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On Generalized Commutative Quaternions with Generalized Tetranacci Number Components

Genelleştirilmiş Tetranacci Sayı Bileşenleri ile Genelleştirilmiş Komutatif Kuaterniyonlar Üzerine

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Abstract

In this research, generalized commutative quaternions with generalized Tetranacci number components are introduced and studied. Then, some algebraic properties of these quaternions, such as a Binet-like formula and the summation formula are presented. Furthermore, a matrix representation is given involving these generalized commutative quaternions.

Keywords: Generalized quaternions, generalized tetranacci numbers, quaternions, tetranacci numbers

Öz

Bu çalışmada, genelleştirilmiş Tetranacci sayı bileşenleri ile genelleştirilmiş komutatif kuaterniyonlar tanımlanmış ve incelenmiştir. Bu kuarternionlara ait Binet-benzeri formül ve toplam formülü gibi bazı cebirsel özellikler sunulmuştur. Ayrıca, bu genelleştirilmiş komutatif kuaterniyonları içeren bir matris temsilcisi verilmiştir.

Anahtar Kelimeler: Genelleştirilmiş kuaterniyonlar, genelleştirilmiş tetranacci sayıları, kuaterniyonlar, tetranacci sayıları

1. Introduction

The Tetranacci numbers $\{M_n\}_{n\in\mathbb{N}}$ were introduced by Waddill (1992), while the generalized Tetranacci numbers were considered by Yeşil Baran and Yetiş (2019). These sequences can be viewed as generalizations of the Fibonacci numbers, which appear frequently in nature and man-made systems.

The generalized Tetranacci sequence $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ is defined with the recurrence relation

$$\mathcal{T}_n = p\mathcal{T}_{n-1} + q\mathcal{T}_{n-2} + r\mathcal{T}_{n-3} + s\mathcal{T}_{n-4},\ n \geq 4\,. \tag{1}$$

Here, $\mathcal{T}_0 = a, \mathcal{T}_1 = b, \mathcal{T}_2 = c, \mathcal{T}_3 = d$ and we have

$$p+q+r+s-1 \neq 0, \tag{2}$$

(Yeşil Baran and Yetiş 2019). When $a=b=0, c=d=1$ and $p=q=r=s=1, \{\mathcal{T}_n\}_{n\in\mathbb{N}}$ becomes the Tetranacci sequence $\{M_n\}_{n\in\mathbb{N}}$. In the case of $a=b=1, c=2, d=4, p=q=r=s=1, \{\mathcal{T}_n\}_{n\in\mathbb{N}}$ transforms into the Tetrabonacci numbers $\{l_n\}_{n\in\mathbb{N}}$ and 4-bonacci numbers $\{\mathcal{F}_n^{(4)}\}_{n\in\mathbb{N}}$ mentioned in Ramírez and Sirvent (2015). We have the Quadrapell numbers $\{D_n\}_{n\in\mathbb{N}}$ considered by Taşcı (2009), for $a = b = c = 1$, $d = 2$ and $p = 0$, $q = r = 1$, $s = 2$. By taking $a = b = 0$, $c = 1$, $d = 3$ and $p = 3$, $q = 0$, $r = -3$, $s = -1$, we get the quadra Fibona-Pell numbers $\{W_n\}_{n\in\mathbb{N}}$, which are presented by Özkoç (2015). $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ becomes the quadra Lucas-Jacobsthal numbers $\{S_n\}_{n\in\mathbb{N}}$ described by Kızılateş (2017), for $a = b = 2$, $c = 4$, $d = 7$ and $p = q = 2$, $r = -3$, $s = -2$. We obtain Gaussian Tetranacci numbers $\{GM_n\}_{n\in\mathbb{N}}$, which are introduced by Taşcı and Acar (2017), for $a = b = 0$, $c = 1$, $d = 1 + i$ and $p = q = r = s = 1$. The case of $a = 1$, $b = 2$, $c = 4$, $d = 9$ and $p = q = 4$, $r = -5$, $s = -1$ gives us the binomial transform of Quadrapell numbers $\{b_n\}_{n\in\mathbb{N}}$ defined by Kızılateş et al. (2017). We get the Tetranacci-Lucas numbers $\{TL_n\}_{n\in\mathbb{N}}$ given in Soykan (2020) for $a = 4$, $b = 1$, $c = 3$, $d = 7$, $p = q = s = r = 1$. While $a = 0$, $b = c = d = 1$ and $p = 0$, $q = s = 1$, $r = 2$, $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ becomes the Pell-Padovan Tetranacci sequence $\{PT_n\}_{n\in\mathbb{N}}$, which is defined by Petroudi et al. (2020).

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Moreover, the sequences $\{N_n\}_{n\in\mathbb{N}}, \{P_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ can be obtained from (1) and (2). The values of *a, b, c, d, p, q, r, s* corresponding to these sequences will be given in the next section. Here, Narayana sequence $\{N_n\}_{n\in\mathbb{N}}$ (sequence A000930 in Sloane), Padovan sequence $\{P_n\}_{n\in\mathbb{N}}$ (sequence A000931 in Sloane) and Perrin sequence $\{R_n\}_{n\in\mathbb{N}}$ (sequence A001608 in Sloane) are available at: http://oeis.org.

