{\lambda}$ are given in Table 1. The simulated values demonstrate that the PSO gives the best results when compared with the GA and QN algorithms. Considering the bias simulation results, it is seen that the least biased values of the shape parameter estimator $\hat{\alpha}$ for nearly all sample sizes are produced by the QN algorithm. With regard to MSE values for the shape parameter estimator $\hat{\alpha}$, it is quite apparent that PSO estimator values perform better than other algorithms for all n values. Also, it's clear that when sample size increases, the MSE values of GA improve to be so close to the PSO values, and in all cases, the PSO and GA MSE have better values than QN. Therefore, the

PSO indicates greater performance in accordance with the Def criteria, with the lowest values for all cases. This enables us to state that the PSO algorithm is efficient in estimating the shape parameter for the Nakagami distribution.

Table 1. *Simulated Mean, Bias, MSE, and Def values for the ML estimators* $\hat{\alpha}$ *and* $\hat{\lambda}$.

		$\hat{\alpha}$			$\hat{\lambda}$			
n	Method $\alpha = 0.6, \lambda = 0.8$	Mean	Bias	MSE	Mean	Bias	MSE	Def
10	PSO	0.5560	-0.0440	0.0672	0.8109	0.0109	0.1094	0.1766
	GA	0.8089	0.2089	0.2241	0.8109	0.0109	0.1094	0.3335
	QN	0.7916	0.1916	0.2297	0.8109	0.0109	0.1094	0.3391
20	PSO	0.5246	-0.0754	0.0176	0.8069	0.0069	0.0537	0.0713
	GA	0.6870	0.0870	0.0471	0.8069	0.0069	0.0537	0.1008
	QN	0.6764	0.0764	0.0502	0.8069	0.0069	0.0537	0.1039
30	PSO	0.5176	-0.0824	0.0133	0.8018	0.0018	0.0356	0.0488
	GA	0.6576	0.0576	0.0281	0.8018	0.0018	0.0356	0.0637
	QN	0.6509	0.0509	0.0299	0.8018	0.0018	0.0356	0.0655
40	PSO	0.5161	-0.0839	0.0113	0.8012	0.0012	0.0261	0.0375
	GA	0.6409	0.0409	0.0164	0.8012	0.0012	0.0261	0.0425
	QN	0.6364	0.0364	0.0176	0.8012	0.0012	0.0261	0.0437
60	PSO	0.5129	-0.0871	0.0102	0.8032	0.0032	0.0185	0.0286
	GA	0.6276	0.0276	0.0108	0.8032	0.0032	0.0185	0.0293
	QN	0.6251	0.0251	0.0114	0.8032	0.0032	0.0185	0.0299

APPLICATIONS

In this section, in order to demonstrate the Nakagami distribution's flexibility, four real datasets are modeled using it. The unknown parameters are estimated via the PSO algorithm. The modeling performance of the Nakagami distribution is evaluated in comparison with the performance of various distinct classical distributions using well-known criteria, including log-likelihood values, the Akaike Information Criterion (AIC), and the corrected AIC (AICc). For more details on these criteria, see [35].

$$AIC = 2P - 2\,log\,L \tag{17}$$

$$AIC_c = AIC + \frac{2P(P+1)}{n-P-1} \tag{18}$$

where $log\ L$, n, and p represent the likelihood function, the number of observations, and the total number of model parameters, respectively. It is stated to be the best-fit model when the probability model satisfies those criteria with the lowest values compared to other probability distributions.

Dataset 1: The Wheaton river data

This dataset consists of 72 exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada, for the years 1958–1984. It was considered for the hydrology field in the literature [27, 36, 37] and is given in Table 2. The Nakagami distribution is compared to several well-known classical distributions in Table 3 below, utilizing the log L, AIC, and AICc criteria for modeling performance. Furthermore, the results show that the Nakagami distribution performs better than other distributions.

Table 2. The Wheaton River data

1.7	1.4	0.6	9.0	5.6	1.5	2.2	18.7	2.2	1.7	30.8	2.5
14.4	8.5	39.0	7.0	13.3	27.4	1.1	25.5	0.3	20.1	4.2	1.0
0.4	11.6	15.0	0.4	25.5	27.1	20.6	14.1	11.0	2.8	3.4	20.2
5.3	22.1	7.3	14.1	11.9	16.8	0.7	1.1	22.9	9.9	21.5	5.3
1.9	2.5	1.7	10.4	27.6	9.7	13.0	14.4	0.1	10.7	36.4	27.5
12.0	1.7	1.1	30.0	2.7	2.5	9.3	37.6	0.6	3.6	64.0	27.0

Table 3. *Parameter estimates, log L, AIC and AICc, values for Wheaton river data.*

	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\lambda}$	*-log L*	AIC	AICc
Nakagami	-	0.3241	298.063	251.2767	506.5534	506.7273
Gamma	-	0.8383	14.5588	251.344	506.688	506.8619
Weibull	-	0.9012	11.6322	251.499	506.998	507.1719
Lognormal	1.7981	-	1.4169	256.215	516.43	516.6039
Log-logistic	1.9107	-	0.8246	257.839	519.678	519.8519
Inverse Gaussian	-	2.2459	12.2042	267.228	538.456	538.6299
logistic	10.6372	-	6.60488	279.958	563.916	564.0899
Normal	12.2042	-	12.2972	282.338	568.676	568.8499
Extreme Value	19.014	-	16.6415	303.92	611.84	612.0139
Rayleigh	-	-	12.2078	302.838	607.676	607.7331

Dataset 2: The tensile strength

This dataset consists of the tensile strength (in GPa) of 69 carbon fibers evaluated under stress at 20-mm gauge lengths. This dataset consists of the tensile strength (in GPa) of 69 carbon fibers evaluated under stress at 20-mm gauge lengths. This dataset was used by Bader and Priest in 1982 [38] and is given in Table 4. The Nakagami distribution is contrasted with a number of well-known distributions in Table 5 below using the considered modeling performance criteria, and the results show that the Nakagami distribution provides a better fit than others.

