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On ρ-Statistically Convergence Defined by a Modulus Function in Fuzzy Difference Sequences

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Bulanık Fark Dizilerinde Bir Modülüs Fonksiyonu Yardımıyla Tanımlı ρ-İstatistiksel Yakınsaklık Üzerine

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Abstract
In this study, we first introduced the definition $\Delta_\rho^m -$statistical convergence for sequences of fuzzy numbers using the generalized difference operator Δ^m. Furthermore, we defined the strong $N_F^\rho(\Delta^m, q) -$summable sequence set and the strong $N_F^\rho(\Delta^m, f, q) -$summable sequence set for fuzzy difference sequences aided by a modulus function f. Subsequently, we provided certain inclusion theorems between these sets and the $S_F^\rho(\Delta^m)$ set.

***Keywords** Statistical covergence; Sequence of fuzzy numbers; Modulus function; Difference sequence*

Öz
Bu çalışmada ilk olarak bulanık sayı dizileri için Δ^m genelleştirilmiş fark operatörünü kullanarak $\rho -$istatistiksel yakınsaklık tanımını verdik. Ayrıca bulanık fark dizileri için kuvvetli $N_F^\rho(\Delta^m, q) -$toplanabilir dizi kümesini ve bir f modülüs fonksiyonu yardımıyla tanımlanan kuvvetli $N_F^\rho(\Delta^m, f, q) -$toplanabilir dizi kümesini tanımladık. Daha sonra bu kümelerle $S_F^\rho(\Delta^m)$ kümesi arasındaki bazı kapsama teoremlerini verdik.

***Anahtar Kelimeler** İstatistiksel yakınsaklık; Bulanık sayı dizisi; Modülüs fonksiyonu; Fark dizisi*

1. Introduction

The concept of statistical convergence was defined independently by Fast (1951) and Steinhaus (1951). Schoenberg (1959) redefined the concept of statistical convergence and provided some of its properties. Subsequently, statistical convergence has been used by many researchers in statistical measurement theory, summability theory, Banach spaces, trigonometric series, and fuzzy set theory. Several researchers, including (Altınok and Yağdıran 2017, Barlak 2022, Karakaş et al. 2014, Şengül et al. 2020, Torgut and Altin 2020), have conducted studies on this concept.

Zadeh (1965) first introduced fuzzy set theory. Matloka (1986) provided the definition of fuzzy number sequences and defined the concepts of boundedness and convergence for sequences of fuzzy numbers, along with some properties. He showed that many properties valid for real number sequences also hold for fuzzy number sequences. Since then, numerous studies have been conducted and continue to be conducted on sequences of fuzzy numbers.

The concept of statistical convergence for sequences of fuzzy numbers was introduced by Nuray and Savaş (1995). Nuray and Savaş (1995) and Kwon (2000) examined the relationship between statistical convergence, convergence, lacunary statistical convergence and strong Cesàro convergence in sequences of fuzzy numbers.

Çakallı (2017) defined $\rho -$statistical convergence for sequences of real numbers. Subsequently, several researchers, including (Aral et al. 2020, Aral 2022, Aral et al. 2022, Çakallı et al. 2020, Gumus 2022, Kandemir 2022), have conducted studies on this topic.

Kızmaz (1981) was firstly introduced the concept of the difference operator in the sequence spaces. Additionally, Et and Çolak (1995) generalized the idea of difference sequence spaces of Kızmaz. Besides this topic was studied by many authors (Bektaş et al. 2004, Et and Esi 2000, Karakaş 2023, Turan 2017). The aim of this study is to generalize the concept of $\rho -$statistically convergence with the help of a f modulus function by using the generalized difference operator Δ^m defined as $\Delta^m Z_k = \Delta^{m-1} Z_k - \Delta^{m-1} Z_{k+1},\ m = (1,2,3, \dots)$ and to fill the

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existing gaps in the generalized statistical convergence theory of fuzzy number sequences in the literature.

2. Definitions and Preliminaries

In this section, we have discussed the fundamental concepts that we will use throughout this study.

A fuzzy number is a fuzzy set that maps from the real numbers $\mathbb{R}$ to the closed interval [0,1], satisfying the following properties:

(i) Z is normal, which means there exists $z_0 \in \mathbb{R}$ such that $Z(z_0) = 1$.

