CONSTRUCTIVE MATHEMATICAL ANALYSIS
6 (2023), No. 3, pp. 184-197
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

Some recent and new fixed point results on orthogonal metric-like space

ÖZLEM ACAR*

ABSTRACT. In this paper, we give some recent and new fixed point results for some contraction mappings on $O-$complete metric-like space and also we give illustrative examples. At the end, we give an application to show the existence of a solution of a differential equation.

Keywords: Fixed point, orthogonal metric-like space, Geraghty contraction.

2020 Mathematics Subject Classification: 54H25, 47H10.

1. INTRODUCTION AND PRELIMINARIES

The Banach contraction principle [7] is one of the fundamental foundations of the metrical fixed theory. Many authors have come up with generalizations, extensions and applications of this principle. They have studied many aspects of the Banach contraction principle and have further developed their findings. One of the most popular topics is studying new classes of spaces and their fundamental properties (see [9, 10, 11, 18, 20, 27]).

Amini-Harandi [6] introduced the concept of metric-like space and gave some fixed point theorems on complete metric-like space. Later, some authors worked on fixed point theorems for various new types of contraction conditions on the metric-like space. For more details, we refer to [1, 5, 16, 21].

We start by recalling some definitions and lemmas about metric-like space.

Definition 1.1 ([6]). *Let X be any non-empty set. A function $\rho : X \times X \to [0, \infty)$ is said to be a metric-like on X if for any $a, b, c \in X$ the following conditions are satisfied:*

(σ_1) $\rho(a, b) = 0 \Rightarrow a = b$,
(σ_2) $\rho(a, b) = \rho(b, a)$,
(σ_3) $\rho(a, b) \leq \rho(a, c) + \rho(c, b)$.

The pair (X, ρ) is called a metric-like space. Each metric-like ρ on X generates a T_0 topology τ_p on X which has as a base the family open ρ-balls

$$\{B_\rho(a, \varepsilon) : a \in X, \varepsilon > 0\},$$

where

$$B_\rho(a, \varepsilon) = \{b \in X : |\rho(a, b) - \rho(a, a)| < \varepsilon\}$$

for all $a \in X$ and $\varepsilon > 0$.

Received: 16.07.2023; Accepted: 03.09.2023; Published Online: 14.09.2023
*Corresponding author: Özlem Acar; acarozlem@ymail.com
DOI: 10.33205/cma.1360402

Definition 1.2 ([6]). *(i) A sequence $\{a_n\}$ in a metric-like space (X, ρ) converges to a point $a \in X$ if and only if $\rho(a, a) = \lim_{n\to\infty} \rho(a, a_n)$.*

(ii) A sequence $\{a_n\}$ in a metric-like space (X, ρ) is called a Cauchy sequence if $\lim_{n,m\to\infty} \rho(a_n, a_m)$ exists (and is finite) .

(iii) A metric-like space (X, ρ) is said to be complete if every Cauchy sequence $\{a_n\}$ in X converges, with respect to τ_p, to a point $a \in X$ such that

$$\lim_{n\to\infty} \rho(a, a_n) = \rho(a, a) = \lim_{n,m\to\infty} \rho(a_n, a_m).$$

Lemma 1.1 ([16]). *Let (X, ρ) be a metric-like space. Let $\{a_n\}$ be a sequence in X such that $a_n \to a$, where $a \in X$ and $\rho(a, a) = 0$. Then for all $b \in X$, we have*

$$\lim_{n\to\infty} \rho(a_n, b) = \rho(a, b).$$

Recently, Gordji et al. [12] extended the literature on metric spaces by introducing the notion of orthogonality. In [12], they proved the Banach fixed point theorem using the orthogonality such as:

Theorem 1.1 ([12]). *Let $(X, \rho, \perp)$ be an O-complete metric space (not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be $\perp$-continuous, $\perp$-contraction (with Lipschitz constant λ) and $\perp$-preserving. Then f has a unique fixed point x^* in X and is a Picard operator, that is, $\lim f^n(x) = x^*$ for all $x \in X$.*

Also, they show that this theorem is a real extension of Banach's contraction principle.

Corollary 1.1 ([12]). *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that, for some $\lambda \in (0, 1]$,*

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$. Then f has a unique fixed point in X.

There are several applications of this new idea of orthogonal sets and also numerous forms of orthogonality. We refer the reader to ([2, 3, 4, 8, 13, 14, 19, 22, 23, 24, 25]) for more details.

In this paper, we give some recent and new results for some contraction mappings on $O-$complete metric-like space and also we give illustrative examples. At the end, we give an application to show the existence of a solution of a differential equation. In order to do this, we will first recall some basic definitions and notations of the orthogonality.

Definition 1.3 ([12]). *Let X be a non-empty set and $\perp$ be a binary relation defined on X. If binary relation $\perp$ fulfills the following criteria:*

$$\exists a_0 (\forall b \in X, b \perp a_0) \text{ or } (\forall b \in X, a_0 \perp b),$$

then pair $(X, \perp)$ known as an orthogonal set. The element a_0 is called an orthogonal element. We denote this O-set or orthogonal set by $(X, \perp)$.

Definition 1.4 ([12]). *Let $(X, \perp)$ be an orthogonal set (O-set). Any two elements $a, b \in X$ such that $a \perp b$, then $a, b \in X$ are said to be orthogonally related.*

Definition 1.5 ([12]). *A sequence $\{a_n\}$ is called an orthogonal sequence (briefly O-sequence) if*

$$(\forall n \in \mathbb{N}, a_n \perp a_{n+1}) \text{ or } (\forall n \in \mathbb{N}, a_{n+1} \perp a_n).$$

Similarly, a Cauchy sequence $\{a_n\}$ is said to be a orthogonally Cauchy sequence (briefly Cauchy O-sequence) if

$$(\forall n \in \mathbb{N}, a_n \perp a_{n+1}) \text{ or } (\forall n \in \mathbb{N}, a_{n+1} \perp a_n).$$

Definition 1.6 ([12]). *Let $(X, \perp)$ be an orthogonal set and ρ be a metric on X. Then $(X, \perp, \rho)$ is called an orthogonal metric space (shortly O-metric space).*

Definition 1.7 ([12]). *Let $(X, \perp, \rho)$ be an orthogonal metric space. Then X is said to be a O-complete if every Cauchy O-sequence is converges in X.*

