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Notes on q-Partial Differential Equations for q-Laguerre Polynomials and Little q-Jacobi Polynomials

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Abstract

This article defines two common q-orthogonal polynomials: homogeneous q-Laguerre polynomials and homogeneous little q-Jacobi polynomials. They can be viewed separately as solutions to two q-partial differential equations. Furthermore, an analytic function satisfies a certain system of q-partial differential equations if and only if it can be expanded in terms of homogeneous q-Laguerre polynomials or homogeneous little q-Jacobi polynomials. As applications, several generalized Ramanujan q-beta integrals and Andrews-Askey integrals are obtained.

1. Introduction

The presence of orthogonal polynomials is ubiquitous in various problems encountered in classical mathematical physics. For instance, the Hermite polynomials manifest in the quantum mechanical treatment of harmonic oscillators, while the Laguerre polynomials arise in the propagation of electromagnetic waves. However, the study of q-orthogonal polynomials is also a crucial study topic and can be found in relevant literature [1, 2, 3, 4, 5].
Throughout the paper, it is supposed that $0<|q|<1$ and denote by $\mathbb{N}$ ($\mathbb{C}$) the set of positive integers (complex numbers, respectively). The q-shifted factorials are defined as

$$(a;q)_0=1,\ (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\ (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)$$

and $(a_1,a_2,\cdots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$, where n is a non-negative integer or ∞. The q-derivative of $f(x)$ with respect to x is defined by

$$\mathcal{D}_q\{f(x)\}=\frac{f(x)-f(qx)}{x}.$$

According to the above definition, it is not difficult to verify

$$\mathcal{D}_q\{f(x)g(x)\} \quad = \quad \mathcal{D}_q\{f(x)\}g(x)+f(qx)\mathcal{D}_q\{g(x)\} \tag{1.1}$$

and the Leibniz rule for the product of two functions

$$\mathcal{D}_q^n\{f(x)g(x)\}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(k-n)}\mathcal{D}_q^k\{f(x)\}\mathcal{D}_q^{n-k}\{g(q^kx)\}, \tag{1.2}$$

where

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},\ 0\le k\le n,\ n\in\mathbb{N} \tag{1.3}$$

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is the Gaussian binomial coefficients, also see [6]. For any real number r, the q-shift operator $\eta_{x_i}^r$ is defined by

$$\eta_{x_i}^r\{f(x_1,\cdots,x_n)\} = f(x_1,\cdots,x_{i-1},q^r x_i,x_{i+1},\cdots,x_n).$$

Generalizing Heine's series, or basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,\cdots,a_r\\ b_1,b_2,\cdots,b_s\end{matrix};q,z\right) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_r;q)_n}{(b_1;q)_n\cdots(b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \tag{1.4}$$

Here and in what follows, $\binom{n}{k}$ represents the standard combination symbol. The series ${}_r\phi_s$ terminates if one of the numerator parameters is of the form q^{-n}, $n\in\mathbb{N}\cup\{0\}$ and $q\neq 0$. If $0<|q|<1$, the series ${}_r\phi_s$ converges absolutely for all x if $r\leq s$ and for $|x|<1$ if $r=s+1$. The famous q-binomial theorem

$${}_1\phi_0\left(\begin{matrix}a\\ -\end{matrix};q,z\right) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty},\ |z|<1, \tag{1.5}$$

is a q-analogue of Newton's binomial series. This theorem can also derive the following two identities

$$\sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty},\ |z|<1, \quad \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}z^n = (z;q)_\infty. \tag{1.6}$$

The theory of basic hypergeometric series has been greatly developed for more than a century, and there are many effective ways to study it, such as the Wilf-Zeilberg algorithm, transformation, inversion and operator, for example, see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. Ten years ago, Liu first introduced the q-partial differential equation method to study q-series. This innovative approach has attracted the attention of numerous mathematicians, For further details, please refer to [26, 27, 28, 29, 31, 32, 33]. To this end, we initially define the q-partial derivative [28].

Definition 1.1. *A q-partial derivative of a function of several variables is its q-derivative with respect to one of those variables, regarding other variables as constants.*

For convenience, the q-partial derivative of a function f with respect to the variable x is denoted by $\mathcal{D}_{q,x}\{f\}$. In [28], Liu proved the following theorem.

Theorem 1.2. *If $f(x,y)$ is a two-variable analytic function at $(0,0)\in\mathbb{C}^2$, then, f can be expanded in terms of homogeneous Rogers-Szegő polynomials (for definition see (5.1)) if and only if f satisfies the q-partial differential equation $\mathcal{D}_{q,x}\{f\} = \mathcal{D}_{q,y}\{f\}$.*

We should point out that the above theorem has developed a new theory for calculating the q-identity and demonstrated its universality when applied to many types of q-orthogonal polynomials, including Rogers-Szegő polynomials, Hahn polynomials, Stieltjes-Wigert polynomials and Askey-Wilson polynomials, as well as classical orthogonal polynomials such as Hermite polynomials (cf. [30]). Later, some related works by Abdlhusein, Arjika, Aslan, Cao, Jia, Li, Mahaman, Niu and Zhang also fall into Liu's theory. Readers interested can see [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47].
Hahn [48] first discovered the q-Laguerre polynomials, according to Koekoek and Swarttouw [49], they are defined by

$$\mathcal{L}_n^{(\alpha)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_1\phi_1\left(\begin{matrix}q^{-n}\\ q^{\alpha+1}\end{matrix};q,-q^{n+\alpha+1}x\right),\alpha>-1. \tag{1.7}$$

Askey pointed out [50] that the q-Laguerre polynomials converge to the Stieltjes-Wigert polynomials for $\alpha\to\infty$ thus the q-Laguerre polynomials are sometimes called the generalized Stieltjes-Wigert polynomials [49]. The explicit form of q-Laguerre polynomials can write as

$$\mathcal{L}_n^{(\alpha)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_k}(-x)^k. \tag{1.8}$$

To study q-Laguerre polynomials from the perspective of q-partial differential equations following Liu's ideas, it is necessary to introduce homogeneous q-Laguerre polynomials

$$L_n^{(\alpha)}(x,y|q) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_k}(-x)^k y^{n-k},\alpha>-1. \tag{1.9}$$

Obviously,

$$L_n^{(\alpha)}(x,y|q) = \frac{(q;q)_n}{(q^{\alpha+1};q)_n}y^n\mathcal{L}_n^{(\alpha)}(x/y|q),\ L_n^{(\alpha)}(x,1|q) = \frac{(q;q)_n}{(q^{\alpha+1};q)_n}\mathcal{L}_n^{(\alpha)}(x|q),\ L_n^{(\alpha)}(0,y|q) = y^n.$$

This paper is organized as follows. Section 2 shows that an analytic function satisfies a system of q-partial differential equations, if and only if it can be expanded in terms of homogeneous q-Laguerre polynomials (see Theorem 2.3). Section 3 is an application of Theorem 2.3, where we use the method of q-partial differential equations to prove the generating functions of homogeneous q-Laguerre polynomials with different weights. Section 4 presents that an analytic function can be expanded in terms of homogeneous little q-Jacobi polynomials (see Theorem 4.2) if and only if it satisfies a system of q-partial differential equations. In section 5, we obtain some identities by applying Theorems 2.3 and 4.2, which generalize famous formulas such as Ramanujan q-beta integrals and Andrews-Askey integrals.

2. Homogeneous q-Laguerre polynomials and q-partial differential equations

Firstly, Proposition 2.1 presents an important property of homogeneous q-Laguerre polynomials.

