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Homoderivations and Their Impact on Lie Ideals in Prime Rings

Research Article

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Abstract

Assume we have a prime ring denoted as R, with a characteristic distinct from two. The concept of a homoderivation refers to an additive map H of a ring R that satisfies the property $\mathrm{H}(r_1r_2) = \mathrm{H}(r_1)r_2 + r_1\mathrm{H}(r_2) + \mathrm{H}(r_1)\mathrm{H}(r_2), \forall r_1, r_2 \in R$. This article aims to obtain results for prime rings, ideals, and Lie ideals by utilizing the concept of homoderivation in conjunction with the established theory of derivations.
Keywords: Prime ring, Homoderivation, Lie ideal, Jordan ideal.

1. INTRODUCTION

Let R be a ring, with $Z(R)$ denoting its center. The commutator of r_1 and r_2 in R is denoted by $[r_1, r_2]$, while the anti-commutator is denoted by $r_1 \circ r_2 = (r_1, r_2)$. A nontrivial additive subset U within the ring R is referred to as a Lie ideal if it satisfies the condition $[U, R] \subset U$. A well-known result due to Bergen et al. (1981) is that if U is a Lie ideal of R such that $U \not\subseteq Z(R)$, then there exists an ideal I in R satisfying $[I, R] \subset U$ but $[I, R] \not\subseteq Z(R)$. Furthermore, in accordance with Sofy (2000), for any subset $A \subseteq R$, a function is $g: R \to R$ considered to preserve A if $g(A) \subseteq A$, and it is termed zero-power valued on A if it meets the criteria of preserving A, with the additional requirement that for each $a \in A$, there exists a positive integer $n(x) > 1$ such that $g^{n(x)} = 0$.

In the ring theory, derivations are additive mappings that satisfy the condition

$$D(r_1r_2) = D(r_1)r_2 + r_1D(r_2)$$

for all $r_1, r_2 \in R$.

The concept of a generalized derivation was introduced by Brēsar ([18]) in the following manner: An additive mapping is referred to as a generalized derivation if there is a derivation ∂ such that

$$G(r_1r_2) = G(r_1)r_2 + r_1\partial(r_2)$$

for all $r_1, r_2 \in R$.

Throughout the last three decades, numerous authors have contributed to the establishment of the most fundamental theorems in ring theory by incorporating automorphisms or derivatives and proving commutation theorems for prime rings, semiprime rings, or suitable subsets. As the definition of derivation evolved, the field of study expanded to include generalized derivations, (α,β)-derivations, and semi-derivations. Recently, El Sofy proposed a new definition of homoderivations, which includes derivations as a special case [8]. By this definition,

$$h(r_1r_2) = h(r_1)h(r_2) + h(r_1)r_2 + r_1h(r_2)$$

is a homoderivation on R as additive mapping $h: R \to R$ for all $r_1, r_2 \in R$. An example of a homoderivation is $h(r) = \rho(r) - r$ for all $r \in R$, where ρ is an endomorphism on R. A recent study by Sofy showed that if $h(r)h(s) = 0$ for all $r, s \in R$, then $h = 0$, and Rehman et al. extended this result to I ideals of R [16].

In recent years, researchers have been exploring various generalizations of homoderivations on rings. One such generalization is the notion of generalized homoderivations, which extend the classical concept of homoderivations to noncommutative rings [1, 6, 14, 16]. As time progresses, one observes that generalizations of this derivative type are also beginning to be introduced [3, 7]. This paper explores implications for homoderivations of Lie ideals that have not been previously examined in the literature. Additionally, we have included some of the results obtained for ideals. Our work is motivated by several fundamental theorems in ring theory that involve automorphisms or derivatives and establish commutation theorems for prime rings, semiprime rings, or suitable subsets. Some notable inspirations for our work include ([15, 11, 12, 4]). We hope that our findings will contribute to the ongoing generalizations of the concept of homoderivations and their applications in ring theory.

2. RESULTS

In this section, we examine some basic properties using homoderivations under the conditions we assume for the ideals of the ring, and in particular for the Lie ideals.

Lemma 2.1. [5] A group cannot be represented as the union of two of its proper subgroups.

Lemma2.2. [13] Let b and ab be in the center of a prime ring R. If b is not zero, then a is in $Z(R)$.

