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RESEARCH ARTICLE

ROGUE WAVES IN DISCRETE KDV EQUATION

Semiha TOMBULOĞLU

Sağlık Hizmetleri MYO, Kırklareli Üniversitesi, Kırklareli, Türkiye

ABSTRACT

This study considers an array of waveguides described by a discrete KdV equation. Rogue wave solutions are numerically derived for the dKdV equation under periodic and non-vanishing boundary conditions. When the dKdV equation is solved numerically under periodic boundary conditions, a discrete rogue wave occurs due to shock front breaking. Furthermore, the dKdV equation has been solved numerically under non-vanishing boundary conditions, and it has been found that the rogue wave amplitude depends on the ρ_0 parameter.

Keywords: Rogue wave, Discrete KdV equation

1. INTRODUCTION

Huge amplitude waves, generally called 'rogue waves' or 'freak waves,' are a hot topic because of their exciting feature [1], [2]. One of its features is unpredictability; it appears without warning and disappears suddenly [3], [4]. In addition, their amplitude exceeds at least twice the surrounding background [5].

Scientists are considering the nonlinearity of the freak waves. They found that modulational instability (MI) (often called Benjamin-Feir instability) plays an essential role in explaining rogue wave structure [7], [8]. Because of periodic perturbation on a plane wave, MI has occurred and is critical in the nonlinear evaluation process. MI occurs in different physical systems such as fiber optic [32], supercontinuum generation [31], and plasmas [33], not only water waves [1]. Although researchers mostly used a numerical approach to investigate MI, the nonlinear Schrödinger (NLS) equation was solved analytically and called Akhmediev breather [10]. Tulin and Waseda conducted another study about MI in 1999 [11]. As can be seen in many studies, rogue waves show nonlinear features [12]. The dynamic becomes more complex when MI develops, and energy exchange is involved. Besides focusing on the NLS equation [13], [14], classical nonlinear evolution equations like the Korteweg de Vries equation [15], [16], [17], Gardner equation [18], Ablowitz Ladik and Hirota equations [19], and similar nonlinear equations can also describe rogue waves [20], [21].

J.S. Russell found the first solitary wave as a nonlinear coherent structure in 1834 [22]. The analysis was made theoretically by Rayleigh in 1876 [23] and Boussinesq in 1871 [24]. Korteweg De Vries formulated the KdV equation for shallow waters in 1895 [25]. It was not easy to find a physical application of the KdV equation until the 1960s. In 1965, Zabusky and Kruskal solved the KdV equation numerically for nonlinear mesh points and determined how long after the system returns to its initial state [26]. It is the pioneering work that reveals the presence of soliton solutions in this equation. We note that there is a connection between the KdV equation and the NLS equation [35]. NLS equation is derived from the KdV equation, or the inverse of this is possible. The general thought that the KdV equation was not able to explain rogue

*Corresponding Author: semihatombuloglu@klu.edu.tr
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wave solutions until recently [36]. This assumption is valid only if the wave described by the KdV equation is purely real. However, if we consider complex-valued solutions to the KdV equation, it becomes possible to obtain rogue wave solutions [35]. The KdV equation has a variety of solutions; indeed, as well as the classical KdV equation, it is possible to get rogue wave solutions in discrete systems.

Suppose that there is an array of waveguides. All the waveguide components are identical and also an equal distance from each other. This study considers such an array of waveguides that a discrete KdV equation can define. We study the dKdV equation numerically under periodic boundary conditions and non-vanishing (constant) boundary conditions to show that rogue wave(s) can occur in an array of waveguides. Firstly, we will numerically derive discrete rogue wave solutions for the dKdV equation under non-vanishing boundary conditions. To our knowledge, it is the first time that the dKdV equation is numerically solved for discrete rogue wave solutions. Then, we numerically solve the dKdV equation under periodic boundary conditions. We show that rogue waves occur not because of MI but because of shock front breaking. This is important since there is a general thought that rogue waves have nonlinear characteristics and occur primarily because of MI. This study can also enable the comparison of discrete rogue wave amplitudes derived from the dKdV and the discrete Schrödinger equations. The results of this comparison can shed light on rogue wave evolution.

