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Wijsman Deferred Invariant Statistical and Strong p-Deferred Invariant Equivalence of Order α

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Abstract

With this work, we present the asymptotical strongly p-deferred invariant and asymptotical deferred invariant statistical equivalence of order α $(0 < \alpha \leq 1)$ for sequences of sets in the Wijsman sense. Furthermore, we investigate the connections between these concepts and conduct their properties.

1. Introduction and backgrounds

One of the convergence concepts for sequences of sets (Ss) is convergence in the Wijsman sense (Ws) (see, [1, 2]). The statistical convergence in Ws was first introduced by Nuray and Rhoades [3]. Then, Ulusu and Nuray [4] studied the lacunary statistical convergence in Ws. Also, Pancaroğlu and Nuray [5] presented the invariant statistical convergence in Ws. Furthermore, Ulusu and Nuray [6] and Pancaroğlu et al. [7] introduced the asymptotical-asymptotical statistical equivalence and asymptotical invariant-asymptotical invariant statistical equivalence in Ws, respectively.
Agnew [8] first introduced the deferred Cesàro mean for real (complex) sequences. Subsequently, the deferred statistical convergence was studied by Küçükaslan and Yılmaztürk [9]. Then, Nuray [10] presented the deferred invariant and deferred invariant statistical convergence.
The deferred statistical convergence in Ws for Ss was introduced by Altınok et al. [11]. Also, Et and Yılmazer [12] studied on this concept. Then, Gülle [13] presented the deferred invariant statistical convergence of order α in Ws. Furthermore, Altınok et al. [14] and Et et al. [15] studied the asymptotical deferred statistical and asymptotical deferred statistical equivalence of order α in Ws, respectively.
In the metric space $(\mathcal{U},d)$, the distance function $\rho(u,C) := \rho_u(C)$ is defined by

$$\rho_u(C) = \inf_{c\in C} d(u,c)$$

for each $u \in \mathcal{U}$ and non-empty $C \subseteq \mathcal{U}$.
For a function $f : \mathbb{N} \to 2^{\mathcal{U}}$ (power set) is defined by $f(j) = C_j \in 2^{\mathcal{U}}$ for each $j \in \mathbb{N}$ (the set of natural numbers), the sequence $\{C_j\} = \{C_1, C_2, \ldots\}$ is called sequence of sets.
Throughout the study, unless otherwise specified, $(\mathcal{U},d)$ is regarded as a metric space and C, C_j, D_j, E_j, F_j as non-empty closed subsets of $\mathcal{U}$.
The Ss $\{C_j\}$ is called convergent in Ws to the set C if for each $u \in \mathcal{U}$

$$\lim_{j\to\infty} \rho_u(C_j) = \rho_u(C)$$

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and it is denoted in $C_j \xrightarrow{W} C$ format.
An invariant mean, also known as a σ-mean, is a continuous linear functional ψ in the bounded sequences space that adhere to the subsequent conditions:

(1) $\psi(x_t) \geq 0$ when the sequence (x_t) consists of non-negative elements for all t,
(2) $\psi(e) = 1$ for $e = (1,1,1,\ldots)$,
(3) $\psi(x_{\sigma(t)}) = \psi(x_t)$ for all the bounded sequences (x_t),

where σ is a mapping from the set of non-negative integers into itself.
The mappings σ are regarded as one-to-one and $\sigma^j(t) \neq t$ (jth iterate of σ) for all positive integers j. Therefore, ψ expands the limit functional on the convergent sequences space c such that $\psi(x_t) = \lim x_t$ for all $(x_t) \in c$.
The Ss $\{C_j\}$ is called;

(i) strongly invariant convergent in Ws to the set C if

$$\lim_{j\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left|\rho_u(C_{\sigma^j(t)}) - \rho_u(C)\right| = 0,$$

(ii) invariant statistically convergent in Ws to the set C if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{ j \leq n : |\rho_u(C_{\sigma^j(t)}) - \rho_u(C)| \geq \varepsilon \right\}\right| = 0$$

for each $u \in \mathcal{U}$ and uniformly in t. These convergences are denoted in $C_j \stackrel{W[V_\sigma]}{\longrightarrow} C$ and $C_j \stackrel{W(S_\sigma)}{\longrightarrow} C$ formats, respectively.
For any non-empty closed subsets $C_j, D_j \in \mathcal{U}$ such that $\rho_u(C_j) > 0$ and $\rho_u(D_j) > 0$ for each $u \in \mathcal{U}$, the Ss $\{C_j\}$ and $\{D_j\}$ are called asymptotically equivalent to multiple η in Ws if for each $u \in \mathcal{U}$

$$\lim_{j\to\infty} \frac{\rho(u,C_j)}{\rho(u,D_j)} = \eta$$

and it is denoted in $C_j \stackrel{W^\eta}{\sim} D_j$ format. These sequences are referred to as asymptotically equivalent in Ws when $\eta = 1$.
For any non-empty closed subsets $C_j, D_j \in \mathcal{U}$ such that $\rho_u(C_j) > 0$ and $\rho_u(D_j) > 0$ for each $u \in \mathcal{U}$, the Ss $\{C_j\}$ and $\{D_j\}$ are called;