Lemma2.3. [8] Let $h: R \to R$ be a homoderivation. If $h^2(R) = 0$ then $h = 0$.

Lemma2.4. Let h_1 and h_2 be two nonzero homoderivations on R. If $h_1h_2(R) = 0$ then $h_1 = 0$ or $h_2 = 0$.

Proof. Let

$$h_1h_2(rs) = 0$$

for all $r, s \in R$. $0 = h_1\big(h_2(rs)\big) = h_1\big(h_2(r)s + rh_2(s) + h_2(r)h_2(s)\big) = h_2(r)h_1(s) + h_1(r)h_2(s)$. Hence, replacing $h_2(r)$ by r,

$$h_2^2(r)h_1(s) = 0 \quad (1)$$

for all $r, s \in R$. By putting st instead of s, we have $h_2^2(r)sh_1(t) = 0$ for all $r, s, t \in R$. Thus,

$$h_2^2(R)Rh_1(R) = 0$$

Since R is a prime ring and from Lemma 2.3 it becomes $h_1 = 0$ or $h_2 = 0$.

Lemma 2.5. [8] Let h be a homoderivation of R. If $[h(R), a] = 0$ then $h = 0$ or $a \in Z(R)$.

Here are two apparent corollaries stemming from this lemma:

Corollary 2.6. Let h be a homoderivation of R. If $h(R) \subset Z(R)$ then $h = 0$ or R is commutative.

Corollary 2.7. Let h be a homoderivation and U be a Lie ideal of R. If $[h(R), U] = 0$ then $h = 0$ or $U \subset Z(R)$.

Lemma 2.8. [8] Assume R is a prime ring with a non-trivial two-sided ideal $I \neq (0)$. If R contains a nonzero homo derivation h that both commutes and exhibits zero power valuation on I, then R must be commutative.

Theorem 2.9. Let I and L be nonzero ideals of R. If h is a zero-power valued nonzero homoderivation on I and L such that $h(I) \subset Z(L)$ then R is commutative.

Proof. Let $h(I) \subset Z(L)$. According to this hypothesis $[h(xr), t] = 0, x \in I, t, r \in L$. Thus

$$0 = [h(x)h(r) + h(x)r + xh(r), t] = h(x)[h(r), t] + h(x)[r, t] + x[h(r), t] + [x, t]h(r) \quad (2)$$

In equation (2) replace r by t to get

$$0 = h(x)[h(t), t] + x[h(t), t] + [x, t]h(t) \quad (3)$$

Now, if we take rx instead of x in the Equation (3) equation, we find $0 = h(rx)[h(t), t] + rx[h(t), t] + [rx, t]h(t) = h(r)x[h(t), t] + rh(x)[h(t), t] + h(r)h(x)[h(t), t] + rx[h(t), t] + r[x, t]h(t) + [r, t]xh(t) = h(r)x[h(t), t] + h(r)h(x)[h(t), t] + [r, t]xh(t)$. If we choose to use t instead of r in this last equation

$$h(t)\big(x + h(x)\big)[h(t), t] = 0 \quad (4)$$

Since h is zero-power valued on I, there exists an integer $n(x) > 1$ such that $h^{n(x)} = 0$, for all $x \in I$. Replacing x by $x - h(x) + h^2(x) + \cdots \ldots + (-1)^{n(x)-1}h^{n(x)-1}$ in Equation (4), for all $x \in I, t \in L$

$$h(t)x[h(t), t] = 0$$

We put $K_1 = \{t \in L | h(t) = 0\}$ and $K_2 = \{t \in L | [h(t), t] = 0\}$. K_1 and K_2 both are additive subgroups of L. Through the Lemma 2.1 we arrive at $K_1 = L$ or $K_2 = L$. It is clear that if $K_1 = L$ then $h(L) = 0$. Since h is a nonzero homoderivation $[h(t), t] = 0$ for all $t \in L$ we see from Lemma 2.8 R is commutative.

Lemma 2.10. Let $h: R \to R$ be a nonzero homoderivation, a be a fixed element of R and $h(R \circ a) = 0$. Then $h(a) = 0$ or $a \in Z(R)$

Proof. Let $h(R \circ a) = 0$. Hence

$$0 = h(rs \circ a) = h(r(s \circ a) - [r,a]s)$$

for all $r, s \in R$. Replacing r by a we have $h(a)(s \circ a) = 0$. Replacing s by sx then $0 = h(a)(sx \circ a) = h(a)s[x,a]$ for all $x, s \in R$. We obtain $h(a) = 0$ or $a \in Z(R)$.

