Black Sea Journal of Engineering and Science
doi: 10.34248/bsengineering.1356036

Open Access Journal
e-ISSN: 2619 – 8991

Research Article
Volume 6 - Issue 4: 639-647 / October 2023

MAXIMUM LIKELIHOOD ESTIMATION FOR THE LOG-LOGISTIC DISTRIBUTION USING WHALE OPTIMIZATION ALGORITHM WITH APPLICATIONS

Adi Omaia FAOURI[1], Pelin KASAP[1]*

[1]*Ondokuz Mayıs University, Faculty of Science, Department of Statistics, 55139, Samsun, Türkiye*

Abstract: The log-logistic distribution has been widely used in several fields, including engineering, survival analysis, and economics. The method of maximum likelihood estimation is used in this study for estimating the shape and scale parameters for the log-logistic distribution, whereas in the case of the log-logistic distribution, likelihood equations lack explicit solutions. Therefore, problems with solving likelihood equations can be solved by using two highly efficient algorithms, which are the whale optimization algorithm and the Nelder-Mead algorithm, as well as by showing the applicability of this distribution by comparing it with other well-known classical distributions. To demonstrate the performance of each algorithm implemented, an extensive Monte Carlo simulation study has been conducted. The performance of maximum likelihood estimators for each algorithm has been evaluated in terms of mean square error and deficiency criteria. It has been seen that the whale optimization algorithm provides the best estimates for the log-logistic distribution parameters according to the simulation data.

Keywords: Maximum likelihood, Log-logistic distribution, Whale optimization, Monte-Carlo simulation

***Corresponding author:** Ondokuz Mayıs University, Faculty of Science, Department of Statistics, 55139, Samsun, Türkiye
E mail: pelin.kasap@omu.edu.tr (P. KASAP)
Adi Omaia FAOURI https://orcid.org/0000-0003-4499-1240
Pelin KASAP https://orcid.org/0000-0002-1106-710X

Received: September 06, 2023
Accepted: October 04, 2023
Published: October 15, 2023

Cite as: Faouri AO, Kasap P. 2023. Maximum likelihood estimation for the log-logistic distribution using whale optimization algorithm with applications. BSJ Eng Sci, 6(4): 639-647.

1. Introduction

In many different fields of research, including medicine, economics and survival analysis, the log-logistic distribution is widely used. The log-logistic distribution can be obtained by applying a logarithmic transform to the logistic distribution. Similar to how the log-normal and normal distributions have a relation to one another, the log-logistic distribution has a relation to the logistic distribution. The log-logistic distribution looks similar to the log-normal distribution in terms of shape but with heavier tails. Burr distribution was introduced by Burr (1942) and log-logistic distribution is considered a special case of the Burr distribution family in Tadikamalla (1980) study. The log-logistic distribution's properties and characteristics were studied for the first time in 1963 (Shah & Dave, 1963). Systems of frequency curves are produced by applying three simple transformations to the logistic distribution (Tadikamalla and Johnson, 1982). Both the studies of Ali and Khan (1987) as well as Balakrishnan and Malik (1987) examined the use of moments of ordered statistics for estimating the unknown parameters of the log-logistic distribution. The initial application was to model the distribution of income and wealth (Fisk, 1961), and this distribution was used for modeling stream flow rates too (Shoukri et al., 1988). Kantam and Srinivasa (2002) in their study regarded maximum likelihood estimators (MLEs) of the scale parameter when its shape parameter already exists and derived the modified maximum likelihood estimation (MLE) of this distribution. A Bayesian approach is applied to estimate the parameters of the log-logistic distribution and compared with the MLEs of the Abbas and Tang (2016) study. Regarding the estimation of the parameters of any distribution, there are several statistical methods that can be used. The maximum likelihood (ML) estimation method is the most popular method due to its great efficiency and well-known asymptotic characteristics for parameter estimators when compared to all other statistical approaches (Yuan and Schuster, 2013). One of the most commonly used techniques for calculating the MLEs of the parameters is the Newton-Raphson algorithm which is a gradient-based search algorithm. The fundamental issue with this technique is the requirement for the second derivatives for all iterations (Kus and Kaya, 2006). In order to avoid such restrictions another kind of classical iterative technique can be employed, such as the Nelder-Mead (NM) algorithm, that doesn't require the gradient information of the fitness function. However, all classical algorithms start from a randomly selected initial point and continue moving to the solution iteratively until the optimum solution is obtained, but the solution may remain at the local optimum with a lack of guarantee that the final result is globally reached (Pratihar, 2012).

The application of non-classical algorithms, such as meta-heuristic algorithms, is more desirable for solving advanced problems, especially when conventional algorithms fail, to avoid such difficulties. In addition, meta-heuristic algorithms, which are more flexible, simple, and derivation-free, ensure global convergence (Sreenivas and Kumar 2015). Recently, several distributions' parameters have been estimated using meta-heuristic algorithms, including an efficient new one known as the whale optimization algorithm (WOA) that was used by Mohammed (2021), Mohammed and Elmasry (2023), Al-Mhairat and Al-Quraan (2022), and many others in the literature. This study's main objective is to demonstrate the log-logistic distribution's usability in various fields and to estimate its scale and shape parameters. The main concern raised by this study is that explicit solutions to the likelihood equations do not exist for the log-logistic distribution. This issue is resolved by using iterative numerical techniques based on the NM algorithm as a conventional technique and the WOA as a meta-heuristic algorithm representing a non-conventional technique. The main contribution of this work is a comprehensive Monte-Carlo simulation study to compare and analyze the two optimization techniques and to provide the estimator's values for the log-logistic distribution parameters based on the WOA and NM algorithms. The remaining sections of this work will be arranged as follows: The log-logistic distribution and its basic properties are addressed in Section 2. The ML estimation method for both NM and WOA is discussed in Section (3). In Section (4), an extensive Monte-Carlo simulation study is performed to compare the performance of the parameter estimators. In Section (5), two real-world dataset applications are implemented. The study presents several conclusions in Section (6).