(ii) Z is fuzzy convex, which means for $z, t \in \mathbb{R}$ and $0 \leq \beta \leq 1$, we have $Z(\beta z + (1-\beta)t) \geq \min[Z(z), Z(t)]$.

(iii) Z is upper semi-continuous.

(iv) The support of Z, denoted by suppZ, is defined as the closure of the set $\{Z \in \mathbb{R}: Z(z) > 0\}$, which is a compact set.

An α-level set of a fuzzy number, denoted as $[Z]^\alpha$, is defined as follows:

$$[Z]^\alpha = \begin{cases} \{z \in \mathbb{R}: Z(z) \geq \alpha\}, & if\ \alpha \in (0,1] \\ \text{supp}Z, & if\ \alpha = 0 \end{cases}$$

For a number Z to be a fuzzy number, the necessary and sufficient condition is that for each $\alpha \in [0,1]$, the set $[Z]^\alpha$ is a closed interval, and $[Z]^1 \neq \emptyset$ is obvious. We will denote the space of all fuzzy numbers with real terms as $L(\mathbb{R})$.

The distance between fuzzy numbers Z and T is calculated using the metric:

$$d(Z,T) = \sup_{0 \leq \alpha \leq 1} d_H([Z]^\alpha, [T]^\alpha)$$

where d_H is the Hausdorff metric and for $Z^\alpha = [\underline{Z}^\alpha, \bar{Z}^\alpha]$ and $T^\alpha = [\underline{T}^\alpha, \bar{T}^\alpha]$, it is defined as:

$$d_H([Z]^\alpha, [T]^\alpha) = \max\{|\underline{Z}^\alpha - \underline{T}^\alpha|, |\bar{Z}^\alpha - \bar{T}^\alpha|\}.$$

The distance d is a metric on $L(\mathbb{R})$ and it is complete.

A sequence $Z = (Z_k)$ of fuzzy number is a function Z from the set $\mathbb{N}$ of all natural numbers into $L(\mathbb{R})$ that is $Z: \mathbb{N} \to L(\mathbb{R})$ (Matloka 1986). In this case, each term of the sequence (Z_k) corresponds to a fuzzy number.

The natural density of a subset E, which is a subset of the set of natural numbers $\mathbb{N}$, is defined as follows:

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n: k \in E\}|.$$

Here, the expression $|\{k \leq n: k \in E\}|$ represents the number of elements in E that are not greater than n.

A sequence $Z = (Z_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number Z_0 if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: d(Z_k, Z_0) \geq \varepsilon\}| = 0.$$

We denote the set of all statistically convergent sequences of fuzzy numbers as $S(F)$.

A sequence $Z = (Z_k)$ of fuzzy numbers is said to be Δ^m −statistically convergent to a fuzzy number Z_0 if for every $\varepsilon > 0$, the following condition holds:

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: d(\Delta^m Z_k, Z_0) \geq \varepsilon\}| = 0.$$

We denote the set of all Δ^m −statistically convergent sequences of fuzzy numbers as $S_F(\Delta^m)$. In this case, it is denoted as $S_F(\Delta^m) - \lim Z_k = Z_0$.

Let $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$ and $0 < \alpha \leq 1$. If for every $\varepsilon > 0$, there exists a fuzzy number Z_0 such that

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n: d(Z_k, Z_0) \geq \varepsilon\}| = 0$$

the sequence $Z = (Z_k)$ is said to be ρ −statistically convergent to the fuzzy number Z_0 Here, $\rho = (\rho_n)$ is an non-decreasing sequence of positive real numbers that approaches to ∞, satisfying $\limsup_{n\to\infty} \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$ and $\Delta\rho_n = \rho_{n+1} - \rho_n$ for every positive integer n. In this case, it is denoted as $S_\rho^\alpha(F) - \lim Z_k = Z_0$.

Throughout this study, let $\rho = (\rho_n)$ be a sequence as given above.