2. DISCRETE ROGUE WAVES

The discrete KdV equation was given in [34] in the form

$$\frac{dw_j}{d1+w_j} = w_{j-1/2} + w_{j+1/2}$$

where w_j is the field variable at site j, and j is the dimensionless variable, t is time. As studied in [34], the equation reduces to the KdV equation in a weakly nonlinear limit. When we perform the differentiation of this equation and denote $\psi = 1 + w$, we get the discrete KdV equation, which was already derived by [29]

$$\frac{d\psi_j}{dt} = \left(\psi_{j-1} - \psi_{j+1}\right)\psi_j^2 \tag{1}$$

where ψ_j is the field amplitude at the jth waveguide and j=1,2,..., N, takes positive integers. Here, N is a number of lattice sites. If we assume that the wave amplitude is too small, we will obtain the KdV equation for which the absence of MI has been proved. Now, we numerically solve equation (1) with the properly given initial condition. Let us now start with the initial wave,

$$\psi_j(t=0) = A(1+\xi j)e^{-iA^2L} \tag{2}$$

where ξ is the very small parameter, $\xi << 1$ is the perturbation term, L is the length of each waveguide in the array, and A is the background intensity. In the case of $\xi \longrightarrow 0$ in equation (2), it defines a constant background whose amplitude equals A. Since these waves arise and recede to the constant background, we can solve equation (1) with non-vanishing boundary conditions. We perform numerical solutions to find the system's time evolution under non-vanishing boundary conditions. We take the non-vanishing boundary conditions as

$$\lim_{j\to\pm\infty} \psi(j,t) = \rho_0 \tag{3}$$

where $\rho_0$is a real constant here. Assume that the initial condition in equation (2) is subject to those non-vanishing boundary conditions. We solve the numerically discrete KdV equation to show that rogue wave occurs under non-vanishing boundary conditions.

Figure 1 shows us the spatial evaluation of the initial wave. The total energy is highly localized in a few waveguides located at the output end of the system. This concentration leads to the amplitude of the maximum value being nearly thirty times higher than the background amplitude when ρ_0= 0,4, accepting that background amplitude, κ, nearly equals 0,1, as seen in Figure 1. a. We see that even small perturbation affects the stability of this uniform solution. The perturbation is very small, beginning from the left side and getting larger while going on the right side. The distribution is perturbated, beginning from the t = 0. It grows in time until its mean peak reaches its maximum value. Then, its amplitude decreases, and finally, we see that this wave disappears at around t = 700. These features are compatible with rogue waves. We show that discrete rogue waves appear in discrete KdV equation under non-vanishing boundary conditions for the first time. Even non-vanishing boundary conditions do not describe the physical system; mathematically, discrete rogue wave evaluation can be found.

When $\rho_0$increases, rogue wave amplitude rises, as resulted in Figure 1. The absolute of the maximum amplitude is nearly sixty times higher than the background amplitude, κ=0,1, when ρ_0= 0,5 as shown in Figure 1. b. In comparison, the absolute of the maximum amplitude is nearly over a hundred times higher than the background amplitude, κ=0,1, when ρ_0= 0,6 as shown in Figure 1. c. We show that the real constant ρ_0, is important in terms of the rogue wave's amplitude, and even a small change in constant ρ_0 leads to a significant effect on it.