Proposition 2.1. *For $n \in \mathbb{N} \cup \{0\}$, the homogeneous q-Laguerre polynomials satisfy the q-partial differential equation*

$$\mathcal{D}_{q,x}(1-q^{\alpha}\eta_x)\left\{L_n^{(\alpha)}(x,y|q)\right\} = -q^{\alpha+1}\eta_x^2\mathcal{D}_{q,y}\left\{L_n^{(\alpha)}(x,y|q)\right\}, \tag{2.1}$$

namely,

$$\mathcal{D}_{q,x}\left\{L_n^{(\alpha)}(x,y|q) - q^{\alpha}L_n^{(\alpha)}(qx,y|q)\right\} = -q^{\alpha+1}\mathcal{D}_{q,y}\left\{L_n^{(\alpha)}(q^2x,y|q)\right\}.$$

Proof. Let LHS denote the left-hand side of the equation (2.1), and by using the formula $\mathcal{D}_{q,x}\{x^n\} = (1-q^n)x^{n-1}$, we can obtain

$$\text{LHS} = \mathcal{D}_{q,x}\left\{\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_{k-1}}x^ky^{n-k}\right\} = \sum_{k=1}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q(1-q^k)\frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_{k-1}}x^{k-1}y^{n-k}.$$

Similarly, use RHS to denote the right-hand side of the equation (2.1). Through simple calculation, we have

$$\begin{aligned}\text{RHS} &= -q^{\alpha+1}\mathcal{D}_{q,y}\left\{\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_k}(q^2x)^ky^{n-k}\right\}\\ &= \sum_{k=0}^{n-1}(-1)^{k+1}\begin{bmatrix}n\\k\end{bmatrix}_q(1-q^{n-k})\frac{q^{(k+1)^2+(k+1)\alpha}}{(q^{\alpha+1};q)_k}x^ky^{n-k-1}\\ &= \sum_{k=1}^{n}(-1)^k\begin{bmatrix}n\\k-1\end{bmatrix}_q(1-q^{n-k+1})\frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_{k-1}}x^{k-1}y^{n-k}.\end{aligned}$$

From the definition of the q-binomial coefficients (1.3), it is easy to verify that

$$\begin{bmatrix}n\\k\end{bmatrix}_q(1-q^k) = \begin{bmatrix}n\\k-1\end{bmatrix}_q(1-q^{n-k+1}). \tag{2.2}$$

It follows from (2.2) that LHS = RHS, which completes the proof. □

In order to prove Theorem 2.3, we need the following proposition (for example, see [51, p.5]).

Proposition 2.2. *If $f(x_1,x_2,\cdots,x_k)$ is analytic at the origin $(0,0,\ldots,0) \in \mathbb{C}^k$, then, f can be expanded in an absolutely and uniformly convergent power series,*

$$f(x_1,x_2,\ldots,x_k) = \sum_{n_1,n_2,\ldots,n_k=0}^{\infty}\lambda_{n_1,n_2,\ldots,n_k}x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}.$$

The main result of this section is Theorem 2.3.

Theorem 2.3. *If $f(x_1,y_1,\cdots,x_k,y_k)$ is a $2k$-variable analytic function at $(0,0,\cdots,0) \in \mathbb{C}^{2k}$, then, f can be expanded*

$$\sum_{n_1,\cdots,n_k=0}^{\infty}\lambda_{n_1,\cdots,n_k}L_{n_1}^{(\alpha_1)}(x_1,y_1|q)\cdots L_{n_k}^{(\alpha_k)}(x_k,y_k|q),$$

where $\lambda_{n_1,\cdots,n_k}$ are independent of $x_1,y_1,\cdots,x_k,y_k$, if and only if f satisfies the q-partial differential equations

$$\mathcal{D}_{q,x_j}(1-q^{\alpha_j}\eta_{x_j})\{f\} = -q^{\alpha_j+1}\eta_{x_j}^2\mathcal{D}_{q,y_j}\{f\} \tag{2.3}$$

for $j \in \{1,2,\ldots,k\}$.

Proof. We employ mathematical induction. When $k=1$, it follows from Proposition 2.2 that f can be expanded in an absolutely and uniformly convergent power series in a neighborhood of $(0,0)$. Therefore, there exists a sequence $\{\lambda_{m,n}\}$ independent of x_1 and y_1 for which

$$f(x_1,y_1)=\sum_{m,n=0}^{\infty}\lambda_{m,n}x_1^m y_1^n=\sum_{m=0}^{\infty}x_1^m\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n. \tag{2.4}$$

Substituting the above equation into the following q-partial differential equation

$$\mathcal{D}_{q,x_1}(1-q^{\alpha_1}\eta_{x_1})\{f(x_1,y_1)\}=-q^{\alpha_1+1}\eta_{x_1}^2\mathcal{D}_{q,y_1}\{f(x_1,y_1)\}, \tag{2.5}$$

we obtain

$$\sum_{m=1}^{\infty}(1-q^{\alpha_1+m})(1-q^m)x_1^{m-1}\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n=-q^{\alpha_1+1}\sum_{m=0}^{\infty}q^{2m}x_1^m\mathcal{D}_{q,y_1}\left\{\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n\right\}. \tag{2.6}$$

Equating the coefficients of x_1^{m-1} in (2.6), we have

$$\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n=\frac{(-q^{\alpha_1+1})q^{2(m-1)}}{(1-q^{\alpha_1+m})(1-q^m)}\mathcal{D}_{q,y_1}\left\{\sum_{n=0}^{\infty}\lambda_{m-1,n}y_1^n\right\}.$$

Iteration $m-1$ times yields

$$\begin{aligned}\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n &= \frac{(-q^{\alpha_1+1})^m q^{m(m-1)}}{(q;q)_m(q^{\alpha_1+1};q)_m}\mathcal{D}_{q,y_1}^m\left\{\sum_{n=0}^{\infty}\lambda_{0,n}y_1^n\right\}\\ &= \frac{(-1)^m q^{m^2+m\alpha_1}}{(q;q)_m(q^{\alpha_1+1};q)_m}\sum_{n=0}^{\infty}\lambda_{0,n}\frac{(q;q)_n}{(q;q)_{n-m}}y_1^{n-m}\\ &= \sum_{n=m}^{\infty}(-1)^m\lambda_{0,n}\begin{bmatrix}n\\m\end{bmatrix}_q\frac{q^{m^2+m\alpha_1}}{(q^{\alpha_1+1};q)_m}y_1^{n-m}.\end{aligned}$$

Noting that the series in (2.4) is an absolutely and uniformly convergent series, substituting the above equation into (2.4) and interchanging the order of the summation, we find

$$\begin{aligned}f(x_1,y_1) &= \sum_{m=0}^{\infty}x_1^m\sum_{n=m}^{\infty}(-1)^m\lambda_{0,n}\begin{bmatrix}n\\m\end{bmatrix}_q\frac{q^{m^2+m\alpha_1}}{(q^{\alpha_1+1};q)_m}y_1^{n-m}\\ &= \sum_{n=0}^{\infty}\lambda_{0,n}\sum_{m=0}^{n}(-1)^m\begin{bmatrix}n\\m\end{bmatrix}_q\frac{q^{m^2+m\alpha_1}}{(q^{\alpha_1+1};q)_m}x_1^m y_1^{n-m}\\ &= \sum_{n=0}^{\infty}\lambda_{0,n}L_n^{(\alpha_1)}(x_1,y_1|q).\end{aligned}$$

The above calculation shows that the sufficiency of Theorem 2.3 is correct. Conversely, if $f(x_1,y_1)$ can be expanded in terms of $L_n^{(\alpha_1)}(x_1,y_1|q)$, then using Proposition 4.1, we find that $f(x_1,y_1)$ satisfies (2.3). So we can prove the case of $k=1$.
Next, we assume that Theorem 2.3 is true for the case $k-1$. Since f is analytic at $(0,0)$ and satisfies (2.5). Thus, there exists a sequence $\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}$ independent of x_1 and y_1 such that

$$f(x_1,y_1,\ldots,x_k,y_k)=\sum_{n_1=0}^{\infty}c_{n_1}(x_2,y_2,\ldots,x_k,y_k)L_{n_1}^{(\alpha_1)}(x_1,y_1|q). \tag{2.7}$$

Putting $x_1=0$ in (2.7) and using $L_{n_1}^{(\alpha_1)}(0,y_1|q)=y_1^{n_1}$, we obtain

$$f(0,y_1,x_2,y_2,\ldots,x_k,y_k)=\sum_{n_1=0}^{\infty}c_{n_1}(x_2,y_2,\ldots,x_k,y_k)y_1^{n_1}.$$

Using the Maclaurin expansion theorem, we have

$$c_{n_1}(x_2,y_2,\ldots,x_k,y_k)=\frac{\partial^{n_1}f(0,y_1,x_2,y_2,\ldots,x_k,y_k)}{n_1!\partial y_1{}^{n_1}}\Big|_{y_1=0}.$$