Recently, general forms of generating functions for sequences of numbers and polynomials are defined by Simsek (2023) as

$$F(w,P(\vec{\varepsilon}_m)) = \frac{1}{1+\sum_{j=1}^{m} P_j(\varepsilon_j)w^j} = \sum_{n=0}^{\infty} \mathbb{Y}_n(P(\vec{\varepsilon}_m))w^n \tag{3}$$

and

$$\begin{aligned} G(w,P(\vec{\varepsilon}_m);Q(\vec{\varepsilon}_k)) &= \frac{\sum_{j=0}^{k} Q_j(\varepsilon_j)w^j}{1+\sum_{j=1}^{m} P_j(\varepsilon_j)w^j} \\ &= \sum_{n=0}^{\infty} \mathbb{S}_n(P(\vec{\varepsilon}_m);Q(\vec{\varepsilon}_k))w^n \end{aligned} \tag{4}$$

$$\begin{aligned} \text{for } P(\vec{\varepsilon}_m) &= (P_1(\varepsilon_1), P_2(\varepsilon_2), ..., P_m(\varepsilon_m)), \\ Q(\vec{\varepsilon}_k) &= (Q_1(\varepsilon_1), Q_2(\varepsilon_2), ..., Q_k(\varepsilon_k)), \\ P_j(\varepsilon_j) &= \sum_{v=0}^{d} a_v \varepsilon_j^v, Q_l(\varepsilon_l) = \sum_{v=0}^{c} b_v \varepsilon_l^v, \end{aligned} \tag{5}$$

$0 \le l \le k, 0 \le j \le m, m \in \mathbb{N}$ and $c,d,k \in \mathbb{N}\cup\{0\}$. By choosing suitable values, we can obtain the generating functions of all the specific cases of $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ that we mentioned above.

Binet formula for $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ is given by the following relationship

$$\begin{aligned} \mathcal{T}_n = &\frac{At_1^n}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} + \frac{Bt_2^n}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} \\ &+ \frac{Ct_3^n}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} + \frac{Dt_4^n}{(t_4-t_1)(t_4-t_2)(t_4-t_3)} \end{aligned} \tag{6}$$

where t_1, t_2, t_3, t_4 are the roots of characteristic equation of (1) and

$$\begin{aligned} A = &\frac{t_1(t_1-t_2)(t_1-t_3)(t_1-t_4)(t_2-t_3)(t_2-t_4) + B(t_1-t_3)(t_1-t_4)}{(t_2-t_3)(t_2-t_4)} + \\ &\frac{-C(t_1-t_2)(t_1-t_4)(t_2-t_4) + D(t_1-t_2)(t_1-t_3)(t_2-t_3)}{(t_2-t_3)(t_2-t_4)(t_3-t_4)}, \end{aligned} \tag{7}$$

$$B = (b-ap)[(t_3-t_2)(t_4-t_2)] - \frac{C(t_2-t_4)+D(t_3-t_2)}{(t_3-t_4)}, \tag{8}$$

$$C = (t_3-t_4)[c-bp-aq]+D, \tag{9}$$

$$D = d - cp - bq - ar, \tag{10}$$

(Yeşil Baran 2021).

The Irish mathematician William Rowan Hamilton defined quaternions in 1843 and they are a four-dimensional hypercomplex number system. Quaternions are widely used in pure and applied mathematics, modern physics and many other fields. Multiplication of quaternions is non-commutative and this property makes it difficult to conduct applications to engineering problems. Commutative quaternions are defined by modifying the definition of quaternions in a way that enabled commutativity in multiplication. They are a number system that has received a lot of attention and are used in applications such as signal processing.

A generalized quaternion x is a vector which can be written as

$$x = x_0 + x_1e_1 + x_2e_2 + x_3e_3. \tag{11}$$

Here,

$$e_1^2 = -\alpha, e_2^2 = -\beta, e_3^2 = -\alpha\beta, \tag{12}$$

$$e_1e_2 = -e_2e_1 = e_3, e_2e_3 = -e_3e_2 = \beta e_1, e_1e_3 = -e_3e_1 = \alpha e_2 \tag{13}$$

and $x_0, x_1, x_2, x_3, \alpha, \beta \in \mathbb{R}$. $\mathbb{H}_{\gamma\beta}$ denotes the family of all generalized quaternions.