Let (z_k) is sequence of complex numbers and $\Delta z = (z_k - z_{k+1})$. The sequence spaces $l_\infty(\Delta), c(\Delta), c_0(\Delta)$ are defined by

$$l_\infty(\Delta) = \{z = (z_k): \Delta z \in l_\infty\},$$

$$c(\Delta) = \{z = (z_k): \Delta z \in c\},$$

$$c_0(\Delta) = \{z = (z_k): \Delta z \in c_0\}.$$

Let $w(F)$ be the set of all sequences of fuzzy numbers. The difference operator order m is defined $\Delta^m: w(F) \to w(F), \Delta^m Z_k = \Delta^{m-1} Z_k - \Delta^{m-1} Z_{k+1}$, for all $m \in \mathbb{N}$. It is clear that the generalized difference operator Δ^m is a linear operator.

The concept of modulus function was first introduced by Nakano [23]. If a function $f: [0,\infty) \to [0,\infty)$ satisfies the following properties:

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) for every $x, y \geq 0$, $f(x+y) \leq f(x) + f(y)$,

(iii) f is right-continuous at, $x = 0$

(iv) f is increasing, then f is called a modulus function.

Let (p_k) be a positive and bounded sequence of real numbers with $\sup_k p_k = N$. Let $K = max(1, 2^{N-1})$ and $a_k, b_k \in \mathbb{C}$. The inequality

$$|a_k + b_k|^{p_k} \le K(|a_k|^{p_k} + |b_k|^{p_k}) \quad (1)$$

given by Maddox [24] will be used throughout this study.

3. Main Results

In this section, we first introduced the definition Δ_ρ^m −statistical convergence for sequences of fuzzy numbers using the generalized difference operator Δ^m. Furthermore, we defined the strong $N_F^\rho(\Delta^m, q)$ −summable sequence set and the strong $N_F^\rho(\Delta^m, f, q)$ −summable sequence set for fuzzy difference sequences aided by a modulus function f. Subsequently, we provided certain inclusion theorems between these sets and the $S_F^\rho(\Delta^m)$ set.

Definition 3.1. Let $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, Δ^m be the generalized difference operator and $\rho = (\rho_n)$ is an non-decreasing sequence of positive real numbers that approaches to ∞, satisfying $\limsup_{n\to\infty} \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$ and $\Delta\rho_n = \rho_{n+1} - \rho_n$ for every positive integer n. If there exists a fuzzy number Z_0 for each $\varepsilon > 0$ and $m = 1,2,3, \dots$ such that

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \le n: d(\Delta^m Z_k, Z_0) \ge \varepsilon\}| = 0$$

then the sequence $Z = (Z_k)$ is said to be Δ_ρ^m −statistically convergent to Z_0 (or $S_F^\rho(\Delta^m)$ − convergent to Z_0). In this case, we write *or* $S_F^\rho(\Delta^m) - \lim Z_k = Z_0$. $S_F^\rho(\Delta^m)$ will denote the set of all Δ_ρ^m −statistically convergent for sequences of fuzzy numbers.

Definition 3.2. Let f be a modulus function, $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, Δ^m be the generalized difference operator and $\rho = (\rho_n)$ is an non-decreasing sequence of positive real numbers that approaches to ∞, satisfying $\limsup_{n\to\infty} \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$ and $\Delta\rho_n = \rho_{n+1} - \rho_n$ for every positive integer n. If there exists a fuzzy number Z_0 for each $\varepsilon > 0$ and $m = 1,2,3, \dots$ such that

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \le n: f[d(\Delta^m Z_k, Z_0)] \ge \varepsilon\}| = 0$$

then the sequence $Z = (Z_k)$ is said to be Δ_ρ^m −statistically convergent to Z_0 by a modulus function f (or $S_F^\rho(\Delta^m, f) -$ convergent to Z_0). In this case, we write $S_F^\rho(\Delta^m, f) - \lim Z_k = Z_0$. $S_F^\rho(\Delta^m, f)$ will denote the set of all Δ_ρ^m −statistically convergent of order α by modulus function for sequences of fuzzy numbers.

Now let's define the strong $N_F^\rho(\Delta^m, q)$ −summable sequence set and the strong $N_F^\rho(\Delta^m, f, q)$ −summable sequence set, respectively.