Since $f(x_1,y_1,\ldots,x_k,y_k)$ is analytic near $(x_1,y_1,\ldots,x_k,y_k)=(0,\ldots,0)\in\mathbb{C}^{2k}$, it follows from the above equation that $c_{n_1}(x_2,y_2,\ldots,x_k,y_k)$ is analytic near $(x_2,y_2,\ldots,x_k,y_k)=(0,\ldots,0)\in\mathbb{C}^{2k-2}$. Substituting (2.7) into (2.3), we find that for $j=2,\ldots,k$,

$$\sum_{n_1=0}^{\infty}\mathcal{D}_{q,x_j}(1-q^{\alpha_j}\eta_{x_j})\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}L_n^{(\alpha_1)}(x_1,y_1|q)=\sum_{n_1=0}^{\infty}(-q^{\alpha_j+1}\eta_{x_j}^2)\mathcal{D}_{q,y_j}\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}L_n^{(\alpha_1)}(x_1,y_1|q).$$

By equating the coefficients of $L_n^{(\alpha_1)}(x_1,y_1|q)$ in the above equation, we obtain

$$\mathcal{D}_{q,x_j}(1-q^{\alpha_j}\eta_{x_j})\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}=-q^{\alpha_j+1}\eta_{x_j}^2\mathcal{D}_{q,y_j}\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}.$$

Therefore, there exists a sequence $\{\lambda_{n_1,n_2,\ldots,n_k}\}$ independent of $x_2,y_2,\ldots,x_k,y_k$ for which

$$c_{n_1}(x_2,y_2,\ldots,x_k,y_k)=\sum_{n_2,\ldots,n_k=0}^{\infty}\lambda_{n_1,n_2,\ldots,n_k}L_{n_2}^{(\alpha_2)}(x_2,y_2|q)\cdots L_{n_k}^{(\alpha_k)}(x_k,y_k|q).$$

Then substituting the above equation into (2.7), we proved the sufficiency of Theorem 2.3. Conversely, if f can be expanded in terms of $L_{n_1}^{(\alpha_1)}(x_1,y_1|q)\cdots L_{n_k}^{(\alpha_k)}(x_k,y_k|q)$, it follows from Proposition 2.1 that f satisfies (2.3). This completes the proof. □

Remark 2.4. *Theorem 2.3 implies that all solutions to q-partial differential equation (2.3) can be represented as linear combinations of homogeneous q-Laguerre polynomials. Its applications are discussed in Sections 3 and 5.*

3. Generating functions for homogeneous q-Laguerre polynomials

Since the Stieltjes and Hamburger moment problems corresponding to the q-Laguerre polynomials are indeterminate there exist many different weight functions, see [2, 52, 53, 54] for details. Theorem 3.2 will use Theorem 2.3 to prove the following generating functions of homogeneous q-Laguerre polynomials with different weights. We often refer to the following Hartog's theorem (see [55, p. 28]) to determine if a given function is an analytic function in several complex variables.

Theorem 3.1. *If a complex valued function $f(z_1,z_2,\cdots,z_n)$ is holomorphic (analytic) in each variable separately in a domain $U\in\mathbb{C}^n$, then, it is holomorphic (analytic) in U.*

Theorem 3.2. *(1) We have*

$$\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}}{(q;q)_n}L_n^{(\alpha)}(x,y|q)t^n=(ty;q)_\infty\,{}_0\phi_2\left(\begin{matrix}-\\q^{\alpha+1},ty\end{matrix};q,-q^{\alpha+1}tx\right).\tag{3.1}$$

(2) For arbitrarily given γ, and for $|ty|<1$, we have

$$\sum_{n=0}^{\infty}\frac{(\gamma;q)_n}{(q;q)_n}L_n^{(\alpha)}(x,y|q)t^n=\frac{(\gamma ty;q)_\infty}{(ty;q)_\infty}\,{}_1\phi_2\left(\begin{matrix}\gamma\\q^{\alpha+1},\gamma ty\end{matrix};q,-q^{\alpha+1}tx\right).\tag{3.2}$$

Proof. For part (1), denote the right-hand side of (3.1) by $f(x,y)$. It follows from Theorem 3.1 that $f(x,y)$ is an analytic function of x and y. Thus $f(x,y)$ is analytic at $(0,0)\in\mathbb{C}^2$. On the one hand, we have

$$\mathcal{D}_{q,x}(1-q^\alpha\eta_x)\{f(x,y)\}=-tq^{\alpha+1}(ty;q)_\infty\sum_{n=0}^{\infty}\frac{[(-1)^{n+1}q^{\binom{n+1}{2}}]^3}{(q,q^{\alpha+1};q)_n(ty;q)_{n+1}}(-q^{\alpha+1}xt)^n.$$

On the other hand, according to (1.1),

$$\mathcal{D}_{q,y}\{f(x,y)\}=(ty;q)_\infty\sum_{n=0}^{\infty}\frac{-tq^n[(-1)^nq^{\binom{n}{2}}]^3}{(q,q^{\alpha+1};q)_n(ty;q)_{n+1}}(-q^{\alpha+1}xt)^n,$$

from which we obtain

$$-q^{\alpha+1}\eta_x^2\mathcal{D}_{q,y}\{f(x,y)\}=tq^{\alpha+1}(ty;q)_\infty\sum_{n=0}^{\infty}\frac{q^{3n}[(-1)^nq^{\binom{n}{2}}]^3}{(q,q^{\alpha+1};q)_n(ty;q)_{n+1}}(-q^{\alpha+1}xt)^n=\mathcal{D}_{q,x}(1-q^\alpha\eta_x)\{f(x,y)\}.$$

Therefore, by Theorem 2.3, there exists a sequence $\{\lambda_n\}$ independent of x and y such that

$$(ty;q)_\infty\,{}_0\phi_2\left(\begin{matrix}-\\q^{\alpha+1},ty\end{matrix};q,-q^{\alpha+1}xt\right)=\sum_{n=0}^{\infty}\lambda_nL_n^{(\alpha)}(x,y|q).\tag{3.3}$$

Putting $x=0$ in the above equation, using $L_n^{(\alpha)}(0,y|q)=y^n$ and (1.6), we find that

$$\sum_{n=0}^{\infty}\lambda_n y^n=(ty;q)_\infty=\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}(ty)^n.$$

Equating the coefficients of y^n in the above equation, we have $\lambda_n=(-1)^n q^{\binom{n}{2}}/[t^n(q;q)_n]$. Then substitute it into (3.3) and equation (3.1) follows.
For part (2), denote the right-hand side of (3.2) by $f(x,y)$. It follows from Theorem 3.1 that $f(x,y)$ is an analytic function of x and y for $|ty|<1$. Thus $f(x,y)$ is analytic at $(0,0)\in\mathbb{C}^2$. On the one hand, we have

$$\begin{aligned}\mathcal{D}_{q,x}(1-q^\alpha\eta_x)\{f(x,y)\} &= \mathcal{D}_{q,x}\left\{\frac{(\gamma ty;q)_\infty}{(ty;q)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma;q)_n[(-1)^n q^{\binom{n}{2}}]^2}{(q^{\alpha+1};q)_{n-1}(q,\gamma ty;q)_n}(-q^{\alpha+1}xt)^n\right\}\\ &= \frac{-tq^{\alpha+1}(\gamma ty;q)_\infty}{(ty;q)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma;q)_{n+1}[(-1)^n q^{\binom{n}{2}}]^2 q^{2n}}{(q,q^{\alpha+1};q)_n(\gamma ty;q)_{n+1}}(-q^{\alpha+1}xt)^n.\end{aligned}$$

On the other hand, according to (1.1),

$$\mathcal{D}_{q,y}\{f(x,y)\}=\frac{t(\gamma ty;q)_\infty}{(ty;q)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma;q)_{n+1}[(-1)^n q^{\binom{n}{2}}]^2}{(q,q^{\alpha+1};q)_n(\gamma ty;q)_{n+1}}(-q^{\alpha+1}xt)^n,$$

from which we obtain

$$-q^{\alpha+1}\eta_x^2\mathcal{D}_{q,y}\{f(x,y)\}=\frac{-tq^{\alpha+1}(\gamma ty;q)_\infty}{(ty;q)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma;q)_{n+1}[(-1)^n q^{\binom{n}{2}}]^2 q^{2n}}{(q,q^{\alpha+1};q)_n(\gamma ty;q)_{n+1}}(-q^{\alpha+1}xt)^n=\mathcal{D}_{q,x}(1-q^\alpha\eta_x)\{f(x,y)\}.$$