Lemma 2.11. Let $h: R \to R$ be a nonzero homoderivation and I be a nonzero ideal of R. If a and b are fixed elements of R such that $bh(I \circ a) = 0$ then $h(a) = 0$ or $b[b,a]$=0.

Proof. If $bh(I \circ a) = 0$ then $0 = bh\big((xa, a)\big) = bh\big((x,a)a\big) = b(x,a)h(a)$. From [10] $h(a) = 0$ or $b[b,a]$=0.

Lemma 2.12. Let h be a nonzero homoderivation, I be a nonzero ideal of R and μ be an automorphism of R. If a is a fixed element of R such that $h\mu(I,a) = 0$ then $a \in Z(R)$ or $h\mu(a) = 0$.

Proof. For all $x \in I$, $h\mu(xa, a) = h\mu\big((x,a)a\big) = h\big(\mu(x,a)\mu(a)\big) = h\big(\mu(x,a)\big)h\mu(a) + h\big(\mu(x,a)\big)\mu(a) + \mu(x,a)h\mu(a)$. So, we have $\mu(I,a)h\mu(a) = 0$. From [10], $a \in Z(R)$ or $h\mu(a) = 0$.

Lemma 2.13. Let $h: R \to R$ be a homoderivation and a be a fixed element of R. If $h([R,a]) = 0$ then $h = 0$ or $a \in Z(R)$.

Proof. If $h([R,a]) = 0$, for all $r \in R$, $h([ar,a]) = h(a)[r,a] = 0$. If we replace r with rx, $x \in R$

$$h(a)R[R,a] = 0$$

Since R is prime, we find that $a \in Z(R)$ or $h(a) = 0$. If $h(a) = 0$ then $0 = h([r,a]) = [h(r),a]$, for all $r \in R$. Thus $[h(R),a] = 0$. We obtain by Lemma 2.5 $h = 0$.

Corollary 2.14. Let $h: R \to R$ be a homoderivation and let U be a Lie ideal of R. If $h([R,U]) = 0$ then $h = 0$ or $U \subset Z(R)$.

Proof. Applying Lemma 2.13 and Corollary 2.7 sequentially makes the proof evident.

Lemma 2.15. Let $h: R \to R$ be a homoderivation. If I is a nonzero right ideal of R and for all $x \in I$, if $h(x) = x$, then $h(t) = 0$ for all $t \in R$.

Proof. By the hypothesis that for any $x \in I$, $t, s \in R$, $h(xst) = xst$, on the other hand, if we apply h homoderivation in this expression $h(xst) = h(xs)t + h(xs)h(t) + xsh(t) = xst$. Again, from our hypothesis we get $h(xs)h(t) + xsh(t) = 0$ and hence $2xsh(t) = 0$.
Since $charR \neq 2$, $xsh(t) = 0$. I is a nonzero ideal from hypothesis then $h(t) = 0$ for all $t \in R$.

Lemma 2.16. Let $h: R \to R$ be a nonzero homoderivation and let U be a Lie ideal on R. If $h(u) = u$ for all $u \in U$, then $U \subset Z(R)$.

Proof. By the hypothesis that $h(u) = u$ for all $u \in U$ we have $h([u,r]) = [u,r]$ for all $r \in R, u \in U$. Hence $h([u,r]) = [h(u),r] + [u,h(r)] + [h(u),h(r)] = [u,r]$. Now we get for all $r \in R, u \in U$

$$2[u,h(r)] = 0.$$

Since $charR \neq 2$we have $[U,h(R)]=0$. Hence from Corollary 2.7, $U\subset Z(R)$.

Lemma 2.17. [9] Let U be a Lie ideal of R. If $h(U)=0$ then $h=0$ or $U\subset Z(R)$.

Lemma 2.18. Let h be a nonzero homo derivation on R, a be a fixed element of R and θ be an automorphism on R. If $ah\theta(I)=0$, then $a=0$.