Definition 1.8 ([12]). *Let $(X, \perp, \rho)$ be an orthogonal metric space. A function $f : X \to X$ is said to be orthogonally continuous ($\perp$-continuous) at a if for each O-sequence $\{a_n\}$ converging to a implies $f(a_n) \to f(a)$ as $n \to \infty$. Also f is $\perp$-continuous on X if f is $\perp$-continuous at every $a \in X$.*

Definition 1.9 ([12]). *Let a pair $(X, \perp)$ be an O-set, where $X(\neq \emptyset)$ be a non-empty set and $\perp$ be a binary relation on set X. A mapping $f : X \to X$ is said to be $\perp$-preserving if $f(a) \perp f(b)$ whenever $a \perp b$ and weakly $\perp$-preserving if $f(a) \perp f(b)$ or $f(b) \perp f(a)$ whenever $a \perp b$.*

Definition 1.10 ([23]). *We say that an O-set is a transitive orthogonal set if $\perp$ is transitive.*

Definition 1.11 ([23]). *Let $(X, \perp)$ be an $O-$set. A path of length k in $\perp$ from a to b is a finite sequence $\{a_0, a_1, ..., a_k\} \subset X$ such that*

$$a_0 = a, a_k = b, a_i \perp a_{i+1} \text{ or } a_{i+1} \perp a_i$$

for all $i = 0, 1, ..., k-1$ and also $\lambda(a, b, \perp)$ be denoted as all path of length k in $\perp$ from a to b .

Before giving our main result, we want to remind some information about Geraghty contraction and also $(\hbar, \phi)$-contraction.

Let Λ be the family of all functions $\eta : [0, \infty) \to [0, 1)$ that satisfy the condition $\lim_{n\to\infty} \eta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

Furthermore, Ψ denotes the class of functions $\varpi : [0, \infty) \to [0, \infty)$ that satisfy the following conditions:

- ϖ is nondecreasing,
- ϖ is continuous,
- $\varpi(t) = 0$ if and ony if $t = 0$.

In [17], the authors proved the following particular result .

Theorem 1.2. *Let (X, ρ) be a complete metric-like space and $f : X \to X$ be a mapping. Suppose that there exists $\eta \in \Lambda$ such that*

$$\rho(fa, fb) \leq \eta((\rho(a, b))\rho(a, b)$$

for all $a, b \in X$. Then f has a unique fixed point $u \in X$ with $\rho(u, u) = 0$.

Recently, in [1], the authors considered a new type of Geraghty contractions in the class of metric-like spaces and proved a fixed point theorem for this type of contractive mapping such that:

Theorem 1.3. *Let (X, ρ) be a complete metric-like space and $f : X \to X$ be a mapping. Suppose that there exists $\eta \in \Lambda$ such that*

$$\rho(fa, fb) \leq \eta((\digamma(a, b))\digamma(a, b) \tag{1.1}$$

for all $a, b \in X$, where

$$\digamma(a, b) = \rho(a, b) + |\rho(a, fa) - \rho(b, fb)| \, .$$

Then f has a unique fixed point $u \in X$ with $\rho(u, u) = 0$.

On the other hand, in [15], Jleli et al. introduced a family $\mathcal{H}$ of functions $\hbar : [0, +\infty[^3 \to [0, +\infty[$ satisfying the following conditions:

(H_1) $\max\{a, b\} \leq \hbar(a, b, c)$ for all $a, b, c \in [0, +\infty[$;

(H_2) $\hbar(0,0,0)=0$;
(H_3) $\hbar$ is continuous.

Some examples of functions belonging to $\mathcal{H}$ are given as follows:

(i) $\hbar(a,b,c)=a+b+c$ for all $a,b,c\in[0,+\infty[$;
(ii) $\hbar(a,b,c)=\max\{a,b\}+c$ for all $a,b,c\in[0,+\infty[$;
(iii) $\hbar(a,b,c)=a+b+ab+c$ for all $a,b,c\in[0,+\infty[$.

Using a function $\hbar\in\mathcal{H}$, the authors of [15] introduced the following notion of $(\hbar,\phi)$-contraction as:

Definition 1.12 ([15]). *Let (M,ρ) be a metric space, $\phi: M\to[0,+\infty[$ be a given function and $\hbar\in\mathcal{H}$. Then, $f: M\to M$ is called a $(\hbar,\phi)$-contraction with respect to the metric ρ if and only if*

$$\hbar(\rho(fa,fb),\phi(fa),\phi(fb))\le k\hbar(\rho(a,b),\phi(a),\phi(b))\quad \text{for all } a,b\in M,$$

for some constant $k\in]0,1[$.

Now, we set

$$Z_\phi := \{a\in M:\phi(a)=0\},$$
$$F_f := \{a\in M: fa=a\}.$$

Furthermore, we say that f is a ϕ-Picard operator if and only if the following condition holds:

$$F_f\cap Z_\phi=\{\varsigma\} \text{ and } f^n a\to\varsigma, \text{ as } n\to+\infty \text{ for each } a\in M.$$

Theorem 1.4 ([15]). *Let (M,ρ) be a C.M.S, $\phi: M\to[0,+\infty[$ be a given function and $\hbar\in\mathcal{H}$. Suppose that the following conditions hold:*

(A_1) *ϕ is lower semi-continuous (l.s.c.);*
(A_2) *$f: M\to M$ is a $(\hbar,\phi)$-contraction with respect to the metric ρ.*

Then

(i) *$F_f\subset Z_\phi$;*
(ii) *f is a ϕ-Picard operator;*
(iii) *for all $a\in M$ and for all $n\in\mathbb{N}$, we have*

$$\rho(f^n a,\varsigma)\le\frac{k^n}{1-k}\hbar(\rho(fa,a),\phi(fa),\phi(a)),$$

where $\{\varsigma\}=F_f\cap Z_\phi=F_f$.

2. Main Results

2.1. A result for orthogonal ϖ_F-Geraghty contraction.

In this section, we give a definition of orthogonal ϖ_F-Geraghty contraction and we aim to obtain some results on O-complete metric-like space $(X,\perp,\rho)$.