(i) asymptotically strongly deferred Cesàro equivalent to multiple η in Ws if

$$\lim_{i\to\infty} \frac{1}{s(i)-r(i)} \sum_{j=r(i)+1}^{s(i)} \left| \frac{\rho(u,C_j)}{\rho(u,D_j)} - \eta \right| = 0,$$

(ii) asymptotically deferred statistical equivalent to multiple η in Ws if for every $\varepsilon > 0$

$$\lim_{i\to\infty} \frac{1}{s(i)-r(i)} \left| \left\{ r(i) < j \leq s(i) : \left| \frac{\rho(u,C_j)}{\rho(u,D_j)} - \eta \right| \geq \varepsilon \right\} \right| = 0$$

for each $u \in \mathcal{U}$, where $(r(i))$ and $(s(i))$ are sequences of non-negative integers satisfying

$$r(i) < s(i) \text{ and } \lim_{i\to\infty} s(i) = \infty. \tag{1.1}$$

These equivalences are denoted in $C_j \stackrel{W_d^\eta}{\sim} D_j$ and $C_j \stackrel{W_d^\eta(S)}{\sim} D_j$ formats, respectively.
Throughout the paper, unless otherwise specified, $(r(i))$ and $(s(i))$ is regarded as non-negative integer sequences satisfying (1.1).
An increasing sequence of integers $\theta = (k_i)$ is called a lacunary sequence when it satisfies two conditions: $k_0 = 0$ and $h_i = k_i - k_{i-1} \to \infty$ as $i \to \infty$.
For more study on the concepts of convergence, invariant summability, deferred mean and asymptotical equivalence for real or set sequences, we refer to [16, 17, 18, 19, 20, 21, 22].
From now on, for short, we will use the term $\rho_u\left(\frac{C_j}{D_j}\right)$ instead of the term $\frac{\rho(u,C_j)}{\rho(u,D_j)}$.

2. Main results

With this section, we present the asymptotical strongly p-deferred invariant and asymptotical deferred invariant statistical equivalence of order α $(0<\alpha\leq 1)$ in Ws for Ss. Furthermore, we investigate the connections between these concepts and conduct their properties.

Definition 2.1. *For any non-empty closed subsets $C_j, D_j \in \mathfrak{U}$ such that $\rho_u(C_j)>0$ and $\rho_u(D_j)>0$ for each $u\in\mathfrak{U}$, the Ss $\{C_j\}$ and $\{D_j\}$ are said to be asymptotically strongly p-deferred invariant equivalent to multiple η of order α in Ws if for each $u\in\mathfrak{U}$*

$$\lim_{i\to\infty}\frac{1}{\left(s(i)-r(i)\right)^{\alpha}}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|^p=0$$

uniformly in t, where $0<p<\infty$ and $0<\alpha\leq 1$. For this case, the notation $C_j \overset{W_d^{\eta}[V_\sigma^{\alpha}]^p}{\sim} D_j$ is used, and these sequences are referred to as asymptotically strongly p-deferred invariant equivalent of order α in Ws when $\eta=1$.

Example 2.2. *Let us take $X=\mathbb{R}^2$ and the Ss $\{C_j\}$ and $\{D_j\}$ as follows:*

$$C_j := \begin{cases} \left\{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+(x_2-1)^2=\frac{1}{j}\right\} & ; \ \textit{if } j \textit{ is a square integer} \\ \{(-1,0)\} & ; \ \textit{if not} \end{cases}$$

and

$$D_j := \begin{cases} \left\{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+(x_2+1)^2=\frac{1}{j}\right\} & ; \ \textit{if } j \textit{ is a square integer} \\ \{(-1,0)\} & ; \ \textit{if not.} \end{cases}$$

Then, the Ss $\{C_j\}$ and $\{D_j\}$ are asymptotically strongly p-deferred invariant equivalent of order α $(0<\alpha\leq 1)$ in Ws.