Hence, by Theorem 2.3, there exists a sequence $\{\lambda_n\}$ independent of x and y such that

$$\frac{(\gamma ty;q)_\infty}{(ty;q)_\infty}{}_1\phi_2\left(\begin{matrix}\gamma\\ q^{\alpha+1},\gamma ty\end{matrix};q,-q^{\alpha+1}xt\right)=\sum_{n=0}^{\infty}\lambda_n L_n^{(\alpha)}(x,y|q). \tag{3.4}$$

Putting $x=0$ in the above equation, using $L_n^{(\alpha)}(0,y|q)=y^n$ and (1.5), we find that

$$\sum_{n=0}^{\infty}\lambda_n y^n=\frac{(\gamma ty;q)_\infty}{(ty;q)_\infty}=\sum_{n=0}^{\infty}\frac{(\gamma;q)_n}{(q;q)_n}(ty)^n.$$

Equating the coefficients of y^n in the above equation, we obtain $\lambda_n=t^n(\gamma;q)_n/(q;q)_n$. Then substitute it into (3.4), which completes the proof of (3.2). □

Remark 3.3. *(1) Taking $y=1$, Theorem 3.2 degenerates into generating functions of q-Laguerre polynomials [49, p.109].*
(2) Taking $\gamma=0$ in (3.2), we can obtain a simpler generating function for $L_n^{(\alpha)}(x,y|q)$:

$$\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x,y|q)}{(q;q)_n}t^n=\frac{1}{(ty;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-q^{\alpha+1}xt\right). \tag{3.5}$$

4. Homogeneous little q-Jacobi polynomials and q-partial differential equations

A q-analogue of Jacobi polynomials was introduced by Hahn [48] and later studied by Andrews and Askey [56, 57], and named by them as little q-Jacobi polynomials:

$$\mathcal{P}_n^{(\alpha,\beta)}(x|q)={}_2\phi_1\left(\begin{matrix}q^{-n},\alpha\beta q^{n+1}\\ \alpha q\end{matrix};q,qx\right). \tag{4.1}$$

As $q\to1$, the little q-Jacobi polynomials tend to a multiple of Jacobi polynomials. The little q-Jacobi polynomials with $\beta=0$ are q-analogs of Laguerre polynomials and are orthogonal with respect to a discrete measure on a countable set, called little q-Laguerre (or Wall) polynomials. Moreover, the little q-Legendre polynomials are little q-Jacobi polynomials with $\alpha=\beta=1$. If we set $\beta\to-\alpha^{-1}q^{-1}\beta$, in the little q-Jacobi polynomials and then take the limit $\alpha\to0$ we obtain the alternative q-Charlier polynomials. For more details about q-Jacobi polynomials, see [49].
To establish the relationship between little q-Jacobi polynomials and q-partial differential equations, similar to Section 2, we naturally introduce homogeneous little q-Jacobi polynomials

$$p_n^{(\alpha,\beta)}(x,y|q)=\sum_{k=0}^{n}q^{k(k+1-2n)/2}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(\alpha\beta q^{n+1};q)_k}{(\alpha q;q)_k}(-x)^k y^{n-k}. \tag{4.2}$$

Evidently,

$$p_n^{(\alpha,\beta)}(x,y|q)=y^n\mathcal{P}_n^{(\alpha,\beta)}(x/y|q),\ p_n^{(\alpha,\beta)}(x,1|q)=\mathcal{P}_n^{(\alpha,\beta)}(x|q),\ p_n^{(\alpha,\beta)}(0,y|q)=y^n. \tag{4.3}$$

Firstly, Proposition 4.1 shows an important property of homogeneous little q-Jacobi polynomials.

Proposition 4.1. *The homogeneous little q-Jacobi polynomials satisfy the q-partial differential equation*

$$\mathcal{D}_{q,x}(1-\alpha\eta_x)\left\{p_n^{(\alpha,\beta)}(x,y|q)\right\}=-q\mathcal{D}_{q,y}(\eta_y^{-1}-q\alpha\beta\eta_x^2)\left\{p_n^{(\alpha,\beta)}(x,y|q)\right\}, \tag{4.4}$$

namely,

$$\mathcal{D}_{q,x}\left\{p_n^{(\alpha,\beta)}(x,y|q)-\alpha p_n^{(\alpha,\beta)}(qx,y|q)\right\}=-q\mathcal{D}_{q,y}\left\{p_n^{(\alpha,\beta)}(x,y/q|q)-q\alpha\beta p_n^{(\alpha,\beta)}(q^2x,y|q)\right\}.$$

Proof. If we use LHS to denote the left-hand side of the equation (4.4), we have

$$\begin{aligned}\text{LHS} &= \mathcal{D}_{q,x}\left\{\sum_{k=0}^{n}(-1)^k q^{k(k+1-2n)/2}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(\alpha\beta q^{n+1};q)_k}{(\alpha q;q)_{k-1}}x^k y^{n-k}\right\}\\ &= \sum_{k=1}^{n}(-1)^k q^{k(k+1-2n)/2}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(1-q^k)(\alpha\beta q^{n+1};q)_k}{(\alpha q;q)_{k-1}}x^{k-1}y^{n-k}.\end{aligned}$$

Similarly, use RHS to denote the right-hand side of the equation (4.4). By simple calculation, we obatin

$$\begin{aligned}\text{RHS} &= \mathcal{D}_{q,y}\left\{\sum_{k=0}^{n}(-1)^{k+1} q^{(k+1)(k+2-2n)/2}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(\alpha\beta q^{n+1};q)_{k+1}}{(\alpha q;q)_{k}}x^k y^{n-k}\right\}\\ &= \sum_{k=0}^{n-1}(-1)^{k+1} q^{(k+1)(k+2-2n)/2}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(1-q^{n-k})(\alpha\beta q^{n+1};q)_{k+1}}{(\alpha q;q)_{k}}x^{k}y^{n-k-1}\\ &= \sum_{k=1}^{n}(-1)^{k} q^{k(k+1-2n)/2}\begin{bmatrix}n\\k-1\end{bmatrix}_q\frac{(1-q^{n-k+1})(\alpha\beta q^{n+1};q)_{k}}{(\alpha q;q)_{k-1}}x^{k-1}y^{n-k}.\end{aligned}$$

It follows from (2.2) that LHS = RHS. □

The main result of this section is Theorem 4.2.

Theorem 4.2. *If $f(x_1,y_1,\cdots,x_k,y_k)$ is a $2k$-variable analytic function at $(0,0,\cdots,0)\in\mathbb{C}^{2k}$, then, f can be expanded*

$$\sum_{n_1,\cdots,n_k=0}^{\infty}\lambda_{n_1,\cdots,n_k}p_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q)\cdots p_{n_k}^{(\alpha_k,\beta_k)}(x_k,y_k|q),$$

where $\lambda_{n_1,\cdots,n_k}$ are independent of $x_1,y_1,\cdots,x_k,y_k$, if and only if f satisfies the q-partial differential equations

$$\mathcal{D}_{q,x_j}(1-\alpha_j\eta_{x_j})\{f\}=-q\mathcal{D}_{q,y_j}(\eta_{y_j}^{-1}-q\alpha_j\beta_j\eta_{x_j}^2)\{f\} \tag{4.5}$$

for $j\in\{1,2,\ldots,k\}$.