Definition 3.3. Let $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, Δ^m be the generalized difference operator and $q > 0$. If there exists a fuzzy number Z_0 for each $\varepsilon > 0$ and $m = 1,2,3, \dots$ such that

$$\lim_{n\to\infty} \frac{1}{\rho_n} \sum_{k=1}^{n} [d(\Delta^m Z_k, Z_0)]^q = 0$$

then the sequence $Z = (Z_k)$ is said to be strongly $N_F^\rho(\Delta^m, q)$ −summable of to Z_0. In this case, we write $N_F^\rho(\Delta^m, q) - \lim Z_k = Z_0$. $N_F^\rho(\Delta^m, q)$ will denote the set of all strongly $N_F^\rho(\Delta^m, q)$ summable for sequences of fuzzy numbers.

Definition 3.4. Let f be a modulus function, $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, Δ^m be the generalized difference operator and $q = (q_k)$ be a sequence of positive real numbers. If there exists a fuzzy number Z_0 such that

$$\lim_{n\to\infty} \frac{1}{\rho_n} \sum_{k=1}^{n} \left[f\left(d(\Delta^m Z_k, Z_0)\right)\right]^{q_k} = 0$$

then the sequence $Z = (Z_k)$ is said to be strongly $N_F^\rho(\Delta^m, f, q)$ −summable to Z_0. In this case, we write $N_F^\rho(\Delta^m, f, q) - \lim Z_k = Z_0$. $N_F^\rho(\Delta^m, f, q)$ will denote the set of all strongly $N_F^\rho(\Delta^m, f, q)$ summable for sequences of fuzzy numbers.

In the following theorems, we will assume that $q = (q_k)$ is a bounded sequence with $0 < r = \inf_k q_k \le q_k \le \sup_k q_k = R < \infty$.

Theorem 3.1. Let (Z_k) and (T_k) be two sequences of points in the fuzzy number set $L(\mathbb{R})$, Δ^m be the generalized difference operator and $\rho = (\rho_n)$ is a non-decreasing sequence for each $n \in \mathbb{Z}^+$ tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$ and $\Delta\rho_n = \rho_{n+1} - \rho_n$ for each $n \in \mathbb{Z}^+$. Then

(i) $Z_k \to Z_0 \left(S_F^\rho(\Delta^m)\right)$ and $c \in \mathbb{C}$ implies $(cZ_k) \to cZ_0 \left(S_F^\rho(\Delta^m)\right)$,

(ii) $Z_k \to Z_0 \left(S_F^\rho(\Delta^m)\right)$ and $T_k \to T_0 \left(S_F^\rho(\Delta^m)\right)$ implies $(Z_k + T_k) \to (Z_0 + T_0) \left(S_F^\rho(\Delta^m)\right)$.

Proof. (i) For $c = 0$, the proof is clear. Let $c \neq 0$, the proof follows from the inequality

$$\frac{1}{\rho_n}|\{k \leq n: d(c\Delta^m Z_k, cZ_0) \geq \varepsilon\}| \leq \frac{1}{\rho_n}\left|\left\{k \leq n: d(\Delta^m Z_k, Z_0) \geq \frac{\varepsilon}{c}\right\}\right|$$

(ii) Let $Z_k \to Z_0\left(S_F^\rho(\Delta^m)\right)$ and $T_k \to T_0\left(S_F^\rho(\Delta^m)\right)$, we can write

$$\frac{1}{\rho_n}|\{k \leq n: d(\Delta^m(Z_k + T_k), (Z_0 + T_0)) \geq \varepsilon\}|$$

$$\leq \frac{1}{\rho_n}\left|\left\{k \leq n: d(\Delta^m Z_k, Z_0) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{\rho_n}\left|\left\{k \leq n: d(\Delta^m T_k, T_0) \geq \frac{\varepsilon}{2}\right\}\right|$$

for each $\varepsilon > 0$ and thus if $Z_k \to Z_0\left(S_F^\rho(\Delta^m)\right)$ and $T_k \to T_0\left(S_F^\rho(\Delta^m)\right)$ then $(Z_k + T_k) \to (Z_0 + T_0)\left(S_F^\rho(\Delta^m)\right)$.