2. Log-logistic Distribution

If we have two random variables, X and Y, whose relationship is represented by Equation 1:

$$Y = \beta \ln\left(\frac{X}{\alpha}\right), \qquad \beta > 0, \qquad \alpha > 0, \tag{1}$$

where β is the shape parameter and α is the scale parameter, and Y is distributed logistically with the following (Equation 2) probability density function (pdf),

$$g(y) = \frac{e^y}{(1+e^y)^2}, \qquad y \in R. \tag{2}$$

Then X follows a two parameter log-logistic distribution, X~*log-logistic* (α, β), with the following (Equation 3) probability density function (pdf),

$$f(x;\alpha,\beta) = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^2}, \qquad x > 0, \beta > 0, \alpha > 0. \tag{3}$$

X's cumulative distribution function (Equation 4) (cdf) is:

$$F(x;\alpha,\beta) = \frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}} \tag{4}$$

The plots of the log-logistic distribution for different values of β with fixed value of α =2 are shown in Figure 1. The log-logistic distribution's k^{th} moment exists only when k is less than β, and its general equation is given in Equation 5:

$$E(X^k) = \alpha^k B\left(1-\frac{k}{\beta}, 1+\frac{k}{\beta}\right) = \frac{\alpha^k\left(\frac{k\pi}{\beta}\right)}{\sin\left(\frac{k\pi}{\beta}\right)} \tag{5}$$

where B is the beta function. By using equation 5, The mean and variance of the random variable X can be calculated as follows (Equation 6-7):

$$E(X) = \frac{\alpha\pi}{\beta\sin\left(\frac{\pi}{\beta}\right)}, \qquad \beta > 1 \tag{6}$$

$$Var\ (X) = \alpha^2\left(\frac{2\left(\frac{\pi}{\beta}\right)}{\sin\left(\frac{2\pi}{\beta}\right)} - \frac{\left(\frac{\pi}{\beta}\right)^2}{sin^2\left(\frac{\pi}{\beta}\right)}\right), \qquad \beta > 2 \tag{7}$$

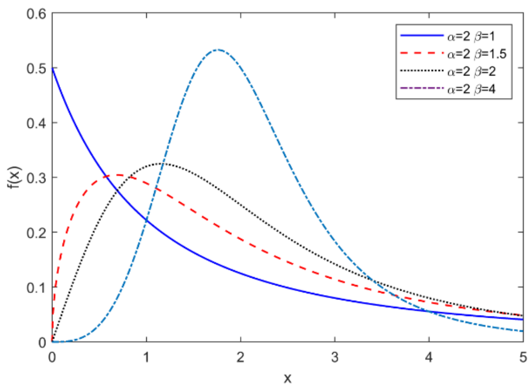

Figure 1. Log-logistic pdf for different parameter values for different values of β with fixed value of α =2

3. Maximum Likelihood Estimation

This method relies on identifying the values that maximize the likelihood function to its maximum; often, the logarithm of the likelihood function is used in order to simplify the calculations. The log-likelihood ($log\ L$) function is provided in Equation 8 to estimate the unknown parameters for the log-logistic distribution in this study.

$$\log L(\alpha,\beta) = nlog(\beta) - n\beta log(\alpha) + (\beta-1)\sum_{i=1}^{n} log(x_i) - 2\sum_{i=1}^{n} log\left[1+\left(\frac{x_i}{\alpha}\right)^{\beta}\right] \tag{8}$$

The partial derivatives corresponding to the considered parameters are obtained and set to zero to estimate the likelihood parameters of the $log\ L$ function for the log-logistic distribution. The likelihood equations are given as in Equation 9 and 10:

$$\frac{\partial \log L(\alpha,\beta)}{\partial \alpha} = \frac{n\beta}{\alpha} + \frac{2\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} + \left[1+\left(\frac{x_i}{\alpha}\right)^{\beta}\right]^{-1} = 0 \quad (9)$$

and

$$\frac{\partial \ln L(\alpha,\beta)}{\partial \beta} = \frac{n}{\beta} - n\log(\alpha) + \sum_{i=1}^{n}\log(x_i) - 2\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\log\left(\frac{x_i}{\alpha}\right)\left[1 + \left(\frac{x_i}{\alpha}\right)^{\beta}\right]^{-1} = 0 \quad (10)$$

Since the likelihood equations include nonlinear functions, as shown by equations (9) and (10), it is unlikely to find explicit solutions to them. Therefore, to solve these equations and obtain ML estimates for α and β, iterative numerical techniques are required. WOA and NM are two highly efficient algorithms that are used in this study as numerical techniques for estimating the likelihood estimators for the log-logistic distribution, and they are briefly introduced in the next few subsections.