The generalized commutative quaternions were introduced and studied in Szynal-Liana and Włoch (2022). A generalized commutative quaternion x is a vector of the form (11), where generalized commutative quaternionic units e_1, e_2, e_3 satisfy the equalities

$$e_1^2 = \alpha, e_2^2 = \beta, e_3^2 = \alpha\beta \tag{14}$$

and

$$e_1e_2 = e_2e_1 = e_3, e_2e_3 = e_3e_2 = \beta e_1, e_1e_3 = e_3e_1 = \alpha e_2 \tag{15}$$

for $x_0, x_1, x_2, x_3, \alpha, \beta \in \mathbb{R}$. The family of all generalized commutative quaternions is denoted by $\mathbb{H}_{\gamma\beta}^c$. The generalized commutative quaternions are generalizations of elliptic quaternions $(\alpha < 0, \beta = 1)$, parabolic quaternions $(\alpha = 0, \beta = 1)$, hyperbolic quaternions $(\alpha > 0, \beta = 1)$, bicomplex numbers $(\alpha = -1, \beta = -1)$, complex hyperbolic numbers $(\alpha = -1, \beta = 1)$ and hyperbolic complex numbers $(\alpha = 1, \beta = -1)$.

The generalized non-commutative Fibonacci quaternions were presented by Horadam (1963). Some properties of generalized non-commutative Fibonacci quaternions were given in Flaut and Shpakivskyi (2013), Akyiğit et al. (2014), Flaut (2014) and Flaut and Savin (2015). As for the generalized commutative quaternions with Fibonacci type number components, they were studied in Szynal-Liana and Włoch (2022), Bród et al. (2022) and Bród and Szynal-Liana (2023), by utilizing Horadam, Jacobsthal and Jacobsthal-Lucas numbers. Then, Szynal-Liana et al. (2023) examined generalized commutative quaternions by using Fibonacci, Lucas, Jacobsthal, Jacobsthal-Lucas, Pell and Pell–Lucas polynomials sequences.

On the other hand, in literature there exist some interesting results about the quaternions defined by using a generalized and recurrent number sequence. To name some examples, generalized Fibonacci quaternions, generalized Fibonacci-Lucas quaternions, Horadam quaternions, generalized Tribonacci quaternions, bicomplex generalized Tribonacci numbers, bicomplex Tetranacci and Tetranacci-Lucas numbers, higher order Fibonacci quaternions, higher order Fibonacci hyper complex numbers, incomplete Fibonacci and Lucas quaternions were defined and examined by Swamy (1973), Flaut and Savin (2015), Halici and Karataş (2017), Cerda-Morales (2017), Kızılateş et al. (2019), Soykan (2020), Kızılateş and Kone (2021a), Kızılateş and Kone (2021b), Kızılateş (2022), respectively.

By taking these studies into account, we will define generalized commutative quaternions with generalized Tetranacci number components in the next section.

2. Generalized Commutative Quaternions with Generalized Tetranacci Number Components

Definition 2.1. For $n \geq 0$, we define the n-th generalized commutative generalized Tetranacci quaternion

$$gc\mathcal{T}_n = \mathcal{T}_n + \mathcal{T}_{n+1}e_1 + \mathcal{T}_{n+2}e_2 + \mathcal{T}_{n+3}e_3 \tag{16}$$

where $\mathcal{T}_n$ is the n-th generalized Tetranacci number and the generalized commutative quaternionic units e_1, e_2, e_3 satisfy (14) and (15). The following are some special cases of this quaternion:

I) $\{gc\mathcal{T}_n\}_{n\in\mathbb{N}}$ become the generalized commutative Fibonacci quaternions $\{gcF_n\}_{n\in\mathbb{N}}$ for $a=0, b=c=1, d=2$ and $p=q=1, r=s=0$.

II) $\{gc\mathcal{T}_n\}_{n\in\mathbb{N}}$ become the generalized commutative Jacobstal quaternions $\{gcJ_n\}_{n\in\mathbb{N}}$ for $a=0, b=c=1, d=3$ and $p=1, q=2, r=s=0$.

III) $\{gc\mathcal{T}_n\}_{n\in\mathbb{N}}$ become the generalized commutative Horadam quaternions $\{gc\mathcal{H}_n\}_{n\in\mathbb{N}}$ for $a,b,c,d,p,q \in \mathbb{R}$ and $q=-q, r=s=0$.

IV) $\{gc\mathcal{T}_n\}_{n\in\mathbb{N}}$ become the generalized commutative Narayana quaternions $\{gcN_n\}_{n\in\mathbb{N}}$ for $a=b=c=1, d=2$ and $p=r=1, q=s=0$.

V) $\{gc\mathcal{T}_n\}_{n\in\mathbb{N}}$ become the generalized commutative Padovan quaternions $\{gcP_n\}_{n\in\mathbb{N}}$ for $a=d=1, b=c=0$ and $p=s=0, q=r=1$.

VI) $\{gc\mathcal{T}_n\}_{n\in\mathbb{N}}$ become the generalized commutative Perrin quaternions $\{gcR_n\}_{n\in\mathbb{N}}$ for $a=d=3, b=0, c=2$ and $p=s=0, q=r=1$.

VII) $\{gc\mathcal{T}_n\}_{n\in\mathbb{N}}$ become the generalized commutative Tetranacci quaternions $\{gcM_n\}_{n\in\mathbb{N}}$ for $a=b=0, c=d=1$ and $p=q=r=s=1$.

