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Review of The Fibonacci Sequences In Finite Groups

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Abstract: In the work of Knox, Steven W. (1992), it is claimed that "A *k-nacci* sequence in a finite group is simply periodic (Knox, 1992)." We provide examples to demonstrate that the claim isn't true ingeneral and we indicate that there is at least a simply periodic *k*-nacci sequence in a finite group.

Keywords: Fibonacci sequences, Period, Binary polyhedral group, Extended triangle group. *Mathematics Subject Classification Number:* **20F05, 20D60, 11B39**

Sonlu Gruplardaki Fibonacci Dizilerinin Yeniden İncelenmesi

Özet: Knox, Steven W. (1992)' deki çalışmasında, *sonlu bir grupdaki bir k-nacci dizisinin basit periodik* (Knox, 1992) olduğunu belirtmiştir. Biz bu ifadenin genelde doğru olmadığını göstermek için örnekler verdik ve sonlu bir grupda en az bir tane basit periodik *k-nacci* dizisinin mevcut olduğunu belirttik.

Anahtar kelimeler: Fibonacci dizileri, Period, Binary polyhedral grup, Genişletilmiş triangle grup: **20F05, 20D60, 11B39**

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Introduction

The study of Fibonacci sequences in groups began with the earlier work of Wall (Wall 1969) where the ordinary Fibonacci sequences in cyclic groups were investigated. In the mid eighties, Wilcox extended the problem to abelian groups (Wilcox 1986). Prolific co-operation of Campbell, Doostie and Robertson expanded the theory to some finite simple groups (Campbell 1990). Aydın and Smith proved in (Aydın and Smith 1994) that the lengths of ordinary 2-step Fibonacci sequences are equal to the lengths of ordinary 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and a prime exponent. The theory has been generalized in Dikici and Smith (1997), Dikici and Smith (1995) to the ordinary 3-step Fibonacci sequences in finite nilpotent groups. Then, it is shown in Aydın and Dikici (1998) that the period of 2-step general Fibonacci sequence is equal to the length of fundamental period of the 2-step general recurrence constructed by two generating elements of the group of exponent p and nilpotency class 2. Karaduman and Yavuz showed that the periods of the 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class *5* and a prime exponent are $p.k(p)$, for $2 < p \leq 2927$, where p is prime and $k(p)$ is the periods of ordinary 2-step Fibonacci sequences (Karaduman and Yavuz 2002). Knox proved that periods of *k*-nacci (*k*-step Fibonacci) sequences in dihedral group were equal to $2k+2$ (Knox 1992). Karaduman and Deveci examined the behaviour period of the k-nacci sequences in the some finite polyhedral groups in (2009). Recently, the works have been done on Fibonacci sequences. See, for example, (Campbell and Campbell 2009, Campbell and Campbell 2005, Campbell et al. 2004, Doostie and Hashemi 2006, Özkan 2003).

Main Results and Proofs

A *k*-nacci sequence in a finite group is a sequence of group elements $x_0, x_1, x_2, \cdots, x_n, \cdots$ for which, given an initial (seed) set $x_0, x_1, x_2, \cdots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \leq n < k \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \geq k \end{cases}.$$

We also require that the initial elements of the sequence, $x_0, x_1, x_2, \cdots, x_{j-1}$, generate the group, thus forcing the *k*-nacci sequence to reflect the structure of the group. The *k*-nacci sequence of a group generated by $x_0, x_1, x_2, \cdots, x_{j-1}$ is denoted by $F_k(G; x_0, x_1, \cdots, x_{j-1})$ and its period is denoted by $P_k(G; x_0, x_1, \cdots, x_{j-1})$.

Definition 2.1. A *2*-step Fibonacci sequence of a group elements is called a Fibonacci sequence of a finite group.

Definition 2.2. A finite group G is *k*-nacci sequenceable if there exists a *k*-nacci sequence of G such that every element of the group appears in the sequence.

Definition 2.3. A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called period of the sequence. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \cdots$ is periodic after the initial element a and has period 4.

Definition 2.4. A sequence of group elements is simply periodic with period *k* if the first *k* elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, f, g, a, b, c, d, e, f, g, a, b, c, d, e, f, g, \cdots$ is simply periodic with period 7.