Proof. If $ah\theta(I)=0$ then $0=ah\theta(xr)=a\theta(x)h\theta(r)$. Thus $a\theta(I)h\theta(R)=0$. Since $\theta(I)$ is a nonzero ideal of R, we obtain $a=0$.

Theorem 2.19. Let h be a nonzero homoderivation a be a fixed element of R and let U be a noncentral Lie ideal on R. If $ah(U)=0$ ($h(U)a=0$) then $a=0$.

Proof. In our hypothesis, we assumed that U is a noncentral Lie ideal. Under this assumption, there exists an $I\neq 0$ ideal in R such that $[I,R]\subset U$ but $[I,R]\nsubseteq Z(R)$ as stated in [4]. Thus, for all $x\in R, m\in I$, the relation $[xm,m]=[x,m]m\in U$ holds true due to $[R,I]\subset U$. Consequently, based on our hypothesis, we can conclude that

$$0=ah([x,m]m)=a[x,m]h(m)=ah([x,m])h(m)=a[x,m]h(m)$$

$$a[x,m]h(m)=0. \tag{5}$$

Taking here $h(u)x$ for x in Equation (5), we have

$$0=a[h(u)x,m]h(m)=ah(u)[x,m]h(m)+a[h(u),m]xh(m)$$

for all $u\in U$, thus

$$a[h(U),m]Rh(m)=0.$$

for all $m\in I$. Let $S_1=\{m\in I|a[h(u),m]=0\}$ and $S_2=\{m\in I|h(m)=0\}$. Utilizing Lemma 2.1, we see either $h(I)=0$ or $a[h(U),I]=0$. But h is a nonzero homoderivation of R we obtain

$$a[h(U),I]=0$$

We have for all $m\in I$, $u\in U$, $a[h(u),m]=0$. Hence, we arrive at $a=0$ or $h(u)=0$. By Lemma 2.17 $a=0$ or $U\subset Z(R)$. Since $U\nsubseteq Z(R)$, we obtain $a=0$.

(If the method described above is used it can also be easily shown that the same result will be obtained when $h(U)a=0$.)

Theorem 2.20. Let $h: R\to R$ be a nonzero homoderivation and U be a noncentral Lie ideal on R. If $h(U)\subset Z(R)$, then $U\subset Z(R)$.

Proof. By Lemma 2.17 $h(U)\neq 0$. Let $r=[u,x]\in U$ for all $u\in U$, $x\in R$. then, $u[u,x]=ur\in U$. Thus

$$0=[h(ur),y]=h(u)[r,y]+[u,y]h(r)$$

$$h(u)[r,y]+[u,y]h(r)=0 \tag{6}$$

for all $u \in U$, $x, y \in R$. We substitute yu for y in this equation to obtain

$$h(u)[r, yu] + [u, yu]h(r) = h(u)y[r, u] + +h(u)[r, y]u + [u, y]uh(r) = 0$$

for all $u \in U$, $x, y \in R$. Since $h(r) \in Z(R)$ using the relation Equation (6) we obtain $h(u)y[r, u] = 0$. Hence

$$h(u)R[r, u] = 0$$

for all $u \in U$, $x \in R$ ($r = [u, x]$). Let $T_1 = \{u \in U | [u, x]u = 0\}$ and $T_2 = \{u \in U | h(u) = 0\}$. Through the T_1 and T_2 additive subgroups of R, from Lemma 2.1 and $h(U) \neq 0$, we arrive at

$$\big[[u, x], u\big] = 0$$

for all $u \in U$. If $U \not\subseteq Z(R)$ then $u \notin Z(R)$ for at least one $u \in U$. Let define a nonzero inner derivation $d_u: R \to R$ induced by u. Hence

$$d_u\big(d_u(x)\big) = \big[u, [u, x]\big] = 0$$

for all $x \in R$. That is $U \subset Z(R)$ by [17].

Example. Let $R = \left\{\begin{bmatrix} k_1 & k_2 \\ 0 & k_3 \end{bmatrix} | k_1, k_2, k_3 \in \mathcal{J}, the\ set\ of\ integers\right\}$ be a ring, on this case $U = \begin{bmatrix} k_1 & k_2 \\ 0 & k_1 \end{bmatrix}$ is a Lie ideal on R.