Definition 2.13. *Let $(X,\perp,\rho)$ be an orthogonal metric-like space and $f: X\to X$ is a mapping. Then we say that f is orthogonal ϖ_F-Geraghty contraction if there exist $\varpi\in\Psi$ and $\eta\in\Lambda$ such that*

$$\varpi(\rho(fa,fb))\le\eta(\varpi(F(a,b)))\varpi(F(a,b)) \tag{2.2}$$

for all $a,b\in X$ with $a\perp b$, where

$$F(a,b)=\rho(a,b)+|\rho(a,fa)-\rho(b,fb)|\,.$$

Theorem 2.5. *Let $(X,\perp,\rho)$ be an O-complete metric-like space, a_0 is an orthogonal element and f is a self mapping on X satisfying the following conditions:*

(i) *$(X,\perp)$ is a transitive orthogonal set,*

(ii) *f is $\perp$-preserving,*
(iii) *f is orthogonal ϖ_F −Geraghty contraction,*
(iv) *f is $\perp$-continuous.*

Then, f has an unique fixed point in X.

Proof. From the definition of the orthogonality, it follows that $a_0 \perp f(a_0)$ or $f(a_0) \perp a_0$. Let

$$a_1 := fa_0, a_2 := fa_1 = f^2a_0, \cdots, a_n := fa_{n-1} = f^n a_0$$

for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $\rho(a_n, a_{n+1}) = 0$ for some n_0, so the proof is completed. Consequently, we assume that

$$\rho(a_n, a_{n+1}) \neq 0$$

for all n. Since f is $\perp$ −preserving, we have

$$a_n \perp a_{n+1} \text{ or } a_{n+1} \perp a_n.$$

This implies that $\{a_n\}$ is an O-sequence. Since f is orthogonal ϖ_F −Geraghty contraction, we have

$$\begin{aligned} \varpi(\rho(a_n, a_{n+1})) &= \varpi(\rho(fa_{n-1}, fa_n)) \\ &\leq \eta(\varpi(F(a_{n-1}, a_n)))\varpi(F(a_{n-1}, a_n)), \quad n \geq 1, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} F(a_{n-1}, a_n) &= \rho(a_{n-1}, a_n) + |\rho(a_{n-1}, fa_{n-1}) - \rho(a_n, fa_n)| \\ &= \rho(a_{n-1}, a_n) + |\rho(a_{n-1}, a_n) - \rho(a_n, a_{n+1})|. \end{aligned}$$

Take $\rho_n = \rho(a_{n-1}, a_n)$ and (2.3) becomes

$$\varpi(\rho_{n+1}) \leq \eta(\varpi(\rho_n + |\rho_n - \rho_{n+1}|))\varpi((\rho_n + |\rho_n - \rho_{n+1}|)). \tag{2.4}$$

Assume that there exists $n > 0$ such that $\rho_n \leq \rho_{n+1}$. From (2.4), we get

$$\varpi(\rho_{n+1}) \leq \eta(\varpi(\rho_{n+1}))\varpi(\rho_{n+1}) < \varpi(\rho_{n+1})$$

which is a contradiction. Thus for all $n > 0$, $\rho_n > \rho_{n+1}$. Hence, we deduce that the sequence $\{\rho_n\}$ is nonincreasing. Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \rho_n = r.$$

Now, we shall prove that $r = 0$. Suppose that $r > 0$. From (2.2), we have

$$\varpi(\rho(a_n, a_{n+1})) \leq \eta(\varpi(F(a_{n-1}, a_n)))\varpi(F(a_{n-1}, a_n))$$

which implies

$$\varpi(\rho_{n+1}) \leq \eta(\varpi(2\rho_n - \rho_{n+1}))\varpi(2\rho_n - \rho_{n+1}).$$

Hence

$$\frac{\varpi(\rho_{n+1})}{\varpi(2\rho_n - \rho_{n+1})} \leq \eta(\varpi(2\rho_n - \rho_{n+1})) < 1.$$

This implies that $\lim_{n\to\infty} \eta(\varpi(2\rho_n - \rho_{n+1})) = 1$. Since $\eta \in \Lambda$, we have

$$\lim_{n\to\infty} \varpi(2\rho_n - \rho_{n+1}) = 0$$

which yields

$$r = \lim_{n\to\infty} \rho(a_n, a_{n+1}) = 0 \tag{2.5}$$

which is a contradiction. So $r = 0$. Now, we shall prove that $\{a_n\}$ is a Cauchy O-sequence. We will prove that

(2.6) $$\lim_{n,m\to\infty} \rho(a_n, a_m) = 0.$$

We argue by contradiction. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{a_{m(k)}\}$ and $\{a_{n(k)}\}$ of $\{a_n\}$ with $m(k) > n(k) > k$ such that for every k

(2.7) $$\rho(a_{m(k)}, a_{n(k)}) \geq \varepsilon.$$

Moreover corresponding to $n(k)$, we can choose $m(k)$ in such a way that is the smallest integer with $m(k) > n(k)$ and satisfying (2.7). Then

(2.8) $$\rho(a_{m(k)-1}, a_{n(k)}) < \varepsilon.$$

Using (2.7) and (2.8)

$$\begin{aligned}\varepsilon &\leq \rho(a_{m(k)}, a_{n(k)}) \\ &\leq \rho(a_{m(k)}, a_{m(k)-1}) + \rho(a_{m(k)-1}, a_{n(k)}) \\ &< \rho(a_{m(k)}, a_{m(k)-1}) + \varepsilon.\end{aligned}$$

By (2.5), we get

(2.9) $$\lim_{k\to\infty} \rho(a_{m(k)}, a_{n(k)}) = \varepsilon.$$

On the other hand, it is easy to see that

$$\left|\rho(a_{m(k)-1}, a_{n(k)-1}) - \rho(a_{m(k)}, a_{n(k)})\right| \leq \left|\rho(a_{m(k)-1}, a_{m(k)}) + \rho(a_{n(k)}, a_{n(k)-1})\right|.$$

Again by (2.5) and (2.9)