Remark 2.3.

(i) For Ss, the asymptotical strongly p-deferred invariant equivalence of order α and asymptotical strongly p-invariant equivalence given in [7] coincide when $r(i)=0$, $s(i)=i$ and $\alpha=1$.
(ii) For Ss, the asymptotical strongly p-deferred invariant equivalence of order α and asymptotical strongly p-lacunary invariant equivalence given in [7] coincide when $r(i)=k_{i-1}$, $s(i)=k_i$ and $\alpha=1$.

Theorem 2.4. *Let $0<p<\infty$ and $0<\alpha\leq\beta\leq 1$. Then,*

$$C_j \overset{W_d^{\eta}[V_\sigma^{\alpha}]^p}{\sim} D_j \Rightarrow C_j \overset{W_d^{\eta}[V_\sigma^{\beta}]^p}{\sim} D_j.$$

Proof. Assume that $0<\alpha\leq\beta\leq 1$ and $C_j \overset{W_d^{\eta}[V_\sigma^{\alpha}]^p}{\sim} D_j$. For each $u\in\mathfrak{U}$, we can write

$$\frac{1}{\left(s(i)-r(i)\right)^{\beta}}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|^p \leq \frac{1}{\left(s(i)-r(i)\right)^{\alpha}}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|^p$$

for all t. Since the right side converges to 0 for $i\to\infty$ based on our assumption, we have $C_j \overset{W_d^{\eta}[V_\sigma^{\beta}]^p}{\sim} D_j$. □

The following corollary is obtained for $\beta=1$ in Theorem 2.4.

Corollary 2.5. *Let $0<p<\infty$ and $0<\alpha\leq 1$. If $C_j \overset{W_d^{\eta}[V_\sigma^{\alpha}]^p}{\sim} D_j$, then $C_j \overset{W_d^{\eta}[V_\sigma]^p}{\sim} D_j$ which this concept has not been studied yet.*

Theorem 2.6. *Let $0<p<q<\infty$ and $0<\alpha\leq 1$. Then,*

$$C_j \overset{W_d^{\eta}[V_\sigma^{\alpha}]^q}{\sim} D_j \Rightarrow C_j \overset{W_d^{\eta}[V_\sigma^{\alpha}]^p}{\sim} D_j.$$

Proof. Assume that $0<p<q<\infty$ and $C_j \overset{W_d^{\eta}[V_\sigma^{\alpha}]^q}{\sim} D_j$. By the Hölder inequality, for each $u\in\mathfrak{U}$, we can write

$$\frac{1}{\left(s(i)-r(i)\right)^{\alpha}}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|^p < \frac{1}{\left(s(i)-r(i)\right)^{\alpha}}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|^q$$

for all t. Since the right side converges to 0 for $i\to\infty$ based on our assumption, we have $C_j \overset{W_d^{\eta}[V_\sigma^{\alpha}]^p}{\sim} D_j$. □

Definition 2.7. *For any non-empty closed subsets $C_j, D_j \in \mathcal{U}$ such that $\rho_u(C_j) > 0$ and $\rho_u(D_j) > 0$ for each $u \in \mathcal{U}$, the Ss $\{C_j\}$ and $\{D_j\}$ are said to be asymptotically deferred invariant statistical equivalent to multiple η of order α in Ws if for every $\varepsilon > 0$ and each $u \in \mathcal{U}$*

$$\lim_{i\to\infty} \frac{1}{\big(s(i)-r(i)\big)^{\alpha}} \left| \left\{ r(i) < j \leq s(i) : \left| \rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big) - \eta \right| \geq \varepsilon \right\} \right| = 0$$

uniformly in t, where $0 < \alpha \leq 1$. For this case, the notation $C_j \overset{W_d^{\eta}(S_\sigma^{\alpha})}{\sim} D_j$ is used, and these sequences are referred to as asymptotically deferred invariant statistical equivalent of order α in Ws when $\eta = 1$.

The set $\{W_d^{\eta}(S_\sigma^{\alpha})\}$ represents all Ss that asymptotically deferred invariant statistical equivalent of order α.

Example 2.8. *Let us take $X = \mathbb{R}^2$ and the Ss $\{C_j\}$ and $\{D_j\}$ as follows:*

$$C_j := \begin{cases} \left\{(x_1,x_2) \in \mathbb{R}^2 : (x_1+j)^2 + x_2^2 = 1\right\} & ; \text{ if } j \text{ is a square integer} \\ \{(1,0)\} & ; \text{ if not} \end{cases}$$

and

$$D_j := \begin{cases} \left\{(x_1,x_2) \in \mathbb{R}^2 : (x_1-j)^2 + x_2^2 = 1\right\} & ; \text{ if } j \text{ is a square integer} \\ \{(1,0)\} & ; \text{ if not.} \end{cases}$$

Then, the Ss $\{C_j\}$ and $\{D_j\}$ are asymptotically deferred invariant statistical equivalent order α $(0 < \alpha \leq 1)$ in Ws.