Proof. We use mathematical induction. When $k=1$, it follows from Proposition 2.2 that f can be expanded in an absolutely and uniformly convergent power series in a neighborhood of $(0,0)$. Therefore, there exists a sequence $\{\lambda_{m,n}\}$ independent of x_1 and y_1 for which

$$f(x_1,y_1)=\sum_{m,n=0}^{\infty}\lambda_{m,n}x_1^m y_1^n=\sum_{m=0}^{\infty}x_1^m\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n. \tag{4.6}$$

Substituting the above equation into following q-partial differential equation

$$\mathcal{D}_{q,x_1}(1-\alpha_1\eta_{x_1})\{f(x_1,y_1)\}=-q\mathcal{D}_{q,y_1}(\eta_{y_1}^{-1}-q\alpha_1\beta_1\eta_{x_1}^2)\{f(x_1,y_1)\}. \tag{4.7}$$

The left-hand side of (4.7) can be written as

$$\sum_{m=1}^{\infty}(1-\alpha_1 q^m)(1-q^m)x_1^{m-1}\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n=\sum_{m=0}^{\infty}(1-\alpha_1 q^{m+1})(1-q^{m+1})x_1^m\sum_{n=0}^{\infty}\lambda_{m+1,n}y_1^n,$$

and right-hand side of (4.7) can be expressed as

$$\mathcal{D}_{q,y_1}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-q)(q^{-n}-\alpha_1\beta_1 q^{2m+1})\lambda_{m,n}x_1^m y_1^n\right\}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-q)(1-q^n)(q^{-n}-\alpha_1\beta_1 q^{2m+1})\lambda_{m,n}x_1^m y_1^{n-1}.$$

Therefore, we obtain

$$\sum_{m=0}^{\infty}(1-\alpha_1 q^{m+1})(1-q^{m+1})x_1^m\sum_{n=0}^{\infty}\lambda_{m+1,n}y_1^n=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-q)(1-q^n)(q^{-n}-\alpha_1\beta_1 q^{2m+1})\lambda_{m,n}x_1^m y_1^{n-1}. \tag{4.8}$$

Equating the coefficients of x_1^m in (4.8), we can easily see that

$$(1-q^m)(1-\alpha_1 q^m)\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n=-q\sum_{n=0}^{\infty}(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-1)+1})\lambda_{m-1,n+1}y_1^n.$$

From the recurrence relation of the above equation, we can derive

$$(1-q^{m-1})(1-\alpha_1 q^{m-1})\sum_{n=0}^{\infty}\lambda_{m-1,n}y_1^n = -q\sum_{n=0}^{\infty}(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-2)+1})\lambda_{m-2,n+1}y_1^n, \tag{4.9}$$

$$(1-q^{m-2})(1-\alpha_1 q^{m-2})\sum_{n=0}^{\infty}\lambda_{m-2,n}y_1^n = -q\sum_{n=0}^{\infty}(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-3)+1})\lambda_{m-3,n+1}y_1^n, \tag{4.10}$$

$$\vdots$$

$$(1-q^2)(1-\alpha_1 q^2)\sum_{n=0}^{\infty}\lambda_{2,n}y_1^n = -q\sum_{n=0}^{\infty}(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2\cdot 1+1})\lambda_{1,n+1}y_1^n, \tag{4.11}$$

$$(1-q)(1-\alpha_1 q)\sum_{n=0}^{\infty}\lambda_{1,n}y_1^n = -q\sum_{n=0}^{\infty}(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2\cdot 0+1})\lambda_{0,n+1}y_1^n. \tag{4.12}$$

By equating the coefficients of y_1^n on both sides of (4.9)-(4.12), we easily deduce that

$$\begin{aligned}
\lambda_{m,n} &= \frac{-q(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-1)+1})}{(1-q^m)(1-\alpha_1 q^m)}\lambda_{m-1,n+1},\\
\lambda_{m-1,n} &= \frac{-q(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-2)+1})}{(1-q^{m-1})(1-\alpha_1 q^{m-1})}\lambda_{m-2,n+1},\\
&\vdots\\
\lambda_{2,n} &= \frac{-q(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2\cdot 1+1})}{(1-q^2)(1-\alpha_1 q^2)}\lambda_{1,n+1},\\
\lambda_{1,n} &= \frac{-q(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2\cdot 0+1})}{(1-q)(1-\alpha_1 q)}\lambda_{0,n+1}.
\end{aligned}$$

By iterating the above equations $m-1$ times, we can deduce that

$$\begin{aligned}
\lambda_{m,n} &= \frac{-q(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-1)+1})}{(1-q^m)(1-\alpha_1 q^m)}\times\frac{-q(1-q^{n+2})(q^{-(n+2)}-\alpha_1\beta_1 q^{2(m-2)+1})}{(1-q^{m-1})(1-\alpha_1 q^{m-1})}\cdots\\
&\quad\times\frac{-q(1-q^{n+m-1})(q^{-(n+m-1)}-\alpha_1\beta_1 q^{2\cdot 1+1})}{(1-q^2)(1-\alpha_1 q^2)}\times\frac{-q(1-q^{n+m})(q^{-(n+m)}-\alpha_1\beta_1 q^{2\cdot 0+1})}{(1-q)(1-\alpha_1 q)}\lambda_{0,n+m}\\
&= \frac{(-q)^m(q^{n+1};q)_m}{(q;q)_m(\alpha_1 q;q)_m}(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-1)+1})\cdots(q^{-(n+m)}-\alpha_1\beta_1 q^{2\cdot 0+1})\lambda_{0,n+m}\\
&= q^{m(1-2n-m)/2}\frac{\lambda_{0,n+m}(-1)^m(q;q)_{m+n}}{(q;q)_m(q;q)_n(\alpha_1 q;q)_m}(1-\alpha_1\beta_1 q^{m+n+1})\cdots(1-\alpha_1\beta_1 q^{2m+n})\\
&= \lambda_{0,n+m}(-1)^m q^{m(1-2n-m)/2}\begin{bmatrix}n+m\\ m\end{bmatrix}_q\frac{(\alpha_1\beta_1 q^{m+n+1};q)_m}{(\alpha_1 q;q)_m}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n &= \sum_{n=0}^{\infty}(-1)^m\lambda_{0,n+m}q^{m(1-2n-m)/2}\begin{bmatrix}n+m\\ m\end{bmatrix}_q\frac{(\alpha_1\beta_1 q^{m+n+1};q)_m}{(\alpha_1 q;q)_m}y_1^n\\
&= \sum_{n=m}^{\infty}(-1)^m\lambda_{0,n}q^{m(1-2n+m)/2}\begin{bmatrix}n\\ m\end{bmatrix}_q\frac{(\alpha_1\beta_1 q^{n+1};q)_m}{(\alpha_1 q;q)_m}y_1^{n-m}.
\end{aligned}$$

Noting that the series in (4.6) is an absolutely and uniformly convergent series, substituting the above equation into (4.6) and interchanging the order of the summation, we obtain

$$\begin{aligned} f(x_1,y_1) &= \sum_{m=0}^{\infty} x_1^m \sum_{n=m}^{\infty} \lambda_{0,n}(-1)^m q^{m(1-2n+m)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\alpha_1\beta_1 q^{n+1};q)_m}{(\alpha_1 q;q)_m} y_1^{n-m} \\ &= \sum_{n=0}^{\infty} \lambda_{0,n} \sum_{m=0}^{n} (-1)^m q^{m(1-2n+m)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\alpha_1\beta_1 q^{n+1};q)_m}{(\alpha_1 q;q)_m} x_1^m y_1^{n-m} \\ &= \sum_{n=0}^{\infty} \lambda_{0,n}\, p_n^{(\alpha_1,\beta_1)}(x_1,y_1|q). \end{aligned}$$

The above calculation shows that the sufficiency of Theorem 4.2 is correct. Conversely, if $f(x_1,y_1)$ can be expanded in terms of $p_n^{(\alpha_1,\beta_1)}(x_1,y_1|q)$, then using Proposition 4.1, we find that $f(x_1,y_1)$ satisfies (4.7). So we can prove the case of $k=1$. Next, we assume that Theorem 4.2 is true for the case $k-1$. Since f is analytic at $(0,0)$. Thus, there exists a sequence $\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}$ independent of x_1 and y_1 such that

$$f(x_1,y_1,\ldots,x_k,y_k) = \sum_{n_1=0}^{\infty} c_{n_1}(x_2,y_2,\ldots,x_k,y_k) p_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q). \tag{4.13}$$

Putting $x_1=0$ in (4.13) and using $p_{n_1}^{(\alpha_1,\beta_1)}(0,y_1|q) = y_1^{n_1}$, we obtain

$$f(0,y_1,x_2,y_2,\ldots,x_k,y_k) = \sum_{n_1=0}^{\infty} c_{n_1}(x_2,y_2,\ldots,x_k,y_k) y_1^{n_1}.$$

Using the Maclaurin expansion theorem, we have

$$c_{n_1}(x_2,y_2,\ldots,x_k,y_k) = \frac{\partial^{n_1} f(0,y_1,x_2,y_2,\ldots,x_k,y_k)}{n_1!\partial {y_1}^{n_1}}\Big|_{y_1=0}.$$