3.1. Whale Optimization Algorithm (WOA)

The WOA is a new intelligent meta-heuristic algorithm that was developed by Mirjalili and Lewis (2016), in their study. It is modeled after the imitation of the humpback whale's bubble-net hunting technique, which involves creating a circle of bubbles around the prey and narrowing it or approaching the target in a spiral pattern while performing a random search, as discussed in Rana et al. (2020) as well as Hu et al. (2016) studies. Encircling the prey, the bubble-net attack mechanism, and the search for prey are the main three phases of this algorithm, and each phase is mathematically modeled (Yan et al., 2018). The population of humpback whales starts their search through a multi-dimensional search space, and their initial positions at the first iteration are represent the initial solutions in WOA. For mathematical modeling, there are many significant parameters should be known first, such as the following:

- Parameter (a), which is an essential parameter, declines linearly for each iteration from 2 to 0. The Equation 11 to obtain this parameter is:

  $$a = 2 * \left(1 - \frac{t}{T_{max}}\right) \quad (11)$$

  where t, is the current iteration and T_{max} is the total number of iterations.
- A and C coefficient vectors, whose equations 12 and 13 are:

  $$A = 2a * r_1 - a \quad (12)$$

  $$C = 2 * r_2 \quad (13)$$

  where r_1 and r_2 are random vectors ranging in the closed interval [0,1].
- Parameter (b), which is a constant that determines how the logarithmic spiral is shaped, Parameter (l) is a number chosen at random from the range [-1, 1], and parameter, p is a chance probability that can be any value between 0 and 1, to give an equal chance of encircling or spiraling movements of whales.

The fitness value for the main study's model, represented by the $log\ L$ function (8), is used to evaluate each whale position, and the best position is then determined and stored. When P<0.5 and $|A|$<1, the currently in progress whale's position is updated using the following Equations 14 and 15:

$$D = \left|C.X_p(t) - X(t)\right| \quad (14)$$

$$X(t+1) = X_p(t) - A * D \quad (15)$$

where the current vector's position at iteration t is represented by $X\ (t)$, and when iterating to the t^{th} time, the best solution's position vector is $X_p(t)$.

However, in case $|A|$>1, one of the whales is picked at random, and the position is updated by applying the Equations 16 and 17 follow as:

$$D = \left|C.X_{rand}(t) - X(t)\right| \quad (16)$$

$$X(t+1) = X_{rand}(t) - A * D \quad (17)$$

where X_{rand} is the position vector of any whale picked at random from the current whale population. On the other hand, the following Equations 18 and 19 update the position of the current whale when P>0.5:

$$D' = \left|X_p(t) - X(t)\right| \quad (18)$$

$$X(t+1) = D'e^{bl}cos(2\pi l) + X_p(t) \quad (19)$$

where D' indicates the distance between both the i^{th} whale and the available current best whale position (prey). Every iteration of WOA involves checking the updated whale's position to make sure it remains within the boundaries of the search space. The final solution positions indicate the values of WOA estimators, and this process continues until it reaches the final iteration needed to achieve convergence. Figure 2 shows the WOA's flowchart.

3.2. Nelder-Mead (NM)

The NM algorithm is a widely used deterministic search technique for locating optimal approximations of a fitness function in a space of multiple dimensions. It had been founded by the study of John Nelder and Roger Mead (1965). The NM algorithm depends on creating a geometric simplex figure with $n+1$ vertex for n-dimensional problems. In this study, the function that needs to be minimized is $f\ (\theta) = -\log L(\theta)$, where $\theta = (\alpha,\beta) \in R \times R^{+}$. At each vertex, the fitness value $f(\theta)$ is calculated and put in ascending order as $\theta_1, \theta_2, \theta_3$, and then a mechanism for generating a new simplex by replacing the vertex that has the highest fitness value is done by applying four main operators. These operators are:

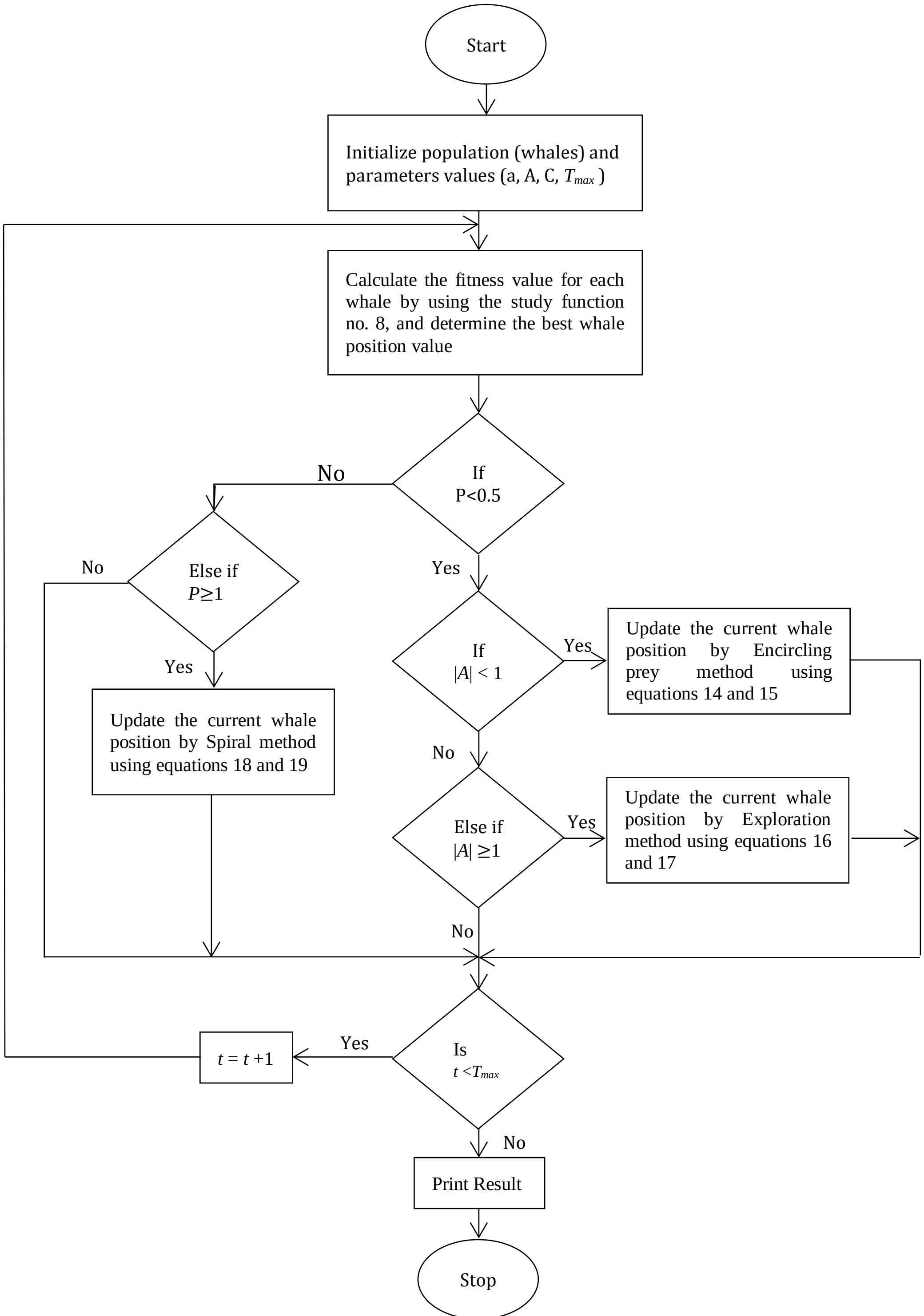

Figure 2. WOA flowchart.

- Reflection, for generating reflection points θ_r, which is defined as $\theta_r = \theta_0 + \alpha(\theta_0 - \theta_3)$, where $\theta_0$is the centroid, and α is the reflection coefficient. When $f(\theta_1) \leq f(\theta_r) \leq f(\theta_2)$ then θ_3 is replaced with θ_r.
- Expansion, for generating the expansion point $\theta_e$and its related equation $\theta_e = \theta_0 + \gamma(\theta_r - \theta_0)$, where γ is the expansion coefficient. The use of expansion points happens when $f(\theta_r) < f(\theta_1)$ then θ_3 is replaced with θ_e if not, θ_0 is replaced with θ_r
- Contraction, for generating the contraction point θ_c, which is defined as $\theta_c = \theta_0 + \rho(\theta_3 - \theta_0)$, where ρ is the contraction coefficient. The use of expansion points happens when $f(\theta_2) \leq f(\theta_r)$, and if $f(\theta_c) < f(\theta_3)$ then a new simplex is generated with θ_3 is replaced with θ_c but if $f(\theta_c) > f(\theta_3)$ then the initial points will be shrinked by applying this equation $\theta_i = \theta_i + \beta(\theta_i - \theta_1)$ to all $i \in \{2,3\}$

This process is continuing until the convergence requirements are satisfied. According to many studies in the literature α, γ, ρ, and β are taken 1, 2, $\frac{1}{2}$, and $\frac{1}{2}$, respectively. The studies by Everitt (1984), Shamir (1987), Gao & Han (2012), and Kucukdeniz & Esnaf (2018) in the literature provide more information.

4. Monte-Carlo Simulations Study

In order to evaluate the efficiency of the ML estimator values for the model parameters using the WOA algorithm with the corresponding ML estimators using the NM algorithm, the numerical results from the Monte Carlo simulation study for various sample sizes are presented in this section of the paper. Matlab R2021a software is used to perform the computations for the simulation study. Every Monte Carlo simulation run is repeated 2,000 times. The scale parameter α remains constant at 1.0, but the shape parameter β is taken to be 1, 1.5, and 2, for a variety of sample sizes n that are assumed to be 10, 20, 30, 50, 100, 250, and 500. The range [0, 20] is chosen as the search space (SS) for the parameters, and as a result, 7×4×2000 = 56000 unique samples are produced. The simulations' final results for the shape and scale parameters are indicated by the symbols $\hat{\alpha}$ and $\hat{\beta}$, respectively. The simulated mean, bias, variance, mean square error (MSE), and deficiency (Def) values provided by the equations 20-24 below are used in order to compare and evaluate the performance of the estimators (Equation 20-24).