(2.10) $$\lim_{k\to\infty} \rho(a_{m(k)-1}, a_{n(k)-1}) = \varepsilon.$$

We go back to (2.2) to have

$$\begin{aligned}\varpi(\varepsilon) &\leq \varpi(\rho(a_{m(k)}, a_{n(k)})) \\ &= \varpi(\rho(fa_{m(k)-1}, fa_{n(k)-1})) \\ &\leq \eta(\varpi(F(a_{m(k)-1}, a_{n(k)-1})))\varpi(F(a_{m(k)-1}, a_{n(k)-1})),\end{aligned}$$

where

$$F(a_{m(k)-1}, a_{n(k)-1}) = \rho(a_{m(k)-1}, a_{n(k)-1}) + \left|\rho(a_{m(k)-1}, fa_{m(k)-1}) - \rho(a_{n(k)-1}, fa_{n(k)-1})\right|.$$

By (2.5) and (2.10)

(2.11) $$\lim_{k\to\infty} F(a_{m(k)-1}, a_{n(k)-1}) = \varepsilon.$$

We deduce

$$\lim_{k\to\infty} \eta\left(\varpi\left(F(a_{m(k)-1}, a_{n(k)-1})\right)\right) = 1.$$

Since $\eta \in \eta$, we have

$$\lim_{k\to\infty} F(a_{m(k)-1}, a_{n(k)-1}) = 0$$

which is a contradiction with respect to (2.11). Thus $\{a_n\}$ is a Cauchy O-sequence in (X, ρ). So there exists $u^* \in X$ such that

$$\lim_{n\to\infty} \rho(a_n, u^*) = \rho(u^*, u^*) = \lim_{n,m\to\infty} \rho(a_n, a_m).$$

By (2.6), we write

$$\lim_{n\to\infty} \rho(a_n, u^*) = \rho(u^*, u^*) = 0$$

because of O-completeness of X. Since f is $\perp -$continuous, we have

$$fu^* = f(\lim_{n\to\infty} fa_n) = \lim_{n\to\infty} a_{n+1} = u^*$$

and so u^* is a fixed point of f.

Now, we can show the uniqueness of the fixed point. We shall prove that such u^* verifying $\rho(u^*, u^*) = 0$ is the unique fixed point of f. We argue by contradiction. Assume that there exists $u^* \neq w^*$ so $\rho(u^*, w^*) > 0$ such that

$$u^* = fu^*,\ w^* = fw^*,\ \rho(u^*, u^*) = \rho(w^*, w^*) = 0.$$

Suppose that there exist two distinct fixed point u^* and w^*. Since $\lambda(a, b, \perp)$ is non-empty for all $a, b \in X$, there exists a path $\{z_0, z_1, ..., z_k\}$ of some finite lenght k in $\perp$ from u^* to w^* such that

$$u_0 = u^*, u_k = w^*, u_i \perp u_{i+1} \text{ or } u_{i+1} \perp u_i.$$

Since $(X, \perp)$ transitive orthogonal set, we get $u^* \perp w^*$ or $w^* \perp u^*$. Then, we have

$$\begin{aligned} F(u^*, w) &=^* \rho(u^*, w^*) + |\rho(u^*, fz) - \rho(w^*, fw^*)| \\ &= \rho(u^*, w^*) + |\rho(u^*, u^*) - \rho(w^*, w^*)| \\ &= \rho(u^*, w^*) \end{aligned}$$

and using this equality in (2.2), we get

$$\begin{aligned} \varpi(\rho(u^*, w^*)) &= \varpi(\rho(fu^*, fw^*)) \\ &\leq \eta(\varpi(F(u^*, w^*)))\varpi(F(u^*, w^*)) \\ &= \eta(\varpi(\rho(u^*, w^*)))\varpi(\rho(u^*, w^*)) \\ &< \varpi(\rho(u^*, w^*)) \end{aligned}$$

which is a contradiction. Thus there exists a unique $u^* \in X$ such that $u^* = fu^*$ with $\rho(u^*, u^*) = 0$. □

Example 2.1. *Let $X = [0, 1]$ and $\rho(a, b) = a + b$. Then (X, ρ) is $O-$complete metric-like space. Define a relation $\perp$ on X by*

$$a \perp b \iff ab \in \{a, b\}.$$

Define $f : X \to X$ by

$$fx = \begin{cases} 0, & x = 1 \\ \frac{x}{2} & x \neq 1 \end{cases}.$$

Then we can see that f is $\perp$-preserving and also $\perp$-continuous. Take $\varpi(t) = \frac{t}{2}$ and $\eta(\alpha) = \frac{1}{2}$, then it is clear that f is a orthogonal ϖ_F-Geraghty contraction and f has a fixed point $u = 0$ with $\sigma(u, u) = 0$.

In Theorem 2.5, if we get $\varpi(t) = t$, then we obtain the following corollary.

Corollary 2.2. *Let $(X, \perp, \rho)$ be an O-complete metric-like space with an orthogonal elements a_0 and f be a self mapping on X satisfying the following conditions:*

(i) *$(X, \perp)$ is a transitive orthogonal set,*
(ii) *f is $\perp$-preserving,*
(iii) *f is orthogonal Geraghty contraction such that*

$$\varpi\rho(fa, fb) \leq \eta((F(a, b)))F(a, b)$$

for all $a, b \in X$ with $a \perp b$, where

$$F(a, b) = \rho(a, b) + |\rho(a, fa) - \rho(b, fb)|,$$

(iv) *f is $\perp$-continuous.*

Then, f has a fixed point in X.

2.2. **A result for orthogonal $(\hbar,\phi)$-contraction.** Now, we give a definition of $(\hbar,\phi)$-contraction on orthogonal metric-like space and prove a fixed point theorem for this type contraction.

Definition 2.14. *Let (X,ρ) be a orthogonal metric-like space and $f: X\to X$ be a mapping. f is called a orthogonal $(\hbar,\phi)$-contraction if there exist $\hbar\in\mathcal{H}$ and $\phi: M\to[0,+\infty[$ s.t.*

$$\hbar(\rho(fa,fb),\phi(fa),\phi(fb)))\leq F(\hbar(\rho(a,b),\phi(a),\phi(b))) \tag{2.12}$$

for all $a,b\in M$ with $a\perp b$.