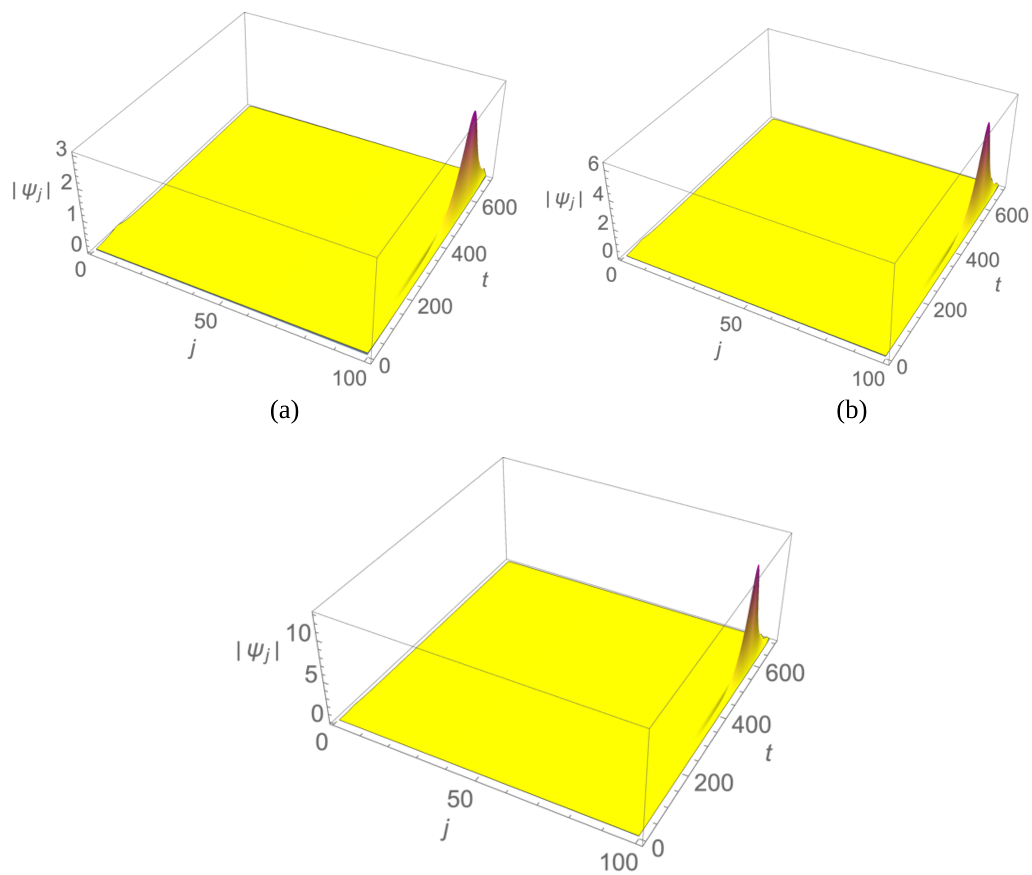
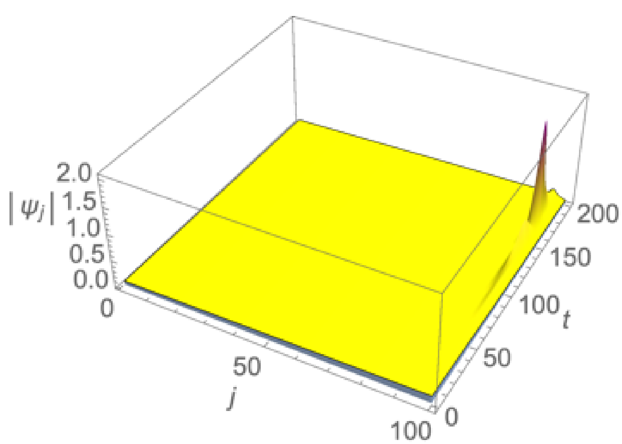


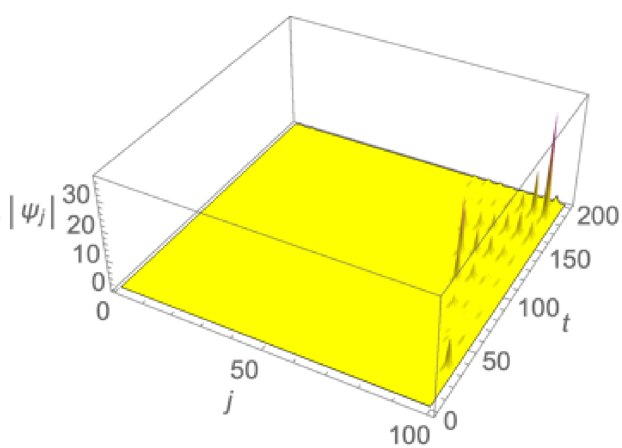
Figure 1: The absolute field amplitude for N=100, A=0.05, ξ= 0,007176, and L=100. The initial wave is given by equation (2), and boundary conditions are provided by equation (3). In Figure 1. a ρ_0= 0,4, Figure 1. b ρ_0= 0,5, Figure 1. c ρ_0= 0,6. The amplitude of discrete rogue waves increases according to ρ_0.