Table 4. *The tensile strength*

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.858	

Table 5. *Parameter estimates, log L, AIC and AICc, values for the tensile strength's data.*

	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\lambda}$	*-log L*	AIC	AICc
Nakagami	-	6.0395	6.2801	50.3933	104.7866	104.9684
Gamma	-	22.8047	0.1077	50.9856	105.9712	106.153
Log-logistic	0.8883	-	0.1187	51.4346	106.8692	107.051
Weibull	-	5.2702	2.6585	51.7165	107.433	107.6148
Lognormal	0.8762	-	0.2161	52.1663	108.3326	108.5144
Inverse Gaussian	-	51.9864	2.4553	52.2804	108.5608	108.7426
Extreme Value	2.7089	-	0.5202	57.5372	119.0744	119.2562
Rayleigh	-	-	1.7720	87.4975	176.995	177.0547

Dataset 3: Failure Times of 84 Aircraft Windshield

This dataset concerns the investigation of 84 windshields' failure times for a certain type of aircraft, which was originally covered by Ramos et al. [39] and is given in Table 6. Using the discussed modeling performance criteria, the Nakagami distribution is compared against a number of common distributions in Table 7 below, and results demonstrate that it has a better fit than others.

Table 6. *Failure times of 84 aircraft windshield*

0.040	1.866	2.385	3.443	0.301	1.876	2.481	3.467	0.309	1.899	2.610	3.478
0.557	1.911	2.625	3.578	0.943	1.912	2.632	3.595	1.070	1.914	2.646	3.699
1.124	1.981	2.661	3.779	1.248	2.010	2.688	3.924	1.281	2.038	2.823	4.035
1.281	2.085	2.890	4.121	1.303	2.089	2.902	4.167	1.432	2.097	2.934	4.240
1.480	2.135	2.962	4.255	1.505	2.154	2.964	4.278	1.506	2.190	3.000	4.305
1.568	2.194	3.103	4.376	1.615	2.223	3.114	4.449	1.619	2.224	3.117	4.485
1.652	2.229	3.166	4.570	1.652	2.300	3.344	4.602	1.757	2.324	3.376	4.663

Table 7. *Parameter estimates, log L, AIC and AICc, values for Failure Times of 84 Aircraft Windshield's data.*

	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\lambda}$	*-log L*	AIC	AIC_C
Nakagami	-	1.1960	7.7774	130.9964	270.996	271.1441
Extreme Value	3.1188	-	1.0633	133.4980	277.874	278.0221
Gamma	-	3.4922	0.7323	136.9370	283.162	283.3101
Log-logistic	0.8718	-	0.3102	139.5810	311.846	311.9941
Lognormal	0.7891	-	0.6910	153.923	369.114	369.2621
Inverse Gaussian	-	2.3595	2.5575	182.557	270.996	271.1441

Dataset 4: Strengths of glass fibers

The fiber glass data, which consists of breaking strengths of 63, is widely used in statistical literature, and this dataset was first given by Smith and Naylor [40] and is given in Table 8. The Nakagami distribution is compared to other common distributions in Table 9 below using the previously used modeling performance criteria, and the findings show that it has greater fit than rivals.

Table 8. *Strengths of glass fibres.*

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64
1.68	1.73	1.81	2	0.74	1.04	1.27	1.39	1.49
1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01	0.77
1.11	1.28	1.42	1.5	1.54	1.6	1.62	1.66	1.69
1.76	1.84	2.24	0.81	1.13	1.29	1.48	1.5	1.55
1.61	1.62	1.66	1.7	1.77	1.84	0.84	1.24	1.3
1.48	1.51	1.55	1.61	1.63	1.67	1.7	1.78	1.89

Table 9. *Parameter estimates, log L, AIC and AICc, values for Strengths of glass fibres's data.*

	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\lambda}$	*-log L*	AIC	AIC_C
Nakagami	-	5.0427	2.3739	20.8435	45.6870	45.8870
Log-logistic	0.4228	-	0.1262	22.7900	49.5800	49.7800
Gamma	-	17.4396	0.0864	23.9515	51.9030	52.1030
Lognormal	0.3811	-	0.2599	28.0089	60.0178	60.2178
Inverse Gaussian	-	21.5608	1.5068	28.6708	61.3416	61.5416
Rayleigh	-	-	1.08948	49.7909	101.5818	101.6474

CONCLUSIONS

In this study, the estimation of the shape and scale parameters of the Nakagami distribution is considered. The ML estimates of the Nakagami distribution are obtained via three iterative algorithms, which are PSO, GA, and QN, because of the complexity of the nonlinear likelihood equation of the shape parameter, and then the performances of the selected algorithms are compared to each other with respect to bias, MSE, and Def criteria by conducting a Monte Carlo simulation study. Simulation results show that the PSO is the most efficient iterative algorithm in terms of considered criteria in all cases, especially for small sample sizes, and as the sample size increases, the performance of GA improves to get close to the PSO algorithm results. To show the flexibility of the Nakagami distribution, four real datasets are employed to fit the Nakagami distribution, and the results demonstrate high performance for the Nakagami distribution in comparison with many well-known statistical distributions such as gamma, Weibull, log-logistic, and others.

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