Lemma 2.2. *Let (X,ρ) be a orthogonal metric-like space and $f: X\to X$ be a $(\hbar,\phi)$-contraction. If $\{a_n\}$ is a sequence of Picard starting at $a_0\in X$, then*

$$\lim_{n\to+\infty}\hbar(\rho(a_{n-1},a_n),\phi(a_{n-1}),\phi(a_n))=0,$$

and hence

$$\lim_{n\to+\infty}\rho(a_{n-1},a_n)=0 \quad \text{and} \quad \lim_{n\to+\infty}\phi(a_n)=0.$$

Proof. By replacing the contradiction in [26, (3.2)] with contradiction (2.12) and following the proof of [26, Lemma 3.1], we immediately have the desired result. □

Theorem 2.6. *Let $(X,\perp,\rho)$ be an O-complete metric-like space and a_0 is an orthogonal element of X and f be a self-mapping on X such that:*
i) For all $a,b\in X$ with $a\perp b$

$$\hbar(\rho(fa,fb),\phi(fa),\phi(fb))\leq k\hbar(\rho(a,b),\phi(a),\phi(b)) \tag{2.13}$$

for some $k\in(0,1)$, $\hbar\in\mathcal{H}$ and $\phi: X\to[0,\infty)$ is lower-semicontinuous function,
ii) f is $\perp$-preserving,
iii) $(X,\perp)$ is transitive orthogonal set.
Then f has a unique fixed point.

Proof. First, we shall prove the uniqueness. Suppose that a^*,b^* are two fixed point of f such that $a^*\neq b^*$. Since $\lambda(a,b,\perp)$ is non-empty for all $a,b\in X$, there exists a path $\{z_0,z_1,\cdots,z_k\}$ of some finite lenght k in $\perp$ from a to b such that

$$z_0=a^*, z_k=b^*, z_{i+1}\perp z_i \text{ for all } i=0,1,2,\cdots,k-1.$$

Since $(X,\perp)$ transitive orthogonal set, we get $a^*\perp b^*$ or $b^*\perp a^*$. From *i)*, we have

$$\begin{aligned}\hbar(\rho(a^*,b^*),\phi(a^*),\phi(b^*)) &= \hbar(\rho(fa^*,fb^*),\phi(fa^*),\phi(fb^*))\\ &\leq k\hbar(\rho(a^*,b^*),\phi(a^*),\phi(b^*))\\ &< \hbar(\rho(a^*,b^*),\phi(a^*),\phi(b^*))\end{aligned}$$

so, this is a contradiction. Then f has a unique fixed point.

Now, assume that $\varsigma\in X$ is a fixed point of f. Applying *(2.13)* with $a=b=\varsigma$, $a\perp b$, we have

$$\hbar(0,\phi(\varsigma),\phi(\varsigma))\leq k\hbar(0,\phi(\varsigma),\phi(\varsigma)),$$

which implies (since $k\in(0,1)$) that

$$\hbar(0,\phi(\varsigma),\phi(\varsigma))=0. \tag{2.14}$$

Moreover, from *(H1)*, we have

$$\phi(\varsigma)\leq\hbar(0,\phi(\varsigma),\phi(\varsigma)). \tag{2.15}$$

Using *(2.14)* and *(2.15)*, we obtain $\phi(\varsigma)=0$.

Now *ii)* from the definition of orthogonality, it follows that

$$a_0 \perp fa_0 \text{ or } fa_0 \perp a_0.$$

Let

$$a_1 = fa_0, a_2 = fa_1 = f^2 a_0, \cdots, a_n = fa_{n-1} = f^n a_0$$

for all $n \in \mathbb{N} \cup \{0\}$. If $a_{n^*} = a_{n^*+1}$ for some $n^* \in \mathbb{N} \cup \{0\}$, then

$$z = a_{n^*} = a_{n^*+1} = fa_{n^*} = fz$$

and z is a fixed point of f such that $\phi(z) = 0$. In fact by Lemma 2.2,

$$\hbar(\rho(a_{n^*-1}, a_{n^*}), \phi(a_{n^*-1}), \phi(a_{n^*})) = 0$$

and by the property *(H1)*, of the function $\hbar$, we have $\phi(z) = 0$. So, we assume that $a_n \neq a_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, we have

$$a_n \perp a_{n+1} \text{ or } a_{n+1} \perp a_n.$$

This implies that $\{a_n\}$ is a O-sequence. Since f is an orthogonal $(\hbar, \phi)$-contraction, we have

$$\begin{aligned}
&\hbar(\rho(a_n, a_{n+1}), \phi(a_n), \phi(a_{n+1})) \\
=&\hbar(\rho(fa_{n-1}, fa_n), \phi(fa_{n-1}), \phi(fa_n)) \\
\leq&k\hbar(\rho(a_{n-1}, a_n), \phi(a_{n-1}), \phi(a_n)) \\
\leq&k^n\hbar(\rho(a_0, a_1), \phi(a_0), \phi(a_1)) \\
=&k^n\hbar(\rho(a_0, fa_0), \phi(a_0), \phi(fa_0)), n \in \mathbb{N} \cup \{0\}
\end{aligned}$$

which implies by property *(H1)* that for all $n \in \mathbb{N} \cup \{0\}$

$$\max\{\rho(a_n, a_{n+1}), \phi(a_n)\} \leq k^n\hbar(\rho(a_0, fa_0), \phi(a_0), \phi(fa_0)).$$

Then, we obtain

$$\rho(a_n, a_{n+1}) \leq k^n\hbar(\rho(a_0, fa_0), \phi(a_0), \phi(fa_0)), n \in \mathbb{N} \cup \{0\}$$

which implies that $\{a_n\}$ is a Cauchy O-sequence. Since X is O-complete then, there exists $a^* \in X$ such that $a_n \to a^*$ as $n \to \infty$.

Since f is orthogonal $(\hbar, \phi)$-contraction, taking into account that ϕ is lower-semicontinuous function, we have

$$0 \leq \phi(a^*) \leq \lim \inf_{n\to\infty} \phi(a_n) = 0,$$

that is, $\phi(a^*) = 0$. Now, show that a^* is a fixed point f.