Let $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in R$. Defining h is follows:

$$h\begin{bmatrix} k_1 & k_2 \\ 0 & k_3 \end{bmatrix} = \begin{bmatrix} 0 & k_3 - k_1 \\ 0 & 0 \end{bmatrix}$$

It can be seen that h is a homoderivation. However, the condition of the above theorem is not satisfied. It is important that the ring is a prime.

Theorem 2.21. Let $h: R \to R$ be a nonzero homoderivation and U be a Lie ideal on R. If h preserves U and $h^2(U) = 0$ then $U \subset Z(R)$.

Proof. Assume that U is a noncentral Lie ideal on the ring R. By [4], $S = [U, U]$ is a noncentral Lie İdeal of R. If we show that $S \subset Z(R)$, we have what we want. From [4], there exists a nonzero ideal I ideal in R which satisfies the condition $[I, R] \subset U$ but at the same time $[I, R] \not\subseteq Z(R)$. For $m \in [I, R] \subset U \cap I$ and $u \in S$ we get $\omega = h(u) \in h(S) \subset U$. By the hypothesis $h(\omega) = 0$. Thus,

$$0 = h^2([m\omega, y]) = h^2(m[\omega, y] + [m, y]\omega)$$

$$= h\{h(m)[\omega, y] + mh[\omega, y] + h(m)h[\omega, y] + h[m, y]\omega + [m, y]h(\omega) + h[m, y]h(\omega)\}$$
$$= 2h(m)h[\omega, y]$$

for all $y \in R$. Since $charR \neq 2$, for all $m \in [I, R]$ gives $h(m)h[\omega, y] = 0$.

$$h([I,R])h[\omega,y]=0$$

for all $y\in R$. Since $[I,R]$ is a noncentral Lie ideal of R, by Lemma 2.19, we get

$$h[\omega,y]=h[h(u),y]=0$$

for all $u\in S$, $y\in R$.

$$0=h[h(u),y]=[h^2(u),y]+[h(u),h(y)]+[h^2(u),h(y)]=[h(u),h(y)]$$

and that is

$$[h(u),h(y)]=0$$

for all $u\in S$, $y\in R$. By Lemma 2.5 we have $h(u)\in Z(R)$, for all $u\in S=[U,U]$. That is $S=[U,U]\subset Z(R)$. Let's remember our acceptance $U\nsubseteq Z(R)$. Then there exist elements a and b in U such that neither of them belongs to the center of R $(Z(R))$. Now let's define two mappings $d_a(x)=[a,x]$ and $d_b(x)=[b,x]$ in R. Since $[a,x]\in U$ $\big[[a,x],b\big]\in Z(R)$. $\big[[a,x],b=\big[a,[x,b]\big]+\big[[a,b],x\big]=\big[a,[x,b]\big]\in Z(R)\in Z(R)\big]\in Z(R)$ and so, we have $d_ad_b(R)\subset Z(R)$. By Lee and Lee [9], we see that R is commutative. This result contradicts of $U\nsubseteq Z(R)$. Therefore, $U\subset Z(R)$.

Theorem 2.22. Let $h: R\to R$ be a nonzero homoderivation and U be a Lie ideal of R. If $[U,h(U)]\subset Z(R)$, then $U\subset Z(R)$.

Proof: Let v be an element of U. For all $u\in U$ we have $[u,h(v)]\in Z(R)$. Thus

$$0=\big[[u,h(v)],r\big]=\big[u,[h(v),r]\big]+\big[[u,r],h(v)\big]\in Z(R)$$

for all $r\in R$. Hence $\big[u,[h(v),r]\big]\in Z(R)$. For two inner derivations $d_1(x)=[u,x]$ and $d_2(x)=[h(v),x]$ by u and $h(v)$ of R, we have $d_1d_2(x)=[u,[h(v),x]]\in Z(R)$ for all $x\in R$. That is by [2] $d_1=0$ or $d_2=0$. Hence Theorem 2.20 yields that $U\subset Z(R)$.

4. CONCLUSION

In this article, algebraic identities are obtained, including homoderivations on prime rings. We also examine algebraic identities involving homoderivations for an ideal of the prime ring or the Lie ideal. We establish that the Lie ideal, conforming to the identities elaborated upon in this section, resides within the core of the prime ring. In future studies, the hypotheses in this study can be examined using homoderivations of the prime ring and Jordan ideals.

5. CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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