$$Mean\left(\hat{\theta}\right) = \frac{\sum_i^n \hat{\theta}_i}{n} \tag{20}$$

$$Bias\left(\hat{\theta}\right) = E\left(\hat{\theta}\right) - \theta \tag{21}$$

$$Var\left(\hat{\theta}\right) = \frac{1}{n-1}\sum_{i=1}^{n}\left(\hat{\theta}_i - Mean\ \hat{\theta}\right)^2 \tag{22}$$

$$MSE\left(\hat{\theta}\right) = Var\left(\hat{\theta}\right) + \left(Bias\left(\hat{\theta}\right)\right)^2 \tag{23}$$

$$Def\left(\hat{\alpha}, \hat{\beta}\right) = MSE(\hat{\alpha}) + MSE\left(\hat{\beta}\right) \tag{24}$$

where, $\theta = (\alpha, \beta) \in$ R×R⁺. The resulting simulated values of mean, bias, MSE, and Def for $\hat{\alpha}$ and $\hat{\beta}$are given in Tables 1-3. In contrast to the NM algorithm, the simulated values demonstrate that the WOA produces the best performance. The values of the shape estimator $\hat{\beta}$ when α = 1 and β = 1, have the least bias values for the NM technique, in accordance with the simulated results, with the exception of when the sample size is large (n = 500), in which case the WOA produces the least biased results as shown in Tables 1. However, tables 2 and 3 show that in all other remaining cases, the WOA produces the least biased results for $\hat{\alpha}$ and $\hat{\beta}$. In terms of MSE values, Tables 1-3 demonstrate that when α = 1 and β = 1, simulated MSE values for $\hat{\beta}$ that belong to the NM algorithm are better than the WOA for all n values; otherwise, MSE values for the WOA outperform the NM algorithm in all other cases for $\hat{\alpha}$ and $\hat{\beta}$ estimators. The WOA shows excellence performance according to the Def criteria with the smallest values in contrast to the NM algorithm for all cases. All of this allows us to say that the WOA is an efficient algorithm for estimating the shape $\hat{\beta}$ and scale $\hat{\alpha}$ parameters for log-logistic distributions, and as a result of the NM algorithm's deficiency criterion values, it is inefficient for this distribution.

Table 1. Simulated mean, bias, variance, MSE, and Def values for the ML estimators $\hat{\alpha}$ and $\hat{\beta}$.

		$\hat{\beta}$			$\hat{\alpha}$			
n	Method	Mean	Bias	MSE	Mean	Bias	MSE	Def
$\alpha = 1, \beta = 1$								
10	WOA	1.2775	0.1537	0.2307	0.9861	0.2690	0.2692	0.4999
	NM	1.2285	0.1301	0.1823	0.5500	0.9439	1.1464	1.3288
20	WOA	1.1874	0.0517	0.0868	0.9256	0.1133	0.1188	0.2056
	NM	1.1473	0.0386	0.0603	0.3715	0.7593	1.1542	1.2145
30	WOA	1.1575	0.0303	0.0551	0.8987	0.0728	0.0830	0.1382
	NM	1.1241	0.0214	0.0368	0.2971	0.7316	1.2257	1.2625
50	WOA	1.1474	0.0166	0.0384	0.8977	0.0401	0.0506	0.0889
	NM	1.1187	0.0134	0.0275	0.2272	0.7250	1.3222	1.3497
100	WOA	1.1315	0.0079	0.0252	0.8861	0.0197	0.0326	0.0578
	NM	1.1081	0.0068	0.0185	0.1800	0.6466	1.3190	1.3375
250	WOA	1.1474	0.0166	0.0384	0.8977	0.0401	0.0506	0.0889
	NM	1.1187	0.0134	0.0275	0.2272	0.7250	1.3222	1.3497
500	WOA	1.1141	0.0015	0.0145	0.8805	0.0041	0.0184	0.0329
	NM	1.0908	0.0029	0.0111	0.0207	0.6256	1.5846	1.5957

Table 2. Simulated mean, bias, variance, MSE, and Def values for the ML estimators $\hat{\alpha}$ and $\hat{\beta}$.

		$\hat{\beta}$			$\hat{\alpha}$			
n	Method	Mean	Bias	MSE	Mean	Bias	MSE	Def
$\alpha = 1, \beta = 1.5$								
10	WOA	1.8014	0.3007	0.3915	1.0390	0.1628	0.1643	0.5558
	NM	1.6377	0.3465	0.3654	0.5746	0.7754	0.9564	1.3218
20	WOA	1.6600	0.1016	0.1272	1.0042	0.0666	0.0666	0.1938
	NM	1.4703	0.1383	0.1392	0.3761	0.7409	1.1302	1.2694
30	WOA	1.6122	0.0592	0.0718	0.9968	0.0427	0.0427	0.1145
	NM	1.4140	0.0927	0.1001	0.2665	0.7246	1.2627	1.3627
50	WOA	1.5932	0.0342	0.0429	0.9899	0.0264	0.0265	0.0694
	NM	1.3913	0.0691	0.0809	0.2288	0.6914	1.2861	1.3670
100	WOA	1.5695	0.0160	0.0209	0.9824	0.0122	0.0125	0.0333
	NM	1.3706	0.0494	0.0661	0.2015	0.6473	1.2850	1.3511
250	WOA	1.5629	0.0060	0.0100	0.9805	0.0051	0.0054	0.0154
	NM	1.3688	0.0424	0.0597	0.1837	0.6156	1.2820	1.3416
500	WOA	1.5564	0.0031	0.0063	0.9813	0.0024	0.0028	0.0091
	NM	1.3386	0.0378	0.0638	0.0752	0.5913	1.4466	1.5104

Table 3. Simulated Mean, Bias, Variance, MSE, and Def values for the ML estimators $\hat{\alpha}$ and $\hat{\beta}$.