If there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} = a^*$ or $fa_{n_k} = fa^*$ for all $k \in \mathbb{N}$, then a^* is a fixed point. Otherwise, we can assume that $a_n \neq a^*$ and $fa_n \neq fa^*$ for all $n \in \mathbb{N}$. So, using f is an $(\hbar, \phi)$-contraction, we deduce that for all $n \in \mathbb{N}$

$$\begin{aligned}
\hbar(\rho(fa_n, fa^*), \phi(fa_n), \phi(fa^*)) &\leq k\hbar(\rho(a_n, a^*), \phi(a_n), \phi(a^*)) \\
&< \hbar(\rho(a_n, a^*), \phi(a_n), \phi(a^*)),
\end{aligned}$$

and so,

$$\begin{aligned}
\rho(a^*, fa^*) &\leq \rho(a^*, a_{n+1}) + \rho(fa_n, fa^*) \\
&\leq \rho(a^*, a_{n+1}) + \hbar(\rho(fa_n, fa^*), \phi(fa_n), \phi(fa^*)) \\
&< \rho(a^*, a_{n+1}) + \hbar(\rho(a_n, a^*), \phi(a_n), \phi(a^*)) \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Finally, letting $n \to \infty$ in the above calculations and using that $\hbar$ is continuous in $(0, 0, 0)$, we deduce that

$$\rho(a^*, fa^*) \leq \hbar(0, 0, 0)$$

that is, $a^* = fa^*$. □

Remark 2.1. *In the Theorem2.6, if we assume that f is $\perp$-continuous, we have*

$$f\mu^* = f\left(\lim_{n\to\infty}\mu_n\right) = \lim_{n\to\infty}\mu_{n+1} = \mu^*$$

and μ^* is a fixed point of f.

2.3. **A result for rational $F-$contraction.** In this part, we modify definition of rational type $F-$contraction using orthogonality and then give some results for this type contraction on $O-$complete metric-like space. But firstly, we want to give some information about $F-$contraction.

Let $\mathcal{F}$ be the set of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying the following conditions:

($F1$) F is strictly increasing, i.e., for all $\alpha,\beta\in(0,\infty)$ such that $\alpha<\beta, F(\alpha)<F(\beta)$,
($F2$) for each sequence $\{a_n\}$ of positive numbers,

$$\lim_{n\to\infty} a_n = 0 \text{ if and only if } \lim_{n\to\infty} F(a_n) = -\infty,$$

($F3$) there exists $k\in(0,1)$ such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha)=0$.

Definition 2.15 ([28])**.** *Let (X,ρ) be a metric space and $f:X\to X$ be a mapping. Given $F\in\mathcal{F}$, f is called as F-contraction if there exists $\tau>0$ such that*

$$a,b\in M,\ \rho(fa,fb)>0\Rightarrow \tau+F(\rho(fa,fb))\le F(\rho(a,b)).$$

Definition 2.16. *Let $(X,\perp,\rho)$ be an orthogonal metric-like space. We say that $f:X\to X$ is an orthogonal rational type $F-$contraction if there are $F\in\mathcal{F}$ and $\tau>0$ such that the following condition holds:*

$$\forall a,b\in X \text{ with } a\perp b;\ \left[\rho(fa,fb)>0\Rightarrow \tau+F(\rho(fa,fb))\le F(M(a,b))\right], \tag{2.16}$$

where

$$M(a,b)=\max\left\{\begin{array}{c}\rho(a,b),\rho(a,fa),\rho(b,fb),\\ \frac{\rho(a,fa)\rho(b,fb)}{1+\rho(a,b)},\frac{\rho(a,fa)\rho(b,fb)}{1+\rho(fa,fb)}\end{array}\right\}.$$

Theorem 2.7. *Let $(X,\perp,\rho)$ be an $O-$complete orthogonal metric-like space, a_0 is an orthogonal element of X and f be a $\perp-$preserving and $\perp-$continuous mapping with satisfying (2.16). Also we assume that $(X,\perp)$ is a transitive orthogonal set, then, f has a unique fixed point in X.*

Proof. Using the definition of the orthogonality, we have $a_0\perp f(a_0)$ or $f(a_0)\perp a_0$. Let

$$a_1:=fa_0, a_2:=fa_1=f^2a_0,\cdots,a_n:=fa_{n-1}=f^na_0$$

for all $n\in\mathbb{N}\cup\{0\}$. Suppose that $\rho(a_n,a_{n+1})=0$ for some n_0, so the proof is completed. Consequently, we assume that

$$\rho(a_n,a_{n+1})\neq 0$$

for all n. Thus, we get $\rho(a_n,a_{n+1})>0$ for all $n\in\mathbb{N}\cup\{0\}$. Then, we obtain

$$a_n\perp a_{n+1} \text{ or } a_{n+1}\perp a_n$$

from $\perp$ $-$preserving of f and then we say that $\{a_n\}$ is an $O-$sequence. From (2.16), for all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
& F(\rho(a_n, a_{n+1})) \\
=& F(\rho(fa_{n-1}, fa_n)) \\
\leq& F(M(a_{n-1}, a_n)) - \tau \\
=& F\left(\max\left\{\begin{array}{c} \rho(a_{n-1}, a_n), \rho(a_{n-1}, fa_{n-1}), \rho(a_n, fa_n), \\ \frac{\rho(a_{n-1}, fa_{n-1})\rho(a_n, fa_n)}{1+\rho(a_{n-1}, a_n)}, \frac{\rho(a_{n-1}, fa_{n-1})\rho(a_n, fa_n)}{1+\rho(fa_{n-1}, fa_n)} \end{array}\right\}\right) - \tau \\
\leq& F(\rho(a_{n-1}, a_n)) - \tau. \qquad (2.17)
\end{aligned}
$$

Let $\alpha_n := \rho(a_n, a_{n+1})$ for all $n \in \mathbb{N}$ and from (2.17), we have

$$F(\alpha_n) \leq F(\alpha_{n-1}) - \tau \leq F(\alpha_{n-2}) - 2\tau \leq \cdots \leq F(\alpha_0) - n\tau. \qquad (2.18)$$

From (2.18), we get $\lim_{n\to\infty} F(\alpha_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} \alpha_n = 0. \qquad (2.19)$$

By the property (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \alpha_n^k F(\alpha_n) = 0. \qquad (2.20)$$

By (2.18), we get

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq -\alpha_n^k n\tau \leq 0 \qquad (2.21)$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.21), we get

$$\lim_{n\to\infty} n\alpha_n^k = 0. \qquad (2.22)$$

From (2.22), there exits $n_1 \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$ for all $n \geq n_1$. So we have

$$\alpha_n \leq \frac{1}{n^{1/k}} \qquad (2.23)$$

for all $n \geq n_1$. In order to show that $\{a_n\}$ is a Cauchy $O-$sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.23), we have

$$
\begin{aligned}
\rho(a_n, a_m) &\leq \rho(a_n, a_{n+1}) + \rho(a_{n+1}, a_{n+2}) + \cdots + \rho(a_{m-1}, a_m) \\
&= a_n + a_{n+1} + \cdots + a_{m-1} \\
&= \sum_{i=n}^{m-1} a_i \\
&\leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}.
\end{aligned}
$$

Hence $\{a_n\}$ is a Cauchy $O-$sequence in M. Because of the $O-$completeness of X, we have $a^* \in M$ such that $a_n \to a^*$ as $n \to \infty$. Using $\perp -$ continuous of f, we have

$$fa^* = f(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} a_{n+1} = a^*$$

and so a^* is a fixed point of f.