Since $f(x_1,y_1,\ldots,x_k,y_k)$ is analytic near $(x_1,y_1,\ldots,x_k,y_k) = (0,\ldots,0) \in \mathbb{C}^{2k}$, it follows from the above equation that $c_{n_1}(x_2,y_2,\ldots,x_k,y_k)$ is analytic near $(x_2,y_2,\ldots,x_k,y_k) = (0,\ldots,0) \in \mathbb{C}^{2k-2}$. Substituting (4.13) into (4.5), we find that for $j=2,\ldots,k$,

$$\begin{aligned} &\sum_{n_1=0}^{\infty} \mathcal{D}_{q,x_j}(1-\alpha\eta_{x_j})\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}\, p_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q) \\ =\; &\sum_{n_1=0}^{\infty} (-q)\mathcal{D}_{q,y_j}(\eta_{y_j}^{-1} - q\alpha\beta\eta_{x_j}^2)\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}\, p_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q). \end{aligned}$$

By equating the coefficients of $p_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q)$ in the above equation, we obtain

$$\mathcal{D}_{q,x_j}(1-\alpha\eta_{x_j})\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\} = -q\mathcal{D}_{q,y_j}(\eta_{y_j}^{-1} - q\alpha\beta\eta_{x_j}^2)\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}.$$

Therefore, by the inductive hypothesis, there exists a sequence $\{\lambda_{n_1,n_2,\ldots,n_k}\}$ independent of x_2, y_2, $\ldots$, x_k, y_k such that

$$c_{n_1}(x_2,y_2,\ldots,x_k,y_k) = \sum_{n_2,\ldots,n_k=0}^{\infty} \lambda_{n_1,n_2,\ldots,n_k} p_{n_2}^{(\alpha_2,\beta_2)}(x_2,y_2|q)\cdots p_{n_k}^{(\alpha_k,\beta_k)}(x_k,y_k|q).$$

Substituting this equation into (4.13), we proved the sufficiency of the theorem. Conversely, if f can be expanded in terms of $p_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q)\cdots p_{n_k}^{(\alpha_k,\beta_k)}(x_k,y_k|q)$, it follows from (4.4) that f satisfies (4.5). This completes the proof of Theorem 4.2. □

Remark 4.3. *Theorem 4.2 implies that all solutions to q-partial differential equation (4.5) can be represented as linear combinations of homogeneous little q-Jacobi polynomials. See Section 5 for the application of this theorem.*

At the end of this section, we will present the generating function of homogeneous little q-Jacobi polynomials.

Proposition 4.4. *Generating function for homogeneous little q-Jacobi polynomials:*

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} t^n}{(q,\beta q;q)_n} p_n^{(\alpha,\beta)}(a,b|q) = {}_0\phi_1\left(\begin{matrix} - \\ \alpha q \end{matrix}; q, -\alpha qat\right) {}_2\phi_1\left(\begin{matrix} b/a, - \\ \beta q \end{matrix}; q, -at\right).$$

Proof. It follows from [49] that

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}t^n}{(q,\beta q;q)_n}\mathcal{P}_n^{(\alpha,\beta)}(a|q)={}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,\alpha qat\right){}_2\phi_1\left(\begin{matrix}1/a,-\\ \beta q\end{matrix};q,at\right).$$

If a is replaced by a/b in the above equation, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}t^n}{(q,\beta q;q)_n}\mathcal{P}_n^{(\alpha,\beta)}(a/b|q)={}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,\alpha qat/b\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,at/b\right).$$

Letting further $t\to -tb$ in the above equation gives

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}t^n}{(q,\beta q;q)_n}b^n\mathcal{P}_n^{(\alpha,\beta)}(a/b|q)={}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha qat\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-at\right).$$

Finally, we can deduce the conclusion by combining the above equation with (4.3). □

By using Proposition 4.1, we can determine that the right-hand side of the equation in Proposition 4.4 satisfies the q-partial differential equation (4.4). Hence, we have the following Corollary 4.5, which will be applied in Section 5.

Corollary 4.5. *We have*

$$\begin{aligned}&\mathcal{D}_{q,a}(1-\alpha\eta_a)\left\{{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha qat\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-at\right)\right\}\\ =\;&-q\mathcal{D}_{q,b}(\eta_b^{-1}-q\alpha\beta\eta_a^2)\left\{{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha qat\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-at\right)\right\}.\end{aligned}$$

5. Applications of Theorems 2.3 and 4.2

The Rogers-Szegő polynomials are famous q-polynomials which play an essential role in the theory of orthogonal polynomials. Liu [28] studied the homogeneous Rogers-Szegő polynomials from the perspective of q-partial differential equations, which are defined as

$$h_n(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q x^k y^{n-k}. \tag{5.1}$$

Further, the homogeneous Hahn polynomials

$$\Psi_n^{(a)}(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q (a;q)_k x^k y^{n-k} \tag{5.2}$$

are a generalization of homogeneous Rogers-Szegő polynomials. They were first studied by Hahn [48], and then by Al-Salam and Carlitz [1]. So they are also called Al-Salam-Carlitz polynomials. The following generating functions will be frequently used (cf. [1, 29])

$$\sum_{n=0}^{\infty}\frac{\Psi_n^{(a)}(x,y|q)}{(q;q)_n}t^n=\frac{(axt;q)_\infty}{(xt,yt;q)_\infty},\ \max\{|xt|,|yt|\}<1. \tag{5.3}$$

When $a=0$, (5.3) degenerates into the generating function of homogeneous Rogers-Szegő polynomials

$$\sum_{n=0}^{\infty}\frac{h_n(x,y|q)}{(q;q)_n}t^n=\frac{1}{(xt,yt;q)_\infty},\ \max\{|xt|,|yt|\}<1. \tag{5.4}$$

We present two famous Ramanujan q-beta integrals [58, 59].

Proposition 5.1. *For* $m\in\mathbb{R}$, $0<q=e^{-2k^2}<1$, *supposing that* $|yzq|<1$, *we have*

$$\int_{-\infty}^{+\infty}\frac{e^{-\theta^2+2m\theta}}{(yq^{1/2}e^{2ki\theta};q)_\infty(zq^{1/2}e^{-2ik\theta};q)_\infty}d\theta=\sqrt{\pi}e^{m^2}\frac{(-yqe^{2mki};q)_\infty(-zqe^{-2mki};q)_\infty}{(yzq;q)_\infty}. \tag{5.5}$$

Supposing that $\max\{|yq^{1/2}e^{2mk}|,|zq^{1/2}e^{-2mk}|\}<1$, *we have*

$$\int_{-\infty}^{+\infty}e^{-\theta^2+2m\theta}(-yqe^{2k\theta};q)_\infty(zqe^{-2k\theta};q)_\infty d\theta=\sqrt{\pi}e^{m^2}\frac{(yzq;q)_\infty}{(yq^{1/2}e^{2mk};q)_\infty(zq^{1/2}e^{-2mk};q)_\infty}. \tag{5.6}$$

The following Theorem 5.2 is a generalization of the Proposition 5.1.

Theorem 5.2. *For $m \in \mathbb{R}$ and $\alpha > -1$, $0 < q = e^{-2k^2} < 1$, supposing that $|yzq| < 1$, we have*

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{e^{-\theta^2+2m\theta}}{(zq^{1/2}e^{-2ik\theta};q)_\infty} \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x,y|q) h_n(-qe^{2mki}, q^{1/2}e^{2ki\theta}|q)}{(q;q)_n} d\theta \\
= \; & \sqrt{\pi} e^{m^2} \frac{(-zqe^{-2mki};q)_\infty}{(yzq;q)_\infty} {}_0\phi_1 \left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix} ; q, -q^{\alpha+2}xz \right). \qquad (5.7)
\end{aligned}$$

Supposing that $\max\{|yq^{1/2}e^{2mk}|, |zq^{1/2}e^{-2mk}|\} < 1$, *we have*

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{-\theta^2+2m\theta} (zqe^{-2k\theta};q)_\infty \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \frac{L_n^{(\alpha)}(x,y|q) g_n(-qe^{2k\theta}, q^{1/2}e^{2mk}|q)}{(q;q)_n} d\theta \\
= \; & \sqrt{\pi} e^{m^2} \frac{(yzq;q)_\infty}{(zq^{1/2}e^{-2mk};q)_\infty} {}_0\phi_2 \left[\begin{matrix} - \\ q^{\alpha+1}, zqy \end{matrix} ; q, -q^{\alpha+2}zx \right], \qquad (5.8)
\end{aligned}$$

where $g_n(x,y|q)$ represent the homogeneous Stieltjes-Wigert polynomials:

$$g_n(x,y|q) = h_n(x,y|q^{-1}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^k y^{n-k}.$$