		$\hat{\beta}$			$\hat{\alpha}$			
n	Method	Mean	Bias	MSE	Mean	Bias	MSE	Def
$\alpha = 1, \beta = 2$								
10	WOA	2.3451	0.5763	0.6954	1.0284	0.0831	0.0840	0.7793
	NM	2.0404	0.7601	0.7617	0.5743	0.6409	0.8221	1.5838
20	WOA	2.1536	0.1758	0.1993	1.0196	0.0396	0.0400	0.2394
	NM	1.8372	0.3546	0.3811	0.4727	0.6469	0.9250	1.3061
30	WOA	2.1083	0.1100	0.1217	1.0072	0.0265	0.0265	0.1483
	NM	1.7689	0.2832	0.3366	0.4049	0.6334	0.9876	1.3242
50	WOA	2.0673	0.0612	0.0657	0.9984	0.0150	0.0150	0.0807
	NM	1.6866	0.2318	0.3300	0.3023	0.6292	1.1160	1.4460
100	WOA	2.0483	0.0300	0.0323	0.9986	0.0076	0.0076	0.0398
	NM	1.6543	0.1981	0.3176	0.2666	0.6149	1.1528	1.4703
250	WOA	2.0356	0.0107	0.0120	0.9976	0.0030	0.0030	0.0150
	NM	1.6459	0.1718	0.2972	0.2711	0.5909	1.1223	1.4195
500	WOA	2.0245	0.0054	0.0060	0.9985	0.0014	0.0014	0.0074
	NM	1.6363	0.1636	0.2958	0.2589	0.5837	1.1329	1.4288

5. Applications

For the sake of illustration of the practical applicability of this model, two real datasets with various sample sizes are modeled using the log-logistic distribution in this section. Well-known criteria concerning log-likelihood values, the Akaike Information Criterion (AIC), and the corrected AIC (AICc) are used for comparing the modeling performance of the log-logistic distribution with the performance of several other different distributions, which include Gamma, Weibull, Lognormal, Nakagami, Inverse Gaussian, logistic, Normal, Rayleigh, and Extreme Value distributions. Anderson et al. (1998) provided in their study more information on these criteria and how they are utilized in practice. Mathematically, these criteria are represented in Equaition 25 and 26:

$$AIC = 2P - 2\log L \quad (25)$$

$$AIC_c = AIC + \frac{2P(P+1)}{n-P-1} \quad (26)$$

where the likelihood function, the number of observations, and the total number of model parameters are represented by $log\ L$, n, and p, respectively. When the probability model has the aforementioned criteria with lower values than other probability distributions, it is said to be the best-fit model.

5.1. Dataset

5.1.1. Remission time of bladders cancer patients

To demonstrate the implementation of the log-logistic distribution, the remission time of bladder cancer patients' dataset is used. Its 128 observations were originally analyzed by Lee and Wang in their study (2003) and used by other studies such as the study of Lemonte and Cordeiro (2011), the studies of Aldeni et al. (2017) as well as Ijaz et al. (2020) and Zea et al. (2012). The values of this dataset are as follows: 0.080, 0.200, 0.400, 0.500, 0.510, 0.810, 0.900, 1.050, 1.190,1.260,1.350, 1.400, 1.460, 1.760, 2.020, 2.020,2.070, 2.090, 2.230, 2.260, 2.460, 2.540, 2.620, 2.640, 2.690, 2.690, 2.750, 2.830, 2.870, 3.020,3.250, 3.310,3.360, 3.360, 3.480, 3.520, 3.570, 3.640, 3.700, 3.820,3.880, 4.180, 4.230, 4.260, 4.330, 4.340, 4.400, 4.500, 4.510, 4.870, 4.980, 5.060, 5.090, 5.170, 5.320, 5.320, 5.340, 5.410, 5.410, 5.490,5.620, 5.710, 5.850, 6.250, 6.540, 6.760, 6.930, 6.940, 6.970, 7.090, 7.260, 7.280, 7.320, 7.390, 7.590, 7.620, 7.630, 7.660, 7.870, 7.930,8.260, 8.370, 8.530, 8.650, 8.660, 9.020, 9.220, 9.470, 9.740, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05. Table 4 provides descriptive statistics, including sample size (n), minimum (min), mean, mode, median, maximum (max), variance (S2), skewness (γ_1), and kurtosis (γ_2).

Table 5 compares the modeling performance of the log-logistic distribution to that of other well-known distributions using the $log\ L$, AIC, and $AICc$ criteria.
The findings in Table 5 demonstrate that, in terms of the criteria taken into consideration, the log-logistic distribution's performance provides a better fit than other distributions.

5.2.2. A relief times (in minutes) of 20 patients receiving an analgesic

A relief time dataset of 20 patients receiving an analgesic was provided by Gross and Clark in their study (1975) and used in other statistical literature such as the studies of Shanker et al. (2016), Shukla (2019), and Marthin and Rao (2020). The dataset values are given as: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2. The descriptive statistics for this dataset are displayed in Table 6 and Table 7 shows the results obtained after fitting this data to the log-logistic distribution model and comparing them to other distributions according to the chosen criteria.

Table 4. The descriptive statistics for the remission time of bladders cancer patient’s data.

n	Min	Mean	Mode	Median	Max	S^2	γ_1	γ_2
128	0.08	9.3656	2.02	6.3950	79.05	1.1042	3.2866	18.4831

Table 5. Parameter estimates, $log\ L$, AIC and AICc, values for bladders cancer patient’s data.