Now, we can show the uniqueness of the fixed point. We shall prove that such a^* verifying $\rho(a^*, a^*) = 0$ is the unique fixed point of f. We argue by contradiction. Assume that there exists $a^* \neq w^*$ so $\rho(a^*, w^*) > 0$ such that

$$a^* = fa^*, \; w^* = fw^*, \; \rho(a^*, a^*) = \rho(w^*, w^*) = 0.$$

Suppose that there exist two distinct fixed point a^* and w^*. Since $\lambda(a, b, \perp)$ is non-empty for all $a, b \in X$, there exists a path $\{z_0, z_1, ..., z_k\}$ of some finite lenght k in $\perp$ from a^* to w^* such that

$$u_0 = a^*, u_k = w^*, u_i \perp u_{i+1} \text{ or } u_{i+1} \perp u_i.$$

Since $(X, \perp)$ transitive orthogonal set, we get $u^* \perp w^*$ or $w^* \perp u^*$. Then, we get

$$\begin{aligned}
&\tau + F(\rho(a^*, w^*)) \\
&= \tau + F(\rho(fa^*, fw^*)) \\
&\leq F(M(a^*, w^*)) \\
&= F\left(\max\left\{\begin{array}{c} \rho(a^*, w^*), \rho(a^*, fa^*), \rho(w^*, fw^*), \\ \frac{\rho(a^*,fa^*)\rho(w^*,fw^*)}{1+\rho(a^*,w^*)}, \frac{\rho(a^*,fa^*)\rho(w^*,fw^*)}{1+\rho(fa^*,fw^*)} \end{array}\right\}\right) \\
&= F(C(a^*, w^*))
\end{aligned}$$

which is a contradiction. Thus there exists a unique $a^* \in X$ such that $a^* = fa^*$ with $\rho(a^*, a^*) = 0$. □

Corollary 2.3. *Let $(X, \perp, \rho)$ be an $O-$complete metric-like space with an orthogonal elements a_0 and f be a $\perp-$preserving and $\perp-$continuous self mapping on M such that*

$$\forall a, b \in X \text{ with } a \perp b; \;\; [\rho(fa, fb) > 0 \Rightarrow \tau + F(\rho(fa, fb)) \leq F(\rho(a, b))].$$

Then, T has a unique fixed point in M.

3. Applications

Recall that, for any $1 \leq p < \infty$, the space $Ł^p(X, F, \mu)$ (or $Ł^p(X)$) consists of all complex-valued measurable functions κ on the underlying space X satisfying

$$\int_M |\kappa(\wp)|^p \, d\mu(\wp),$$

where F is the σ-algebra of measurable sets and μ is the measure. When $p = 1$, the space $Ł^p(X)$ consists of all integrable functions κ on X and we define the $Ł^1$-norm of κ by

$$\|\kappa\|_1 = \int_M |\kappa(\wp)| \, d\mu(\wp).$$

In the section, using Theorem 2.7, we show the existence of a solution of the following differential equation:

$$\left\{\begin{array}{ll} u'(t) = \mathrm{f}(t, u(t)), & a.e.\ t \in I := [0, T] \\ u(0) = a, & a \geq 1, \end{array}\right. \tag{3.24}$$

where $\mathrm{f} : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function satisfying the following conditions:
(i) $\mathrm{f}(s, \eta) \geq 0$ for all $\eta \geq 0$ and $s \in I$;

(ii) for each $\wp, \gamma \in Ł^1(I)$ with $\wp(s)\gamma(s) \geq \wp(s)$ or $\wp(s)\gamma(s) \geq \gamma(s)$ for all $s \in I$, there exist $\kappa \in Ł^1(I)$ and $\tau > 0$ such that

$$|f(s, \wp(s)) - f(s, \gamma(s))| \leq \frac{\kappa(s)}{\left(1 + \tau\sqrt{\kappa(s)}\right)^2} |\wp(s) - \gamma(s)| \tag{3.25}$$

and

$$|\wp(s) - \gamma(s)| \leq \kappa(s) e^{A(s)}$$

for all $s \in I$, where $A(s) := \int_0^s |\kappa(w)| dw$.

Theorem 3.8. *Consider the differential equation (3.24). If (i) and (ii) are satisfied, then the differential equation (3.24) has a unique positive solution.*

Proof. Let $X = \{u \in C(I, \mathbb{R}) : u(t) > 0 \text{ for all } t \in I\}$. Define the orthogonality relation $\perp$ on X by

$$\wp \perp \gamma \Longleftrightarrow \wp(s)\gamma(s) \geq \wp(s) \text{ or } \wp(s)\gamma(s) \geq \gamma(s) \text{ for all } t \in I.$$

Since $A(t) = \int_0^t |\kappa(s)| ds$, we have $A'(t) = |\kappa(t)|$ for almost everywhere $t \in I$.

Define a mapping $d(\wp, \gamma) = \|\wp - \gamma\|_A = \sup_{t \in I} e^{-A(t)} |\wp(s) - \gamma(s)|$ for all $\wp, \gamma \in X$. Thus, (X, d) is a metric-like space and also a complete metric-like space. Define a mapping $\not\prec : X \to X$ by

$$(\not\prec\wp)(t) = a + \int_0^t f(s, \wp(s)) ds.$$

Then, we see that $\not\prec$ is $\perp$-continuous.