We note that cases I, II and III are considered by Szynal-Liana and Włoch (2022).

Theorem 2.1. Let $n \geq 0$ be an integer. Then, a Binet-like formula for generalized commutative quaternions with generalized Tetranacci components can be written as

$$gc\mathcal{T}_n = \frac{At_1^n\left[1+t_1e_1+t_1^2e_2+t_1^3e_3\right]}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} + \frac{Bt_2^n\left[1+t_2e_1+t_2^2e_2+t_2^3e_3\right]}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} + \frac{Ct_3^n\left[1+t_3e_1+t_3^2e_2+t_3^3e_3\right]}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} + \frac{Dt_4^n\left[1+t_4e_1+t_4^2e_2+t_4^3e_3\right]}{(t_4-t_1)(t_4-t_2)(t_4-t_3)}, \tag{17}$$

where t_1, t_2, t_3, t_4 are the roots of characteristic equation of (1) and A, B, C, D are given with equalities (7) - (10).

Proof. By (6) and (16), we have

$$
\begin{aligned}
gc\mathcal{T}_n &= \mathcal{T}_n + \mathcal{T}_{n+1}e_1 + \mathcal{T}_{n+2}e_2 + \mathcal{T}_{n+3}e_3 = \frac{At_1^n}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} + \frac{Bt_2^n}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} \\
&+ \frac{Ct_3^n}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} + \frac{Dt_4^n}{(t_4-t_1)(t_4-t_2)(t_4-t_3)} + \left(\frac{At_1^{n+1}}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} + \frac{Bt_2^{n+1}}{(t_2-t_1)(t_2-t_3)(t_2-t_4)}\right. \\
&\left.+ \frac{Ct_3^{n+1}}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} + \frac{Dt_4^{n+1}}{(t_4-t_1)(t_4-t_2)(t_4-t_3)}\right)e_1 + \left(\frac{At_1^{n+2}}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} + \frac{Bt_2^{n+2}}{(t_2-t_1)(t_2-t_3)(t_2-t_4)}\right. \\
&\left.+ \frac{Ct_3^{n+2}}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} + \frac{Dt_4^{n+2}}{(t_4-t_1)(t_4-t_2)(t_4-t_3)}\right)e_2 + \left(\frac{At_1^{n+3}}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} + \frac{Bt_2^{n+3}}{(t_2-t_1)(t_2-t_3)(t_2-t_4)}\right. \\
&\left.+ \frac{Ct_3^{n+3}}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} + \frac{Dt_4^{n+3}}{(t_4-t_1)(t_4-t_2)(t_4-t_3)}\right)e_3 = \frac{At_1^n[1+t_1e_1+t_1^2e_2+t_1^3e_3]}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} + \frac{Bt_2^n[1+t_2e_1+t_2^2e_2+t_2^3e_3]}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} \\
&+ \frac{Ct_3^n[1+t_3e_1+t_3^2e_2+t_3^3e_3]}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} + \frac{Dt_4^n[1+t_4e_1+t_4^2e_2+t_4^3e_3]}{(t_4-t_1)(t_4-t_2)(t_4-t_3)},
\end{aligned}
\tag{18}
$$

which completes the proof.

For simplicity, we can write

$$\widehat{t}_1 = \frac{1+t_1e_1+t_1^2e_2+t_1^3e_3}{(t_1-t_2)(t_1-t_3)(t_1-t_4)}, \tag{19}$$

$$\widehat{t}_2 = \frac{1+t_2e_1+t_2^2e_2+t_2^3e_3}{(t_2-t_1)(t_2-t_3)(t_2-t_4)}, \tag{20}$$

$$\widehat{t}_3 = \frac{1+t_3e_1+t_3^2e_2+t_3^3e_3}{(t_3-t_1)(t_3-t_2)(t_3-t_4)}, \tag{21}$$

$$\widehat{t}_4 = \frac{1+t_4e_1+t_4^2e_2+t_4^3e_3}{(t_4-t_1)(t_4-t_2)(t_4-t_3)}. \tag{22}$$

Then, using (19) - (22), equality (17) becomes

$$gc\mathcal{T}_n = At_1^n\widehat{t}_1 + Bt_2^n\widehat{t}_2 + Ct_3^n\widehat{t}_3 + Dt_4^n\widehat{t}_4. \tag{23}$$

Theorem 2.2. The following relations are valid for $gc\mathcal{T}_n$:

$$gc\mathcal{T}_n = p.gc\mathcal{T}_{n-1} + q\cdot gc\mathcal{T}_{n-2} + r\cdot gc\mathcal{T}_{n-3} + s\cdot gc\mathcal{T}_{n-4}, (n\geq 4), \tag{24}$$

$$gc\mathcal{T}_n - gc\mathcal{T}_{n+1}e_1 + gc\mathcal{T}_{n+2}e_2 - gc\mathcal{T}_{n+3}e_3 = (\mathcal{T}_n - \alpha\mathcal{T}_{n+2} + \beta\mathcal{T}_{n+4} - \alpha\beta\mathcal{T}_{n+6}) + 2(\mathcal{T}_{n+2} - \alpha\mathcal{T}_{n+4})e_2, (n\geq 0). \tag{25}$$