We assume the system is subject to periodic boundary conditions, i.e., $\psi(N+1) = \psi 1$. We aim to solve equation (1) numerically under periodic boundary conditions to show that discrete rogue waves occur in a discrete KdV equation. We will use the complex solution of the initial condition mentioned in equation (2). The perturbation is given very small on the left side (j=1) and gets larger when going on the right side (j=100). As a result of the periodic boundary conditions, the discontinuity inevitably breaks up into waves due to dispersion, causing the waves to arise on the right edge. Note that the waves occur not because of MI but because of the shock front breaking.

As seen from Figure 2. a, the amplitude of a wave is nearly twenty times higher than the average surrounding background, κ=0,1. We note that such a huge amplitude growth is much higher than that of a discrete rogue wave for the discrete nonlinear Schrödinger equation [28]. The exciting point of view is that the study allows comparing the amplitude of discrete rogue waves calculated by the discrete KdV equation and the discrete Schrödinger equation. Also, when the discrete KdV equation is solved numerically under periodic boundary conditions, the rogue wave occurs at about t=180; instead, the equation is solved numerically under non-vanishing boundary conditions, and the rogue wave occurs at about t=600. We observe that such a different boundary condition results in a shift of freak wave peak. Indeed, it is well known that rogue waves sensitively depend on the initial condition. Even small changes can make all the features of rogue waves different and even unstable [28], [30]. If the constant A increases, the discrete rogue waves become more chaotic, as seen in Figure 2. b. We numerically observe that if A gets more significant, more rogue waves occur, and their amplitude increases. Figures 2. a and 2. b show us that discrete rogue waves can happen in a discrete KdV equation, and the numerousness of discrete rogue waves and their amplitude depends on the A parameter. One question came up. What is the long-time behavior of the system after the rogue wave disappears? We see that the discrete system fluctuations begin, leading to other random rogue waves for very long times.



(a)



(b)

Figure 2. The absolute field amplitude for N=100, A=0.1, ξ = 0,0065, and L=100. We study equation (1), and the initial form (for t=0) is given by equation (2). A wave has occurred, and its amplitude is nearly twenty times higher than the average background, as shown in Figure 2. a. In this study, Figure 2. b shows the absolute field amplitude for N=100, A=0.2, ξ = 0,0065, and L=100. We study equation (1), and the initial form (for t=0) is given by equation (2). We see that more rogue waves have occurred. Their amplitude increases as well.

In general relief, rogue wave evolution depends on only initial condition or small experimental imperfections [28], [30]. However, this study discusses that it can also depend on assumed boundary conditions in theoretical studies. We study the discrete KdV equation under periodic and non-vanishing boundary conditions with the initial wave given in equation (2). We see that a very small perturbation leads to strong localization, and a large amplitude discrete rogue wave has occurred. Moreover, we see that the localization of discrete rogue waves under non-vanishing boundary conditions differs from under periodic boundary conditions, as seen in Figure 1. a and Figure 2. Also, the amplitude of these waves depends on parameters.

3. CONCLUSION

As rogue waves are rare, it is crucial to understand the mathematical explanation of rogue waves. In this study, we numerically solve the discrete KdV equation for the first time with the proper initial condition under non-vanishing and periodic boundary conditions. We show that discrete rogue wave(s) occur in these types of boundary conditions, and we examine discrete rogue wave features under these boundary conditions. Even in the NLS equation, MI is mostly one step ahead of explaining rogue waves; we have discussed that when we solve the dKdV equation under periodic boundary conditions, discrete rogue wave occurs because of shock front breaking. We observe that rogue wave amplitude rises when the A parameter increases; more freak waves can also occur. We have solved the dKdV equation numerically under non-vanishing boundary conditions. We show that ρ_0 parameter is also important in enhancing the rogue wave's amplitude.

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CONFLICT OF INTEREST

The author have no conflicts of interest to declare that they are relevant to the content of this article.

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