Now, we show that $\not\prec$ is $\perp$- preserving. For each $\wp, \gamma \in X$ with $\wp \perp \gamma$ and $t \in I$, we have

$$(\not\prec\wp)(t) = a + \int_0^t f(s, \wp(s)) ds \geq 1.$$

It follows that $[(\not\prec\wp)(t)][(\not\prec\gamma)(t)] \geq (\not\prec\gamma)(t)$ and so $(\not\prec\wp)(t) \perp (\not\prec\gamma)(t)$. Then $\not\prec$ is $\perp$-preserving.

Now, we can say that $\not\prec$ satisfies Corollary 2.3 with $F(\alpha) = \frac{-1}{\sqrt{\alpha}}$. Hence the differential equation (3.24) has a unique positive solution. □

Acknowledgement. Some parts of this paper were presented at IWCMA - 2nd International Workshop: Constructive Mathematical Analysis.

REFERENCES

[1] H. Aydi, A. Felhic and H.Afsharid: *New Geraghty type contractions on metric-like spaces*, J. Nonlinear Sci. Appl., **10** (2017), 780-788.
[2] Ö. Acar, A. S. Özkapu: *Multivalued rational type $F-$contraction on orthogonal metric space*, Math. Found. Comput., **6** (3) (2023), 303–312.
[3] Ö. Acar, E. Erdoğan: *Some fixed point results for almost contraction on orthogonal metric space*, Creat. Math. Inform., **31** (2) (2022), 147–153.
[4] Ö. Acar, A. S. Özkapu and E. Erdoğan: *Some Fixed Point Results on Orthogonal Metric Space*, Bull. Comput. Appl. Math., **50** (1) (2023), 53–59.

[5] S. S. Mohammed, M. Alansari, A. Azam and S. Kanwal: *Fixed points of* (φ, F) *-weak contractions on metric-like spaces with applications to integral equations on time scales*, Bol. Soc. Mat. Mex., **27** (2) (2021), ARTICLE ID: 39.

[6] A. Amini-Harandi: *Metric-like spaces, partial metric spaces and fixed points*, Fixed Point Theory Appl., 2012 (2012), 10 pages.

[7] S. Banach, *Sur les opérations dans les ensembles abstraits et leurs applicationsauxéquations int égrales*, Fund. Math., **3** (1992), 133–181.

[8] S. Kanokwan, W. Sintunavarat and Y. J. Cho: *Fixed point theorems for orthogonal* $F-$*contraction mappings on O-complete metric space*, J. Fixed Point Theory Appl., (2020) 22:10.

[9] D. O'Regan: *Equilibria for abstract economies in Hausdorff topological vector spaces*, Constr. Math. Anal., **5** (2) (2022), 54–59.

[10] E. Karapınar: *A Short Survey on the Recent Fixed Point Results on* $b-$*Metric Spaces*, Constr. Math. Anal., **1** (1) (2018), 15–44.

[11] R. Heckmann: *Approximation of metric spaces by partial metric spaces*, Appl. Categ. Structures, **7** (1999) 71–83.

[12] M.E. Gordji, M. Rameani, M. De La Sen and Y. J. Cho: *On orthogonal sets and Banach fixed point theorem*, Fixed Point Theory **18** (2017), 569–578.

[13] M. E. Gordji, H. Habibi, *Fixed point theory in generalized orthogonal metric space*, Journal of Linear and Topological Algebra (JLTA), **6** (3), 251–260.

[14] N. B. Gungor: *Extensions of Orthogonal* $p-$*Contraction on Orthogonal Metric Spaces*, Symmetry, **14** 2022, 746.

[15] M. Jleli, B. Samet and C. Vetro: *Fixed point theory in partial metric spaces via* ϕ*-fixed point's concept in metric spaces*, J. Inequal. Appl., **2014**:426 (2014), 9 pp.

[16] E. Karapınar, P. Salimi: *Dislocated metric space to metric spaces with some fixed point theorems*, Fixed Point Theory Appl., **2013** (2013), 19 pages.

[17] E. Karapınar, H. H. Alsulami and M. Noorwali: *Some extensions for Geragthy type contractive mappings*, Fixed Point Theory Appl., **2015** (2015), 22 pages.

[18] S. G. Matthews: Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., vol. 728, 1994, pp. 183–197.

[19] M. Nazam, H. Aydi and A. Hussain: *Existence theorems for* $(\Psi, \Phi)-$*orthogonal interpolative contractions and an application to fractional differential equations*, Optimization, **72** (7) (2023), 1899–1929.

[20] M. Nazam, C. Park and M. Arshad, *Fixed point problems for generalized contractions with applications*, Adv. Differential Equations, **2021**:247 (2021).

[21] S. Som, A. Dey Petruşel and K. Lakshmi: *Some remarks on the metrizability of some metric-like structures*, Carpathian J. Math., **37** (2) (2021), 265–272.

[22] K., Sawangsup, W., Sintunavarat and Y. J., Cho: *Fixed point theorems for orthogonal* $F-$*contraction mappings on* $O-$*complete metric spaces*, J. Fixed Point Theorey Appl., **22**:10 (2020).

[23] K. Sawangsup, W. Sintunavarat: *Fixed Point Results for Orthogonal* $Z-$*Contraction Mappings in* $O-$*Complete Metric Spaces*, Int. J. Appl. Physics Math., **10** (1) (2020), 33–40.

[24] B. Singh, V. Singh, I. Uddin and Ö. Acar: *Fixed point theorems on an orthogonal metric space using Matkowski type contraction*, Carpathian Math. Publ., **14** (1) (2022), 127–134.

[25] A. Alsaadi, B. Singh, V. Singh and I. Uddin: *Meir-Keeler type contraction in orthogonal* $M-$*metric spaces*, Symmetry, **14** (9) (2022), 1856.

[26] C. Vetro: *A fixed-point problem with mixed-type contractive condition*, Constr. Math. Anal., **3** (1) (2020), 45–52.

[27] Z. Kadelburg, S. Radenovic: *Notes on Some Recent Papers Concerning* $F-$*Contractions in b-Metric Spaces*, Constr. Math. Anal., **1** (2) (2018), 108–112.

[28] D. Wardowski: *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012** (2012), ARTICLE ID: 94.

ÖZLEM ACAR
SELÇUK UNIVERSITY
DEPARTMENT OF MATHEMATICS
42003, KONYA, TURKEY
ORCID: 0000-0001-6052-4357
Email address: acarozlem@ymail.com