Proof. In order to examine the first equality, we write

$$
\begin{aligned}
&p\cdot gc\mathcal{T}_{n-1} + q\cdot gc\mathcal{T}_{n-2} + r\cdot gc\mathcal{T}_{n-3} + s\cdot gc\mathcal{T}_{n-4} = p(\mathcal{T}_{n-1} + \mathcal{T}_n e_1 + \mathcal{T}_{n+1}e_2 + \mathcal{T}_{n+2}e_3) + q(\mathcal{T}_{n-2} + \mathcal{T}_{n-1}e_1 \\
&+\mathcal{T}_n e_2 + \mathcal{T}_{n+1}e_3) + r(\mathcal{T}_{n-3} + \mathcal{T}_{n-2}e_1 + \mathcal{T}_{n-1}e_2 + \mathcal{T}_n e_3) + s(\mathcal{T}_{n-4} + \mathcal{T}_{n-3}e_1 + \mathcal{T}_{n-2}e_2 + \mathcal{T}_{n-1}e_3) \\
&= (p\mathcal{T}_{n-1} + q\mathcal{T}_{n-2} + r\mathcal{T}_{n-3} + s\mathcal{T}_{n-4}) + (p\mathcal{T}_n + q\mathcal{T}_{n-1} + r\mathcal{T}_{n-2} + s\mathcal{T}_{n-3})e_1 \\
&+ (p\mathcal{T}_{n+1} + q\mathcal{T}_n + r\mathcal{T}_{n-1} + s\mathcal{T}_{n-2})e_2 + (p\mathcal{T}_{n+2} + q\mathcal{T}_{n+1} + r\mathcal{T}_n + s\mathcal{T}_{n-1})e_3 = \mathcal{T}_n + \mathcal{T}_{n+1}e_1 + \mathcal{T}_{n+2}e_2 + \mathcal{T}_{n+3}e_3 = gc\mathcal{T}_n,
\end{aligned}
\tag{26}
$$

by using (1), (16) and we obtain (24). As for (25), by (14) - (16), we have

$$
\begin{aligned}
&gc\mathcal{T}_n - gc\mathcal{T}_{n+1}e_1 + gc\mathcal{T}_{n+2}e_2 - gc\mathcal{T}_{n+3}e_3 = (\mathcal{T}_n + \mathcal{T}_{n+1}e_1 + \mathcal{T}_{n+2}e_2 + \mathcal{T}_{n+3}e_3) - (\mathcal{T}_{n+1} + \mathcal{T}_{n+2}e_1 + \mathcal{T}_{n+3}e_2 + \mathcal{T}_{n+4}e_3)e_1 \\
&+ (\mathcal{T}_{n+2} + \mathcal{T}_{n+3}e_1 + \mathcal{T}_{n+4}e_2 + \mathcal{T}_{n+5}e_3)e_2 - (\mathcal{T}_{n+3} + \mathcal{T}_{n+4}e_1 + \mathcal{T}_{n+5}e_2 + \mathcal{T}_{n+6}e_3)e_3 = (\mathcal{T}_n + \mathcal{T}_{n+1}e_1 + \mathcal{T}_{n+2}e_2 + \mathcal{T}_{n+3}e_3) \\
&+ (-\mathcal{T}_{n+1}e_1 - \mathcal{T}_{n+2}\alpha - \mathcal{T}_{n+3}e_3 - \mathcal{T}_{n+4}\alpha e_2) + (\mathcal{T}_{n+2}e_2 + \mathcal{T}_{n+3}e_3 + \mathcal{T}_{n+4}\beta + \mathcal{T}_{n+5}\beta e_1) + (-\mathcal{T}_{n+3}e_3 - \mathcal{T}_{n+4}\alpha e_2 - \mathcal{T}_{n+5}\beta e_1 - \mathcal{T}_{n+6}\alpha\beta) \\
&= (\mathcal{T}_n - \alpha\mathcal{T}_{n+2} + \beta\mathcal{T}_{n+4} - \alpha\beta\mathcal{T}_{n+6}) + 2(\mathcal{T}_{n+2} - \alpha\mathcal{T}_{n+4})e_2.
\end{aligned}
\tag{27}
$$

Thus, the proof is completed.

With the next theorem, we will present a summation formula for the generalized commutative quaternions with generalized Tetranacci number components.