	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\beta}$	$log\ L$	AIC	AIC_C
Log-logistic	-	6.0887	1.7254	411.4575	826.9150	827.011
Gamma	-	7.9877	1.1725	413.3680	830.7360	830.832
Weibull	-	9.5607	1.0478	414.0870	832.1740	832.27
Lognormal	1.7535	1.0773	-	415.0960	834.1920	834.288
Nakagami	-	197.2770	0.37420	426.6020	857.2040	857.3
I-G	-	9.3656	3.3820	440.3050	884.6100	884.706
logistic	7.5857	4.4825	-	456.6650	917.3300	917.426
Normal	9.3656	10.5083	-	486.2020	976.4040	976.5
Rayleigh	-	9.9317	-	491.2660	986.5320	986.628
E-V	15.8369	19.1518	-	549.1570	1102.3140	1102.41

I-G= inverse gaussian t, E-V = extreme value.

Table 6. The descriptive statistics for the relief times of 20 patients receiving an analgesic data.

n	Min	Mean	Mode	Median	Max	S^2	γ_1	γ_2
20	1.1	1.9	1.7	1.7	4.1	0.4958	1.7197	5.9241

Table 7. Parameter estimates, $log\ L$, AIC and AICc, values for bladders cancer patient’s data.

	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\beta}$	$-ln\ L$	AIC	AIC_C
Log-logistic	-	1.7525	5.8895	16.4766	36.9532	37.65908
I-G	-	1.9000	18.6978	16.7723	37.5446	38.25048
Lognormal	0.5893	0.3185	-	16.7806	37.5612	38.26708
Gamma	-	0.1965	9.6695	17.8186	39.6372	40.34308
Nakagami	-	4.0810	2.3478	19.1701	42.3402	43.04608
logistic	1.7905	0.3390	-	19.2433	42.4866	43.19248
Weibull	-	2.1300	2.7870	20.5864	45.1728	45.87868
Normal	1.9000	0.7041	-	20.8627	45.7254	46.43128
Rayleigh	-	1.4285	-	22.4788	48.9576	49.66348
E-V	2.2913	0.9163	-	26.7927	57.5854	58.29128

I-G= inverse gaussian t, E-V = extreme value.

The findings demonstrate that, in terms of modeling performance, the log-logistic distribution is better than all other distributions.

6. Conclusion

In this study, estimation of the shape and scale parameters of the log-logistic distribution is considered. The ML estimates of the log-logistic distribution parameters via two iterative algorithms which are NM and WOA are obtained and then their performances compared to each other with respect to bias, MSE and Def criteria by conducting a Monte Carlo simulation study. Simulation results show that the WOA is more efficient than NM iterative algorithm in terms of these criteria in all cases. Two real datasets are employed to fit the log-logistic distribution, and the results demonstrate high performance for the log-logistic distribution in comparison with many well-known statistical distributions.

Author Contributions

The percentage of the author(s) contributions is presented below. All authors reviewed and approved the final version of the manuscript.

	A.O.F.	P.K.
C	50	50
D	50	50
S	50	50
DCP	50	50
DAI	50	50
L	50	50
W	50	50
CR	50	50
SR	50	50
PM	50	50
FA	50	50

C=Concept, D= design, S= supervision, DCP= data collection and/or processing, DAI= data analysis and/or interpretation, L= literature search, W= writing, CR= critical review, SR= submission and revision, PM= project management, FA= funding acquisition.

Conflict of Interest

The authors declared that there is no conflict of interest.

Ethical Consideration

Ethics committee approval was not required for this study because of there was no study on animals or humans.

References

Abbas K, Tang Y. 2016. Objective Bayesian analysis for log-logistic distribution. Commun Stat-Simul Comput, 45(8): 2782-2791. DOI: 10.1080/03610918.2014.925925.

Aldeni M, Lee C, Famoye F. 2017. Families of distributions arising from the quantile of generalized lambda distribution. J Stat Distribut Appl, 4: 1-18. DOI: 10.1186/s40488-017-0081-4.

Ali M, Khan A. 1987. On order statistics from the log-logistic distribution. J Stat Plan Infer, 17: 103-108. DOı: 10.1016/0378-3758(87)90104-2.

Al-Mhairat B, Al-Quraan A. 2022. Assessment of wind energy resources in Jordan using different optimization techniques. Processes, 10(1): 105. DOI: 10.3390/pr10010105.

Anderson D, Burnham K, White G. 1998. Comparison of Akaike information criterion and consistent Akaike information criterion for model selection and statistical inference from capture-recapture studies. J Appl Stat, 25(2): 263-282. DOI: 10.1080/02664769823250.

Balakrishnan N, Malik H. 1987. Moments of order statistics from truncated log-logistic distribution. J Stat Plann Infer, 17: 251-267. DOI: 10.1016/0378-3758(87)90117-0.

Burr I. 1942. Cumulative frequency functions. Annals Math Stat, 13(2): 215-232. DOI: 10.1214/aoms/1177731607.

Everitt B. 1984. Maximum likelihood estimation of the parameters in a mixture of two univariate normal distributions; a comparison of different algorithms. J Royal Stat Soc Series D: The Statistician, 33(2): 205-215. DOI: 10.2307/2987851.