The following appears in [Knox, 1992, Theorem 1].

Theorem 2.5. A k-nacci sequence in a finite group is simply periodic (Knox 1992).

Definition 2.6. The *binary polyhedral group* $< l,m,n >$, for $l,m,n > 1$ is defined by the presentation

$$< x,y,z : x^l = y^m = z^n = xyz > .$$

The binary polyhedral group $< l,m,n >$, is finite if, and only if, the number

$$\mu = lmn\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}-1\right) = mn+nl+lm-lmn$$

is positive. Its order is $4lmn/\mu$.

For more information on these groups see (Campbell and Campbell 2009).

Definition 2.7. The extended triangle group $E(p,q,r)$, for $p,q,r>1$, is defined by the presentation

$$\left\langle x,y,z : x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^r = 1 \right\rangle$$

The triangle group (polyhedral group) (p,q,r), is finite if, and only if, the number

$$\mu = lmn\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}-1\right) = mn+nl+lm-lmn$$

is positive. Its order is $2lmn/\mu$. The triangle groups (polyhedral groups) (p,q,r), are index two subgroups of the extended triangle groups.

For more information on these groups see (Campbell and Campbell 2009, Conder 2003).

Now, we will give examples satisfying the condition of [Knox, 1992, Theorem 1], but the 2-nacci sequence in a finite group is not to be necessary *simply periodic*.

Example 2.8. Let us consider the *binary polyhedral group* $< 2,2,n >$,for $n > 2$, defined by the presentation

$$< x,y,z : x^2 = y^2 = z^n = xyz > .$$

The order the group defined by this presentation is $4n$ and the order of z is $2n$ and the orders of x and y are 4. By the definition of a k-nacci sequence in a finite group, we obtain *2-nacci* sequence in the group $< x,y,z : x^2 = y^2 = z^n = xyz >$ as follow, with respect generating set $\{x,y,z\}$,

$$\begin{aligned}
x_0 &= x,\\
x_1 &= y,\\
x_2 &= z,\\
x_3 &= yz = x,\\
x_4 &= zx = y,\\
x_5 &= xy,\\
x_6 &= yxy,\\
x_7 &= xyyxy = x^4y = y = x_1,\\
x_8 &= yxyy = yx^3 = z = x_2,\\
x_9 &= yz = x = x_3,\\
x_{10} &= zx = y = x_4,\\
x_{11} &= xy = x_5,\\
x_{12} &= yxy = x_6,\\
&\vdots
\end{aligned}$$

It is clear that *2-nacci* sequence in the group $< 2,2,n >$ is periodic with period 6 but not *simply periodic*.

Example 2.9. Let $E(2,2,2)$ be a extended triangle group defined by the presentation

$$\left\langle x,y,z : x^2 = y^2 = z^2 = (xy)^2 = (yz)^2 = (zx)^2 = 1 \right\rangle .$$

The order the group defined by this presentation is 8 and the order of x, y and z are 2. Thus, from relations in the group we have

$$xy = yx,$$

$$xz = zx,$$

$$yz = zy.$$

Using definition of a k-nacci sequence in a finite group, we obtain 2-nacci sequence in the group defined by this presentation as follow:

$$x_0 = x,$$

$$x_1 = y,$$

$$x_2 = z$$

$$x_3 = yz,$$

$$x_4 = zyz = y = x_1,$$

$$x_5 = yzy = z = x_2,$$

$$x_6 = yz = x_3,$$

$$\vdots$$

It is clear that *2*-nacci sequence in the group $E(2,2,2)$ is periodic with period 3 but not *simply periodic* with respect the initial elements x, y, z.

Result and Discussion

From the definition of a *k*-nacci sequence in a finite group, the initial elements of the sequence, $x_0, x_1, x_2, \cdots x_{j-1}$, is require to generate the group.

In the examples that are given above it is obvious that the initial elements x, y, z of the *2*-nacci sequences are generate the finite groups in these examples, but none of *2*-nacci sequences are not simply periodic according to the initial elements x, y, z.

We notice that being simply periodic of a *2*-nacci sequence in a finite group is related initial elements that are chosen for this sequence. So the body of Theorem 5 should be "there is at least a simply periodic *k*-nacci sequence in a finite group".

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