Theorem 2.3. Suppose that $n \geq 4$ is an integer. Then, we have

$$\sum_{k=0}^{n} gc\mathcal{T}_k = \frac{(p+q+r+s)gc\mathcal{T}_n + (q+r+s)gc\mathcal{T}_{n-1} + (r+s)gc\mathcal{T}_{n-2} + sgc\mathcal{T}_{n-3} + (p+q+r-1)gc\mathcal{T}_0 + (p+q-1)gc\mathcal{T}_1}{p+q+r+s-1} + \frac{(p-1)gc\mathcal{T}_2 - gc\mathcal{T}_3}{p+q+r+s-1}. \tag{28}$$

Proof. Since we have equality (24) for $n \geq 4$ and by considering (2), we get

$$\begin{aligned}\sum_{k=0}^{n} gc\mathcal{T}_k &= gc\mathcal{T}_0 + gc\mathcal{T}_1 + gc\mathcal{T}_2 + gc\mathcal{T}_3 + p\sum_{k=3}^{n-1} gc\mathcal{T}_k + q\sum_{k=2}^{n-2} gc\mathcal{T}_k + r\sum_{k=1}^{n-3} gc\mathcal{T}_k + \\ &s\sum_{k=0}^{n-4} gc\mathcal{T}_k = (p+q+r+s)\sum_{k=0}^{n} gc\mathcal{T}_k - (p+q+r+s)gc\mathcal{T}_n - (q+r+s)gc\mathcal{T}_{n-1} - \\ &(r+s)gc\mathcal{T}_{n-2} - sgc\mathcal{T}_{n-3} - (p+q+r-1)gc\mathcal{T}_0 - (p+q-1)gc\mathcal{T}_1 - (p-1)gc\mathcal{T}_2 + gc\mathcal{T}_3.\end{aligned} \tag{29}$$

Therefore, the proof is completed.

3. Matrix Representation of Generalized Commutative Quaternions with Generalized Tetranacci Number Components

Now, we give the matrix generator of the numbers $gc\mathcal{T}_n$.

Theorem 3.1. Suppose that $n \geq 1$ is an integer. Then, matrix formulation of $gc\mathcal{T}_n$ can be given as

$$\begin{bmatrix} gc\mathcal{T}_{n+5} & gc\mathcal{T}_{n+4} & gc\mathcal{T}_{n+3} & gc\mathcal{T}_{n+2} \\ gc\mathcal{T}_{n+4} & gc\mathcal{T}_{n+3} & gc\mathcal{T}_{n+2} & gc\mathcal{T}_{n+1} \\ gc\mathcal{T}_{n+3} & gc\mathcal{T}_{n+2} & gc\mathcal{T}_{n+1} & gc\mathcal{T}_{n} \\ gc\mathcal{T}_{n+2} & gc\mathcal{T}_{n+1} & gc\mathcal{T}_{n} & gc\mathcal{T}_{n-1} \end{bmatrix} = \begin{bmatrix} gc\mathcal{T}_6 & gc\mathcal{T}_5 & gc\mathcal{T}_4 & gc\mathcal{T}_3 \\ gc\mathcal{T}_5 & gc\mathcal{T}_4 & gc\mathcal{T}_3 & gc\mathcal{T}_2 \\ gc\mathcal{T}_4 & gc\mathcal{T}_3 & gc\mathcal{T}_2 & gc\mathcal{T}_1 \\ gc\mathcal{T}_3 & gc\mathcal{T}_2 & gc\mathcal{T}_1 & gc\mathcal{T}_0 \end{bmatrix} \cdot \begin{bmatrix} p & 1 & 0 & 0 \\ q & 0 & 1 & 0 \\ r & 0 & 0 & 1 \\ s & 0 & 0 & 0 \end{bmatrix}^{n-1}. \tag{30}$$

Proof. The proof was done using the principle of mathematical induction. The result is obvious for $n=1$, which can be easily seen. We will show the equality is true for $n+1$, by assuming that the formula (30) holds for $n \geq 1$. We get

$$\begin{aligned}&\begin{bmatrix} gc\mathcal{T}_6 & gc\mathcal{T}_5 & gc\mathcal{T}_4 & gc\mathcal{T}_3 \\ gc\mathcal{T}_5 & gc\mathcal{T}_4 & gc\mathcal{T}_3 & gc\mathcal{T}_2 \\ gc\mathcal{T}_4 & gc\mathcal{T}_3 & gc\mathcal{T}_2 & gc\mathcal{T}_1 \\ gc\mathcal{T}_3 & gc\mathcal{T}_2 & gc\mathcal{T}_1 & gc\mathcal{T}_0 \end{bmatrix} \cdot \begin{bmatrix} p & 1 & 0 & 0 \\ q & 0 & 1 & 0 \\ r & 0 & 0 & 1 \\ s & 0 & 0 & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} p & 1 & 0 & 0 \\ q & 0 & 1 & 0 \\ r & 0 & 0 & 1 \\ s & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} gc\mathcal{T}_{n+5} & gc\mathcal{T}_{n+4} & gc\mathcal{T}_{n+3} & gc\mathcal{T}_{n+2} \\ gc\mathcal{T}_{n+4} & gc\mathcal{T}_{n+3} & gc\mathcal{T}_{n+2} & gc\mathcal{T}_{n+1} \\ gc\mathcal{T}_{n+3} & gc\mathcal{T}_{n+2} & gc\mathcal{T}_{n+1} & gc\mathcal{T}_{n} \\ gc\mathcal{T}_{n+2} & gc\mathcal{T}_{n+1} & gc\mathcal{T}_{n} & gc\mathcal{T}_{n-1} \end{bmatrix} \cdot \begin{bmatrix} p & 1 & 0 & 0 \\ q & 0 & 1 & 0 \\ r & 0 & 0 & 1 \\ s & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} gc\mathcal{T}_{n+6} & gc\mathcal{T}_{n+5} & gc\mathcal{T}_{n+4} & gc\mathcal{T}_{n+3} \\ gc\mathcal{T}_{n+5} & gc\mathcal{T}_{n+4} & gc\mathcal{T}_{n+3} & gc\mathcal{T}_{n+2} \\ gc\mathcal{T}_{n+4} & gc\mathcal{T}_{n+3} & gc\mathcal{T}_{n+2} & gc\mathcal{T}_{n+1} \\ gc\mathcal{T}_{n+3} & gc\mathcal{T}_{n+2} & gc\mathcal{T}_{n+1} & gc\mathcal{T}_{n} \end{bmatrix},\end{aligned} \tag{31}$$