Theorem 3.2. Let $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, Δ^m be the generalized difference operator and $\rho = (\rho_n)$ be as in Definition 3.1. If for each $n \in \mathbb{N}$, $\liminf\left(\frac{\rho_n}{n}\right) \geq 1$, then $S_F(\Delta^m) \subset S_F^\rho(\Delta^m)$.

Proof. Suppose that $Z_k \to Z_0(S_F(\Delta^m))$, the proof is obtained from the following inequality, for every $\varepsilon > 0$,

$$\frac{1}{n}|\{k \leq n: d(\Delta^m Z_k, Z_0) \geq \varepsilon\}|$$

$$= \frac{\rho_n}{n}\frac{1}{\rho_n}|\{k \leq n: d(\Delta^m Z_k, Z_0) \geq \varepsilon\}|$$

$$\geq \frac{1}{\rho_n}|\{k \leq n: d(\Delta^m Z_k, Z_0) \geq \varepsilon\}|.$$

Another coverage theorem is as follows:

Theorem 3.3. Let $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences such that $\rho_n \leq \tau_n$ for all $n \in \mathbb{N}$, Δ^m be the generalized difference operator. If $\lim\left(\frac{\rho_n}{\tau_n}\right) > 0$, then $S_F^\rho(\Delta^m) \subset S_F^\tau(\Delta^m)$.

Proof. Suppose that $Z_k \to Z_0 S_F^\rho(\Delta^m))$, the proof is obtained from the following inequality, for every $\varepsilon > 0$,

$$\frac{1}{\tau_n}|\{k \leq n: d(\Delta^m Z_k, Z_0) \geq \varepsilon\}|$$

$$< \frac{\rho_n}{\tau_n}\frac{1}{\rho_n}|\{k \leq n: d(\Delta^m Z_k, Z_0) \geq \varepsilon\}|.$$

A corollary of Theorem 3.2 and Theorem 3.3 follows.

Corollary 3.1. Let $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences such that $\rho_n \leq \tau_n$ for all $n \in \mathbb{N}$, Δ^m be the generalized difference operator. If $\lim\left(\frac{\rho_n}{\tau_n}\right) > 0$, then $S_F(\Delta^m) \subset S_F^\rho(\Delta^m) \subset S_F^\tau(\Delta^m)$.

Theorem 3.4. Let f_1, f_2 be any two modulus function, $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, Δ^m be the generalized difference operator and $\rho = (\rho_n)$ be as in Definition 3.1. Then

$$S_F^\rho(\Delta^m, f_1) \cap S_F^\rho(\Delta^m, f_2) \subseteq S_F^\rho(\Delta^m, f_1 + f_2).$$

Proof. Suppose that $Z = (Z_k) \in S_F^\rho(\Delta^m, f_1) \cap S_F^\rho(\Delta^m, f_2)$, for each $\varepsilon > 0$, we can write

$$\lim_{n\to\infty}\frac{1}{\rho_n}|\{k \leq n: f_1[d(\Delta^m Z_k, Z_0)] \geq \varepsilon\}| = 0$$

and

$$\lim_{n\to\infty}\frac{1}{\rho_n}|\{k \leq n: f_2[d(\Delta^m Z_k, Z_0)] \geq \varepsilon\}| = 0.$$

Since $(f_1 + f_2)[d(\Delta^m Z_k, Z_0)] = f_1[d(\Delta^m Z_k, Z_0)] + f_2[d(\Delta^m Z_k, Z_0)]$, the following equation can be written:

$$\lim_{n\to\infty}\frac{1}{\rho_n}|\{k \leq n: (f_1 + f_2)[d(\Delta^m Z_k, Z_0)] \geq \varepsilon\}| = 0.$$

So $(Z_k) \in S_F^\rho(\Delta^m, f_1 + f_2)$ and $S_F^\rho(\Delta^m, f_1) \cap S_F^\rho(\Delta^m, f_2) \subseteq S_F^\rho(\Delta^m, f_1 + f_2)$ is obtained.

Theorem 3.5. Let f_1, f_2 be any two modulus function such that $f_1(u) \leq f_2(u)$, for each $u \in [0, \infty)$, $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, Δ^m be the generalized difference operator and $\rho = (\rho_n)$ be as in Definition 3.1. Then $S_F^\rho(\Delta^m, f_2) \subseteq S_F^\rho(\Delta^m, f_1)$.