Proof. (1) We use $f(x,y)$ to represent the right-hand side of (5.7). Obviously, $f(x,y)$ is analytic near $(0,0) \in \mathbb{C}^2$. It is evident from (3.5) that $f(x,y)$ satisfies

$$\mathcal{D}_{q,x}(1-q^\alpha \eta_x)\{f(x,y)\} = -q^{\alpha+1}\eta_x^2 \mathcal{D}_{q,y}\{f(x,y)\}.$$

According to Theorem 2.3, there exists a sequence $\{\lambda_n\}$ independent of x and y such that

$$\sqrt{\pi} e^{m^2} \frac{(-zqe^{-2mki};q)_\infty}{(yzq;q)_\infty} {}_0\phi_1 \left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix} ; q, -q^{\alpha+2}xz \right) = \sum_{n=0}^{\infty} \lambda_n L_n^{(\alpha)}(x,y|q). \qquad (5.9)$$

By letting $x = 0$ in the above equation and using $L_n^{(\alpha)}(0,y|q) = y^n$, we can derive that

$$\sqrt{\pi} e^{m^2} \frac{(-zqe^{-2mki};q)_\infty}{(yzq;q)_\infty} = \sum_{n=0}^{\infty} \lambda_n y^n. \qquad (5.10)$$

Next, by using equations (5.4) and (5.5),

$$\begin{aligned}
\sqrt{\pi} e^{m^2} \frac{(-zqe^{-2mki};q)_\infty}{(yzq;q)_\infty} &= \frac{1}{(-yqe^{2mki};q)_\infty} \int_{-\infty}^{+\infty} \frac{e^{-\theta^2+2m\theta}}{(yq^{1/2}e^{2ik\theta}, zq^{1/2}e^{-2ik\theta};q)_\infty} d\theta \\
&= \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \frac{e^{-\theta^2+2m\theta}}{(zq^{1/2}e^{-2ik\theta};q)_\infty} \frac{h_n(-qe^{2mki}, q^{1/2}e^{2ki\theta}|q)}{(q;q)_n} d\theta y^n. \qquad (5.11)
\end{aligned}$$

Then comparing the y^n coefficients of (5.10) and (5.11), we can obtain

$$\lambda_n = \int_{-\infty}^{+\infty} \frac{e^{-\theta^2+2m\theta}}{(zq^{1/2}e^{-2ik\theta};q)_\infty} \frac{h_n(-qe^{2mki}, q^{1/2}e^{2ki\theta}|q)}{(q;q)_n} d\theta.$$

Finally, substitute the above equation into (5.9) to complete the proof.
(2) Similarly, we use $f(x,y)$ to represent the right-hand side of (5.8). Obviously, $f(x,y)$ is analytic near $(0,0) \in \mathbb{C}^2$. It is evident from (3.1) that $f(x,y)$ satisfies

$$\mathcal{D}_{q,x}(1-q^\alpha \eta_x)\{f(x,y)\} = -q^{\alpha+1}\eta_x^2 \mathcal{D}_{q,y}\{f(x,y)\}.$$

According to Theorem 2.3, there exists a sequence $\{\lambda_n\}$ independent of x and y such that

$$\frac{\sqrt{\pi} e^{m^2} (yzq;q)_\infty}{(zq^{1/2}e^{-2mk};q)_\infty} {}_0\phi_2 \left[\begin{matrix} - \\ q^{\alpha+1}, zqy \end{matrix} ; q, -q^{\alpha+2}zx \right] = \sum_{n=0}^{\infty} \lambda_n L_n^{(\alpha)}(x,y|q). \qquad (5.12)$$

By letting $x=0$ in the above equation and using $L_n^{(\alpha)}(0,y|q)=y^n$, we can derive that

$$\sqrt{\pi}e^{m^2}\frac{(yzq;q)_\infty}{(zq^{1/2}e^{-2mk};q)_\infty}=\sum_{n=0}^{\infty}\lambda_n y^n. \tag{5.13}$$

Next, by using equations (5.6) and [28, Theorem 3.1]:

$$(sy,ty;q)_\infty=\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}g_n(s,t|q)\frac{y^n}{(q;q)_n}.$$

So the left-side of (5.13) can be rewritten as

$$\begin{aligned}\sqrt{\pi}e^{m^2}\frac{(yzq;q)_\infty}{(zq^{1/2}e^{-2mk};q)_\infty} &= (yq^{1/2}e^{2mk};q)_\infty\int_{-\infty}^{+\infty}e^{-\theta^2+2m\theta}(-yqe^{2k\theta};q)_\infty(zqe^{-2k\theta};q)_\infty d\theta\\ &= \int_{-\infty}^{+\infty}e^{-\theta^2+2m\theta}(zqe^{-2k\theta};q)_\infty(yq^{1/2}e^{2mk};q)_\infty(-yqe^{2k\theta};q)_\infty d\theta\\ &= \int_{-\infty}^{+\infty}e^{-\theta^2+2m\theta}(zqe^{-2k\theta};q)_\infty\sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}\frac{g_n(q^{1/2}e^{2mk},-qe^{2k\theta}|q)y^n}{(q;q)_n}d\theta\\ &= \sum_{n=0}^{\infty}\int_{-\infty}^{+\infty}e^{-\theta^2+2m\theta}(zqe^{-2k\theta};q)_\infty(-1)^nq^{\binom{n}{2}}\frac{g_n(q^{1/2}e^{2mk},-qe^{2k\theta}|q)}{(q;q)_n}d\theta y^n.\end{aligned}$$

Then comparing the y^n coefficients of (5.13) and the above equation, we can obtain

$$\lambda_n=\int_{-\infty}^{+\infty}e^{-\theta^2+2m\theta}(zqe^{-2k\theta};q)_\infty(-1)^nq^{\binom{n}{2}}\frac{g_n(q^{1/2}e^{2mk},-qe^{2k\theta}|q)}{(q;q)_n}d\theta.$$

Finally, substitute the above equation into (5.12) to complete the proof. □

Remark 5.3. *When $x=0$, (5.7) and (5.8) degenerate to (5.5) and (5.6), respectively. In the later Theorem 6.4, we will provide an equivalent form of (5.7), and (5.8) is similar, which we leave for interested readers.*

Now, we will present some applications of Theorems 2.3 and 4.2 in q-integral. The Jackson q-integral of the function $f(x)$ from a to b is defined as

$$\int_a^b f(x)d_qx=(1-q)\sum_{n=0}^{\infty}[bf(bq^n)-af(aq^n)]q^n. \tag{5.14}$$

If f is continuous on (a,b), then it is easily seen that

$$\lim_{q\to1^-}\int_a^b f(x)d_qx=\int_a^b f(x)dx.$$

The famous Andrews-Askey integral formula [60, Theorem 1] can be stated in the following proposition.

Proposition 5.4. *For $\max\{|bu|,|bv|,|cu|,|cv|\}<1$, we have*

$$\int_u^v\frac{(qx/u,qx/v;q)_\infty}{(bx,cx;q)_\infty}d_qx=\frac{(1-q)v(q,u/v,qv/u,bcuv;q)_\infty}{(bu,bv,cu,cv;q)_\infty}.$$

In [29, Theorem 4.4], Liu extended Proposition 5.4 and proved the following q-integral formula.

Proposition 5.5. *If there are no zero factors in the denominator of the integral, we have*

$$\int_u^v\frac{(qx/u,qx/v,\beta ax;q)_\infty}{(ax,bx,cx,dx;q)_\infty}d_qx=\frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}\sum_{n=0}^{\infty}\frac{W_n(c,d,u,v|q)\Psi_n^{(\beta)}(a,b|q)}{(q;q)_n}$$

with

$$W_n(a,b,u,v|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(av,bv;q)_k}{(abuv;q)_k}u^kv^{n-k}. \tag{5.15}$$

The main results of this section is the following Theorems 5.6 and 5.9.

Theorem 5.6. *For* $\max\{|cu|,|cv|,|du|,|dv|,|bzu|,|bzv|\}<1$, *we have*

$$\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}\mathbb{T}(\beta,y;\mathcal{D}_{q,z})\left\{\frac{1}{(bzx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-q^{\alpha+1}azx\right)\right\}d_qx$$
$$= \frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}\sum_{n=0}^{\infty}\frac{W_n(c,d,u,v|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}L_n^{(\alpha)}(a,b|q)$$

with

$$\mathbb{T}(\beta,y;\mathcal{D}_{q,z})=\sum_{k=0}^{\infty}\frac{(\beta;q)_k}{(q;q)_k}(y\mathcal{D}_{q,z})^k,$$

it is called the Cauchy augmentation operator [61, (1.2)].