by considering induction's hypothesis and equality (24) and this ends the proof.

Moreover, we have

$$\begin{bmatrix} gc\mathcal{T}_6 & gc\mathcal{T}_5 & gc\mathcal{T}_4 & gc\mathcal{T}_3 \\ gc\mathcal{T}_5 & gc\mathcal{T}_4 & gc\mathcal{T}_3 & gc\mathcal{T}_2 \\ gc\mathcal{T}_4 & gc\mathcal{T}_3 & gc\mathcal{T}_2 & gc\mathcal{T}_1 \\ gc\mathcal{T}_3 & gc\mathcal{T}_2 & gc\mathcal{T}_1 & gc\mathcal{T}_0 \end{bmatrix} \cdot \begin{bmatrix} p & 1 & 0 & 0 \\ q & 0 & 1 & 0 \\ r & 0 & 0 & 1 \\ s & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} p & 1 & 0 & 0 \\ q & 0 & 1 & 0 \\ r & 0 & 0 & 1 \\ s & 0 & 0 & 0 \end{bmatrix}^{T} \cdot \begin{bmatrix} gc\mathcal{T}_6 & gc\mathcal{T}_5 & gc\mathcal{T}_4 & gc\mathcal{T}_3 \\ gc\mathcal{T}_5 & gc\mathcal{T}_4 & gc\mathcal{T}_3 & gc\mathcal{T}_2 \\ gc\mathcal{T}_4 & gc\mathcal{T}_3 & gc\mathcal{T}_2 & gc\mathcal{T}_1 \\ gc\mathcal{T}_3 & gc\mathcal{T}_2 & gc\mathcal{T}_1 & gc\mathcal{T}_0 \end{bmatrix} \tag{32}$$

and

$$\det\begin{bmatrix} p & 1 & 0 & 0 \\ q & 0 & 1 & 0 \\ r & 0 & 0 & 1 \\ s & 0 & 0 & 0 \end{bmatrix} = \det\begin{bmatrix} p & 1 & 0 & 0 \\ q & 0 & 1 & 0 \\ r & 0 & 0 & 1 \\ s & 0 & 0 & 0 \end{bmatrix}^T = -r. \tag{33}$$

4. Conclusion

We defined the generalized commutative quaternions by using a number sequence which is defined with a generalized recurrence relation. This new definition generalizes the quaternions introduced by Szynal-Liana and Włoch (2022) and further investigated by Bród et al. (2022) and Bród and Szynal-Liana (2023). Some properties involving the sequence $\{gc\mathcal{T}_n\}_{n\in\mathbb{N}}$ were presented, including the Binet type formula and the summation formula. In addition, a specific matrix, whose elements are the generalized commutative quaternions with generalized Tetranacci number components, was given as an alternative way to acquire the n-th term of the sequence $\{gc\mathcal{T}_n\}_{n\in\mathbb{N}}$. We believe that the quaternions considered in this article can be extended to generalize other quaternion families and the results given in this article could be useful for further research on this topic.