Proof. Suppose that $Z = (Z_k) \in S_F^\rho(\Delta^m, f_2)$, in this case, there exists a fuzzy number Z_0 for each $\varepsilon > 0$ such that

$$\lim_{n\to\infty}\frac{1}{\rho_n}|\{k \leq n: f_2[d(\Delta^m Z_k, Z_0)] \geq \varepsilon\}| = 0.$$

Since $|\{k \leq n: f_1[d(\Delta^m Z_k, Z_0)] \geq \varepsilon\}|$

$$\leq |\{k \leq n: f_2[d(\Delta^m Z_k, Z_0)] \geq \varepsilon\}|$$

inequality is provided for any $n \in \mathbb{N}$, we can write

$$\lim_{n\to\infty}\frac{1}{\rho_n}|\{k \leq n: f_1[d(\Delta^m Z_k, Z_0)] \geq \varepsilon\}| = 0.$$

So $(Z_k) \in S_F^\rho(\Delta^m, f_1)$ and $S_F^\rho(\Delta^m, f_2) \subseteq S_F^\rho(\Delta^m, f_1)$ is obtained.

Theorem 3.6. Let f_1, f_2 be any two modulus function, $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, Δ^m be the generalized difference operator and $\rho = (\rho_n)$ be as in Definition 3.1. Then $S_F^\rho(\Delta^m, f_1) \subseteq S_F^\rho(\Delta^m, f_2 \circ f_1)$.

Proof. Let $Z_k \in S_F^{\rho}(\Delta^m, f_2)$. Since f_2 is continuous, there is a number $\delta > 0$ for each $\varepsilon > 0$ such that $f_2(\delta) = \varepsilon$. On the other hand, for $\delta > 0$, there is exists fuzzy number Z_0 such that

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n: f_1[d(\Delta^m Z_k, Z_0)] \geq \delta\}| = 0.$$

In this case,

$$f_2(f_1[d(\Delta^m Z_k, Z_0)]) \geq f_2(\delta) = \varepsilon,$$

$$(f_2 \text{ o } f_1)\ [d(\Delta^m Z_k, Z_0)] \geq \varepsilon$$

can be written for $k \leq n$. Thus, since

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n: (f_2 \text{ o } f_1)[d(\Delta^m Z_k, Z_0)] \geq \delta\}| = 0,$$

$Z = (Z_k) \in S_F^{\rho}(\Delta^m, f_2 \text{ o } f_1)$. This requires $S_F^{\rho}(\Delta^m, f_1) \subseteq S_F^{\rho}(\Delta^m, f_2 \text{ o } f_1)$.

The following are the theorems regarding the strong $N_F^{\rho}(\Delta^m, q)$ −summable sequence set and the strong $N_F^{\rho}(\Delta^m, f, q)$ −summable sequence set.

Theorem 3.7. Let f be a modulus function $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, Δ^m be the generalized difference operator and $q > 1$. If $\liminf_{u\to\infty} \frac{f(u)}{u} > 0$, for each $u \in [0, \infty)$, then $N_F^{\rho}(\Delta^m, f, q) \subset N_F^{\rho}(\Delta^m, q)$.

Proof. When $\liminf_{u\to\infty} \frac{f(u)}{u} > 0$, for $u > 0$, it means that there exists a positive number c such that $f(u) > cu$ holds for $u > 0$. Let $Z = (Z_k) \in N_F^{\rho}(\Delta^m, f, q)$. Therefore, we have

$$\frac{1}{\rho_n} \sum_{k=1}^{n} [f(d(\Delta^m Z_k, Z_0))]^q \geq \frac{1}{\rho_n} \sum_{k=1}^{n} [c d(\Delta^m Z_k, Z_0)]^q$$
$$= \frac{c}{\rho_n} \sum_{k=1}^{n} [d(\Delta^m Z_k, Z_0)]^q$$

Consequently, we have obtained $N_F^{\rho}(\Delta^m, f, q) \subset N_F^{\rho}(\Delta^m, q)$.