Fisk P. 1961. The graduation of income distributions. Econometrica, 1961: 171-185. DOI: 10.2307/1909287.

Gao F, Han L. 2012. Implementing the Nelder-Mead simplex algorithm with adaptive parameters. Comput Optimiz Appl, 51(1): 259-277. DOI: 10.1007/s10589-010-9329-3.

Gross A, Clark V. 1975. Survival distributions: Reliability applications in the biometrical sciences. John Wiley, New York, US.

Hu H, Bai Y, Xu T. 2016. A whale optimization algorithm with inertia weight. WSEAS Trans Comput, 15: 319-326.

Ijaz M, Asim S, Alamgir, Farooq M, Khan S, Manzoor S. 2020. A Gull Alpha Power Weibull distribution with applications to real and simulated data. Plos One, 15(6): e0233080. DOI: 10.1371/journal.pone.0233080.

Kantam R, Srinivasa R. 2002. Log-logistic distribution: modified maximum likelihood estimation. Gujarat Stat Rev, 29(1): 25-36.

Kucukdeniz T, Esnaf S. 2018. Hybrid revised weighted fuzzy c-means clustering with Nelder-Mead simplex algorithm for generalized multisource Weber problem. J Enterp Info Manag, 31(6): 908-924. DOI: 10.1108/JEIM-01-2018-0002.

Kus C, Kaya M. 2006. Estimation of parameters of the loglogistic distribution based on progressive censoring using the EM algorithm. Hacettepe J Math Stat, 35(2): 203-211.

Lee E, Wang J. 2003. Statistical methods for survival data analysis, John Wiley & Sons, New York, US, pp: 476.

Lemonte A, Cordeiro G. 2011. An extended Lomax distribution. Statistics, 47(4): 800-816. DOI: 10.1080/02331888.2011.568119.

Marthin P, Rao G. 2020. Generalized Weibull–Lindley (GWL) distribution in modeling lifetime data. J Math, 2020: 2049501. DOI: 10.1155/2020/2049501.

Mirjalili S, Lewis A. 2016. The whale optimization algorithm. Adv Eng Software, 95: 51-67. DOI: 10.1016/j.advengsoft.2016.01.008.

Mohammed W, Elmasry W. 2023. A comparative assessment of five different distributions based on five different optimization methods for modeling wind speed distribution. Gazi Univ J Sci, 2023: 1-1. DOI: 10.35378/gujs.1026834.

Mohammed W. 2021. Five different distributions and metaheuristics to model wind speed distribution. J Thermal Eng, 7(14): 1898-1920. DOI: 10.14744/jten.2021.xxxx.

Nelder J, Mead R. 1965. A simplex method for function minimization. Comput J, 7(4): 308-313. DOI:

10.1093/comjnl/7.4.308.

Pratihar D. 2012. Traditional vs non-traditional optimization tools. In: Basu K (ed) Computational Optimization and Applications, Narosa Publishing House Pvt. Ltd, New Delhi, India, pp: 25-33.

Rana N, Latiff M, Abdulhamid S, Chiroma H. 2020. Whale optimization algorithm: a systematic review of contemporary applications, modifications and developments. Neural Comput Appl, 32(20): 16245-16277. DOI: 10.1007/s00521-020-04849-z.

Shah B, Dave P. 1963. A note on log-logistic distribution. J MS Univ Baroda, 12: 15-20.

Shamir R. 1987. The efficiency of the simplex method: a survey. Manag Sci, 33(3): 301-334. DOI: 10.1287/mnsc.33.3.301.

Shanker R, Shukla K, Shanker R, Leonida T. 2016. On modeling of lifetime data using two-parameter Gamma and Weibull distributions. Biomet Biostat Int J, 4(5): 201-206. DOI: 10.15406/bbij.2016.04.00107.

Shoukri M, Mian I, Tracy D. 1988. Sampling properties of estimators of the log-logistic distribution with application to Canadian precipitation data. Canadian J Stat, 16(3): 223-236. DOI: 10.2307/3314729.

Shukla K. 2019. A comparative study of one parameter lifetime distributions. Biomet Biostat Int J, 8(4): 111-123. DOI: 10.15406/bbij.2019.08.00280.

Sreenivas P, Kumar S. 2015. A review on non-traditional optimization algorithm for simultaneous scheduling problems. J Mech Civil Eng, 12(2): 50-53. DOI: 10.9790/1684-12225053.

Tadikamalla P, Johnson N. 1982. Systems of frequency curves generated by transformations of logistic variables. Biometrika, 69(2): 461-465. DOI: 10.1093/biomet/69.2.461.

Tadikamalla P. 1980. A look at the Burr and related distributions. Int Stat Rev, 337-344. DOI: 10.2307/1402945.

Yan Z, Wang S, Liu B, Li X. 2018. Application of whale optimization algorithm in optimal allocation of water resources. E3S Web of Conferences, 53: 04019. EDP Sciences. DOI: 10.1051/e3sconf/20185304019.

Yuan K, Schuster C. 2013. Overview of statistical estimation methods. The Oxford handbook of quantitative methods. Oxford, UK. DOI: 10.1093/oxfordhb/9780199934874.013.0018.

Zea L, Silva R, Bourguignon M, Santos A, Cordeiro G. 2012. The beta exponentiated Pareto distribution with application to bladder cancer susceptibility. Int J Stat Probab, 1(2): 8. DOI: 10.5539/ijsp.v1n2p8.