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5. References

Akyiğit, M., Kosal, H. H., Tosun, M. 2014. Fibonacci generalized quaternions. Adv. Appl. Clifford Algebr., 24: 631-641. Doi: 10.1007/s00006-014-0458-0

Bród, D., Szynal-Liana, A., Włoch, I. 2022. On some combinatorial properties of generalized commutative Jacobsthal quaternions and generalized commutative Jacobsthal-Lucas quaternions. Czechoslov. Math. J., 72: 1239-1248. Doi: 10.21136/CMJ.2022.0174-22

Bród, D., Szynal-Liana, A. 2023. Generalized commutative Jacobsthal quaternions and some matrices. Examples and Counterexamples, 3: 100102. Doi: 10.1016/j.exco.2023.100102

Cerda-Morales, G. 2017. On a generalization for Tribonacci quaternions. Mediterr. J. Math., 14: 239. Doi: 10.1007/s00009-017-1042-3

Flaut, C. 2014. A Clifford algebra associated to generalized Fibonacci quaternions. Adv. Differ. Equ., 279. Doi: 10.1186/1687-1847-2014-279

Flaut, C., Savin, D. 2015. Quaternion algebras and generalized Fibonacci–Lucas quaternions. Adv. Appl. Clifford Algebr., 25: 853-862. Doi: 10.1007/s00006-015-0542-0

Flaut, C., Shpakivskyi, V. 2013. On generalized Fibonacci quaternions and Fibonacci-Narayana quaternions. Adv. Appl. Clifford Algebr., 23: 673–688. Doi: 10.1007/s00006-013-0388-2

Halici, S., Karataş, A. 2017. On a generalization for Fibonacci quaternions. Chaos, Solitons & Fractals, 98: 178-182. Doi: 10.1016/j.chaos.2017.03.03

Horadam, A. F. 1963. Complex Fibonacci numbers and Fibonacci quaternions. American Mathematical Monthly, 70: 289-291. Doi: 10.2307/2313129

Kızılateş, C. 2017. On the Quadra Lucas-Jacobsthal Numbers. Karaelmas Science and Engineering Journal, 7(2): 619-621.

Kızılateş, C. 2022. On quaternions with incomplete Fibonacci and Lucas numbers components. Util. Math., 110: 263-269.

Kızılateş, C., Catarino, P., Tuğlu, N. 2019. On the bicomplex generalized Tribonacci quaternions. Mathematics, 7(1): 80. Doi: 10.3390/math7010080

Kızılateş, C., Kone, T. 2021a. On higher order Fibonacci quaternions. J. Anal. 29: 1071-1082. Doi: 10.1007/s41478-020-00295-1

Kızılateş, C., Kone, T. 2021b. On higher order Fibonacci hyper complex numbers. Chaos, Solitons & Fractals, 148, 111044. Doi: 10.1016/j.chaos.2021.111044

Kızılateş, C., Tuglu, N., Çekim, B. 2017. Binomial transform of quadrapell sequences and quadrapell matrix sequences. J. Sci. Arts, 1(38): 69-80.

Özkoç, A. 2015. Some algebraic identities on quadra Fibona-Pell integer sequence. Adv. Differ. Equ., 148(2015): 1-10. Doi: 10.1186/s13662-015-0486-7

Petroudi, S. H. J., Pirouz, M., Ozkoc, A. 2020. On some properties of particular Tetranacci sequences. J. Int. Math. Virtual Inst., 10(2): 361-376. Doi: 10.7251/JIMVI2002361P

Ramírez, J. L., Sirvent, V. F. 2015. A generalization of the k-bonacci sequence from Riordan arrays. Electron. J. Comb., 22(1), P1.38: 1-20.

Simsek, Y. 2023. Construction of general forms of ordinary generating functions for more families of numbers and multiple variables polynomials. Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., 117, 130: 1-26. Doi: 10.1007/s13398-023-01464-0

Soykan, Y. 2020. Bicomplex Tetranacci and Tetranacci-Lucas quaternions. Commun. Math. Appl., 11(1): 95-112. Doi: 10.26713/cma.v11i1.1212

Swamy, M. N. S. 1973. On generalized Fibonacci quaternions, Fibonacci Q., 11(5): 547-549.

Szynal-Liana, A., Włoch, I. 2022. Generalized commutative quaternions of the Fibonacci type. Bol. Soc. Mat. Mex. 28: 1. Doi: 10.1007/s40590-021-00386-4

Szynal-Liana, A., Włoch, I., Liana, M. 2023. Generalized commutative quaternion polynomials of the Fibonacci type. Ann. Univ. Mariae Curie-Skłodowska Lub.-Pol., A- Mathematica, 76(2): 33-44. Doi: 10.17951/a.2022.76.2.33-44

Taşcı, D. 2009. On Quadrapell numbers and Quadrapell polynomials. Hacet. J. Math. Stat., 38(3): 265-275.

Taşcı, D., Acar, H. 2017. Gaussian Tetranacci numbers. Commun. Math. Appl., 8(3): 379-386. Doi: 10.26713/cma.v8i3.615

Waddill, M. E. 1992. The Tetranacci sequence and generalizations. Fibonacci Q., 30(1): 9-20.

Yeşil Baran, F., Yetiş, T. 2019. On the norms of circulant matrices via generalized Tetranacci numbers. Bilecik Seyh Edebali University Journal of Science, 6(2): 444-454. Doi: 10.35193/bseufbd.662239

Yeşil Baran, F. 2021. The eigenvalues of circulant matrices with generalized Tetranacci numbers. Gümüşhane Üniversitesi Fen Bilimleri Dergisi, 11(2): 417-423. Doi: 10.17714/gumusfenbil.830575