Remark 2.9.

(i) For Ss, the asymptotical deferred invariant statistical equivalence of order α and asymptotical invariant statistical equivalence given in [7] coincide when $r(i) = 0$, $s(i) = i$ and $\alpha = 1$.

(ii) For Ss, the asymptotical deferred invariant statistical equivalence of order α and asymptotical lacunary invariant statistical equivalence given in [7] coincide when $r(i) = k_{i-1}$, $s(i) = k_i$ and $\alpha = 1$.

Theorem 2.10. *Let $0 < \alpha \leq \beta \leq 1$. Then*

$$C_j \overset{W_d^{\eta}(S_\sigma^{\alpha})}{\sim} D_j \Rightarrow C_j \overset{W_d^{\eta}(S_\sigma^{\beta})}{\sim} D_j.$$

Proof. Assume that $0 < \alpha \leq \beta \leq 1$ and $C_j \overset{W_d^{\eta}(S_\sigma^{\alpha})}{\sim} D_j$. For every $\varepsilon > 0$ and each $u \in \mathcal{U}$, we can write

$$\frac{1}{\big(s(i)-r(i)\big)^{\beta}} \left| \left\{ r(i) < j \leq s(i) : \left| \rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big) - \eta \right| \geq \varepsilon \right\} \right| \leq \frac{1}{\big(s(i)-r(i)\big)^{\alpha}} \left| \left\{ r(i) < j \leq s(i) : \left| \rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big) - \eta \right| \geq \varepsilon \right\} \right|$$

for all t. Since the right side converges to 0 for $i \to \infty$ based on our assumption, we have $C_j \overset{W_d^{\eta}(S_\sigma^{\beta})}{\sim} D_j$. □

The following corollary is obtained for $\beta = 1$ in Theorem 2.10.

Corollary 2.11. *Let $0 < \alpha \leq 1$. If $C_j \overset{W_d^{\eta}(S_\sigma^{\alpha})}{\sim} D_j$, then $C_j \overset{W_d^{\eta}(S_\sigma)}{\sim} D_j$ which this concept has not been studied yet.*

Theorem 2.12. *If the Ss $\{C_j\}$ and $\{D_j\}$ are asymptotically strongly p-deferred invariant equivalent to multiple η of order α in Ws, then the sequences are asymptotically deferred invariant statistical equivalent to multiple η of order α in Ws, where $0 < \alpha \leq 1$.*

Proof. Assume that $0 < \alpha \leq 1$ and $C_j \overset{W_d^{\eta}[V_\sigma^{\alpha}]^p}{\sim} D_j$. For every $\varepsilon > 0$ and each $u \in \mathcal{U}$, we can write

$$\begin{aligned} \sum_{j=r(i)+1}^{s(i)} \left| \rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big) - \eta \right|^p &\geq \sum_{\substack{j=r(i)+1 \\ \left| \rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right) - \eta \right| \geq \varepsilon}}^{s(i)} \left| \rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big) - \eta \right|^p \\ &\geq \varepsilon^p \left| \left\{ r(i) < j \leq s(i) : \left| \rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big) - \eta \right| \geq \varepsilon \right\} \right| \end{aligned}$$

and so,

$$\frac{1}{\varepsilon^p\left(s(i)-r(i)\right)^{\alpha}}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^p \geq \frac{1}{\left(s(i)-r(i)\right)^{\alpha}}\left|\left\{r(i)<j\leq s(i):\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|\geq\varepsilon\right\}\right|$$

for all t. Since the left side converges to 0 for $i\to\infty$ based on our assumption, we have $C_j \overset{W_d^{\eta}(S_\sigma^\alpha)}{\sim} D_j$. □

In the case of $\alpha=1$, the opposite of Theorem 2.12 is provided.