Theorem 3.8. Let f be a modulus function, $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, Δ^m be the generalized difference operator and $\lim q_k > 0$. If the sequence $Z = (Z_k)$ is strongly $N_F^{\rho}(\Delta^m, f, q)$ −summable to the fuzzy number Z_0, then the limit is unique.

İspat. Let $N_F^{\rho}(\Delta^m, f, q)$ −lim$Z_k = Z_0$, $N_F^{\rho}(\Delta^m, f, q)$ −lim$Z_k = Z_0'$ and $\lim q_k = t > 0$. In this case, we can obtain the following:

$$\lim_{n\to\infty} \frac{1}{\rho_n} \sum_{k=1}^{n} [f(d(\Delta^m Z_k, Z_0))]^{q_k} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{\rho_n} \sum_{k=1}^{n} [f(d(\Delta^m Z_k, Z_0'))]^{q_k} = 0.$$

Thus, from the definition of f and (1), for $\sup_k q_k = K$, $0 < \alpha \leq \beta \leq 1$ ve $N = \max(1, 2^{K-1})$, we can write:

$$\frac{1}{\rho_n} \sum_{k=1}^{n} [f(d(\Delta^m Z_k, Z_0'))]^{q_k}$$
$$\leq \frac{N}{\rho_n} (\sum_{k=1}^{n} [f(d(\Delta^m Z_k, Z_0))]^{q_k} + \sum_{k=1}^{n} [f(d(\Delta^m Z_k, Z_0'))]^{q_k})$$

$$\leq \frac{N}{\rho_n} (\sum_{k=1}^{n} [f(d(\Delta^m Z_k, Z_0))]^{q_k}) + \frac{N}{\rho_n} (\sum_{k=1}^{n} [f(d(\Delta^m Z_k, Z_0'))]^{q_k})$$

Therefore, we have:

$$\frac{1}{\rho_n} (\sum_{k=1}^{n} [f(d(\Delta^m Z_k, Z_0'))]^{q_k}) = 0$$

Since $\lim q_k = t$, we can conclude that $Z_0 - Z_0' = 0$. Thus, the limit is unique.

Theorem 3.9. Let $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, f be a modulus function, Δ^m be the generalized difference operator and $\liminf q_k > 0$. If the sequence $Z = (Z_k)$ is convergent to a fuzzy number Z_0, then the sequence $Z = (Z_k)$ is strongly $N_F^{\rho}(\Delta^m, f, q)$ −summable.

Proof. Let $S_F(\Delta^m) - \lim Z_k = Z_0$. Since f is a modulus function, $f(d(\Delta^m Z_k, Z_0)) \to \bar{0}$. As $\liminf q_k > 0$, $[f(d(\Delta^m Z_k, Z_0))]^{q_k} \to \bar{0}$. Thus, we have $N_F^{\rho}(\Delta^m, f, q) - \lim Z_k = Z_0$. This is what we wanted to prove.

Theorem 3.10. Let $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$, f be a modulus function, $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences such that $\rho_n \leq \tau_n$ for all $n \in \mathbb{N}$ and Δ^m be the generalized difference operator. If $\liminf \frac{\rho_n}{\tau_n} > 0$, then $N_F^{\tau}(\Delta^m, f, q) \subset N_F^{\rho}(\Delta^m, f, q)$.

Proof. Let $Z = (Z_k) \in N_F^{\tau}(\Delta^m, f, q)$ and $\liminf \frac{\rho_n}{\tau_n} > 0$. In this case:

$$\frac{1}{\tau_n}\left(\sum_{k=1}^{n}\left[f\left(d(\Delta^m Z_k, Z_0)\right)\right]^{q_k}\right) \geq \frac{\rho_n}{\tau_n}\frac{1}{\rho_n}\left(\sum_{k=1}^{n}\left[f\left(d(\Delta^m Z_k, Z_0)\right)\right]^{q_k}\right)$$

Thus, if $Z \in N_F^{\tau}(\Delta^m, f, q)$, then $Z \in N_F^{\rho}(\Delta^m, f, q)$.

Declaration of Ethical Standards
The author declares that she complies with all ethical standards.

Declaration of Competing Interest
The author has no conflict of interest to declare regarding the content of this article.

Data Availability Statement
The author declares that the main data supporting the findings of this work are available within the article.

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