Theorem 2.13. *Let $\rho_u(C_j)\,\mathcal{O}\,\rho_u(D_j)$. If the Ss $\{C_j\}$ and $\{D_j\}$ are asymptotically deferred invariant statistical equivalent to multiple η in Ws, then the sequences are asymptotically strongly p-deferred invariant equivalent to multiple η in Ws.*

Proof. Suppose that $\rho_u(C_j)\,\mathcal{O}\,\rho_u(D_j)$ and $C_j \overset{W_d^{\eta}(S_\sigma)}{\sim} D_j$. Since $\rho_u(C_j)\,\mathcal{O}\,\rho_u(D_j)$, then there exists an $M>0$ such that

$$\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|\leq M$$

for all t and each $u\in\mathcal{U}$. For every $\varepsilon>0$, we can write

$$\begin{aligned}\frac{1}{s(i)-r(i)}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^p &= \frac{1}{s(i)-r(i)}\sum_{\substack{j=r(i)+1\\ \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\geq\varepsilon}}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^p\\ &\quad+\frac{1}{s(i)-r(i)}\sum_{\substack{j=r(i)+1\\ \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|<\varepsilon}}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^p\\ &\leq \frac{M^p}{s(i)-r(i)}\left|\left\{r(i)<j\leq s(i):\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|\geq\varepsilon\right\}\right|+\varepsilon^p\end{aligned}$$

for all t. Since the left side converges to 0 for $i\to\infty$ based on our assumption, we have $C_j \overset{W_d^{\eta}[V_\sigma]^p}{\sim} D_j$. □

3. Auxiliary results

With this section, first of all, we define the asymptotical invariant statistical equivalence to multiple η of order α in Ws for Ss, then we examine the relationship between this concept and the asymptotical deferred invariant statistical equivalence to multiple η of order α.

Definition 3.1. *For any non-empty closed subsets $C_j, D_j\in\mathcal{U}$ such that $\rho_u(C_j)>0$ and $\rho_u(D_j)>0$ for each $u\in\mathcal{U}$, the Ss $\{C_j\}$ and $\{D_j\}$ are said to be asymptotically invariant statistical equivalent to multiple η of order α in Ws if for every $\varepsilon>0$ and each $u\in\mathcal{U}$*

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\left\{j\leq n:\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|\geq\varepsilon\right\}\right|=0$$

uniformly in t, where $0<\alpha\leq 1$. For this case, the notation $C_j \overset{W^{\eta}(S_\sigma^\alpha)}{\sim} D_j$ is used, and these sequences are referred to as asymptotically invariant statistical equivalent of order α in Ws when $\eta=1$.

The set $\{W^{\eta}(S_\sigma^\alpha)\}$ represents all Ss that asymptotically invariant statistical equivalent of order α.

Theorem 3.2. *If* $\left\{\frac{r(i)}{s(i)-r(i)}\right\}$ *is bounded, then* $\{W^{\eta}(S^{\alpha}_{\sigma})\} \subset \{W^{\eta}_{d}(S^{\alpha}_{\sigma})\}$*, where* $0<\alpha\leq 1$.

Proof. Suppose that $0<\alpha\leq 1$ and $C_j \overset{W^{\eta}(S^{\alpha}_{\sigma})}{\sim} D_j$. Then, for every $\varepsilon>0$ and each $u\in\mathfrak{U}$, we have

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\left\{j\leq n:\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\geq\varepsilon\right\}\right|=0$$

uniformly in t. Here using the well-known fact,

$$\lim_{i\to\infty}\frac{1}{(s(i))^{\alpha}}\left|\left\{j\leq s(i):\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\geq\varepsilon\right\}\right|=0$$

is hold uniformly in t. Also, since

$$\left\{r(i)<j\leq s(i):\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\geq\varepsilon\right\}\subset\left\{0<j\leq s(i):\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\geq\varepsilon\right\},$$

we can write

$$\left|\left\{r(i)<j\leq s(i):\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\geq\varepsilon\right\}\right|\leq\left|\left\{0<j\leq s(i):\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\geq\varepsilon\right\}\right|$$

for all t. Thus, the inequality is handled:

$$\frac{1}{(s(i)-r(i))^{\alpha}}\left|\left\{r(i)<j\leq s(i):\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\geq\varepsilon\right\}\right|\leq\left(1+\frac{r(i)}{s(i)-r(i)}\right)^{\alpha}\frac{1}{(s(i))^{\alpha}}\left|\left\{0<j\leq s(i):\left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\geq\varepsilon\right\}\right|.$$

If $\left\{\frac{r(i)}{s(i)-r(i)}\right\}$ is bounded in above inequality, then the desired result is obtained for $i\to\infty$. □

4. Conclusion

In this study, as a combination of asymptotical equivalence, deferred statistical convergence, invariant summability and order α, we defined new concepts for sequences of sets and obtained noteworthy results.

Declarations

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