Proof. We use $I(a,b)$ to represent the left-hand side of the equation in Theorem 5.3, then we have

$$I(a,b)=\mathbb{T}(\beta,y;\mathcal{D}_{q,z})\left\{\int_u^v\frac{(qx/u,qx/v;q)_\infty}{(cx,dx,bzx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-q^{\alpha+1}azx\right)d_qx\right\}. \tag{5.16}$$

It is evident that the function in braces in (5.16) is analytic near $(0,0)\in\mathbb{C}^2$ for $\max\{|cu|,|cv|,|du|,|dv|,|bzu|,|bzv|\}<1$, therefore $I(a,b)$ is also analytic. By using

$$\mathbb{T}(\beta,y;\mathcal{D}_{q,z})\{z^n\}=\sum_{k=0}^{\infty}\frac{(\beta;q)_k}{(q;q)_k}y^k\mathcal{D}_{q,z}^k\{z^n\}=\Psi_n^{(\beta)}(y,z|q)$$

and (3.5), then (5.16) can be rewritten as

$$\begin{aligned}I(a,b) &= \int_u^v\frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}\mathbb{T}(\beta,y;\mathcal{D}_{q,z})\left\{\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(a,b|q)}{(q;q)_n}(xz)^n\right\}d_qx\\ &= \int_u^v\frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(a,b|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}x^nd_qx.\end{aligned} \tag{5.17}$$

According to the definition of q-integral, it can be seen that (5.17) is a linear combination of $L_n^{(\alpha)}(a,b|q)$, namely,

$$\begin{aligned}I(a,b) &= (1-q)\sum_{m=0}^{\infty}\left[\frac{vq^m(vq^{m+1}/u,q^{m+1};q)_\infty}{(cvq^m,dvq^m;q)_\infty}\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(a,b|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}(vq^m)^n\right.\\ &\left.-\frac{uq^m(q^{m+1},uq^{m+1}/v;q)_\infty}{(cuq^m,duq^m;q)_\infty}\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(a,b|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}(uq^m)^n\right].\end{aligned}$$

Since $\mathcal{D}_q$ is a difference operator, it follows from the above equation and Proposition 2.1 that

$$\mathcal{D}_{q,a}(1-q^\alpha\eta_a)\{I(a,b)\}=-q^{\alpha+1}\eta_a^2\mathcal{D}_{q,b}\{I(a,b)\}.$$

Then by Theorem 2.3, there exists a sequence $\{\lambda_n\}$ independent of a and b such that

$$\int_u^v\frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(a,b|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}x^nd_qx=\sum_{n=0}^{\infty}\lambda_nL_n^{(\alpha)}(a,b|q). \tag{5.18}$$

Putting $a=0$ in the above equation, using $L_n^{(\alpha)}(0,b|q)=b^n$ and (5.3), we find that

$$I(0,b)=\int_u^v\frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}\sum_{n=0}^{\infty}\frac{\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}(bx)^nd_qx=\int_u^v\frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}\frac{(\beta ybx;q)_\infty}{(ybx,zbx;q)_\infty}d_qx=\sum_{n=0}^{\infty}\lambda_nb^n. \tag{5.19}$$

Substituting $a\to yb$ and $b\to zb$ in Proposition 5.5 yields the following result

$$\int_u^v\frac{(qx/u,qx/v,\beta ybx;q)_\infty}{(ybx,zbx,cx,dx;q)_\infty}d_qx=\frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}\sum_{n=0}^{\infty}\frac{W_n(c,d,u,v|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}b^n.$$

By combining the above q-integral with (5.19) and equating the coefficients of b^n, we can obtain

$$\lambda_n=\frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}\frac{W_n(c,d,u,v|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}.$$

Substituting the above equation into (5.18), Theorem 5.6 follows. □

Remark 5.7. *(1) When $a=b=y=z=0$, Theorem 5.6 immediately reduces to the Proposition 5.4, so Theorem 5.6 is really an extension of the Andrews-Askey integral.*
(2) When $a=0$ and $b=1$, Theorem 5.6 becomes Proposition 5.5.
(3) When $y=0$, $z=1$ and combining (3.5), we obtain

$$\begin{aligned}&\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(bx,cx,dx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-q^{\alpha+1}ax\right)d_qx\\ =&\ \frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}\sum_{n=0}^\infty\frac{W_n(c,d,u,v|q)}{(q;q)_n}L_n^{(\alpha)}(a,b|q).\end{aligned}\tag{5.20}$$

(4) Setting $d=0$ in (5.20) and noticing that $W_n(c,0,u,v|q)=\Psi_n^{(cv)}(u,v|q)$. We immediately obtain following corollary.

Corollary 5.8. *For* $\max\{|cu|,|cv|,|bu|,|bv|\}<1$, *we have*

$$\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(bx,cx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-q^{\alpha+1}ax\right)d_qx=\frac{(1-q)v(q,u/v,qv/u;q)_\infty}{(cu,cv;q)_\infty}\sum_{n=0}^\infty\frac{\Psi_n^{(cv)}(u,v|q)L_n^{(\alpha)}(a,b|q)}{(q;q)_n}.$$

Theorem 5.9. *For* $\max\{|au|,|av|,|cu|,|cv|,|du|,|dv|,|\alpha q|,|\beta q|\}<1$, *we have*

$$\begin{aligned}&\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha qax\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-ax\right)d_qx\\ =&\ \frac{(1-q)v(q,u/v,qv/u,dcuv;q)_\infty}{(du,dv,cu,cv;q)_\infty}\sum_{n=0}^\infty\frac{q^{n(n-1)/2}W_n(d,c,u,v|q)}{(q,\beta q;q)_n}p_n^{(\alpha,\beta)}(a,b|q).\end{aligned}$$

Proof. We use $I(a,b)$ to represent the left-hand side of the equation in Theorem 5.9. Clearly, $I(a,b)$ is analytic near $(0,0)\in\mathbb{C}^2$. According to the definition of q-integral, we have

$$\begin{aligned}I(a,b)=&\ (1-q)\sum_{n=0}^\infty\left[\frac{vq^n(vq^{n+1}/u,q^{n+1};q)_\infty}{(cvq^n,dvq^n;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha avq^{n+1}\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-avq^n\right)\right.\\ &\left.-\frac{uq^n(uq^{n+1}/v,q^{n+1};q)_\infty}{(cuq^n,duq^n;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha auq^{n+1}\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-auq^n\right)\right].\end{aligned}\tag{5.21}$$

By setting $t=vq^n$ in Corollary 4.5, we obtain

$$\begin{aligned}&\mathcal{D}_{q,a}(1-\alpha\eta_a)\left\{{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha avq^{n+1}\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-avq^n\right)\right\}\\ =&\ -q\mathcal{D}_{q,b}(\eta_b^{-1}-q\alpha\beta\eta_a^2)\left\{{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha avq^{n+1}\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-avq^n\right)\right\}.\end{aligned}\tag{5.22}$$

Similarly,

$$\begin{aligned}&\mathcal{D}_{q,a}(1-\alpha\eta_a)\left\{{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha auq^{n+1}\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-auq^n\right)\right\}\\ =&\ -q\mathcal{D}_{q,b}(\eta_b^{-1}-q\alpha\beta\eta_a^2)\left\{{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha auq^{n+1}\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-auq^n\right)\right\}.\end{aligned}\tag{5.23}$$

Since $\mathcal{D}_q$ is a difference operator, it follows from equations (5.21)-(5.23) that

$$\mathcal{D}_{q,a}(1-\alpha\eta_a)\{I(a,b)\}=-q\mathcal{D}_{q,b}(\eta_b^{-1}-q\alpha\beta\eta_a^2)\{I(a,b)\}.$$

By Theorem 4.2, there exists a sequence $\{\lambda_n\}$ independent of a and b such that

$$\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha qax\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-ax\right)d_qx=\sum_{n=0}^\infty\lambda_np_n^{(\alpha,\beta)}(a,b|q).\tag{5.24}$$

Letting $a=0$ into (5.24) and using $p_n^{(\alpha,\beta)}(0,b|q)=b^n$, we can find

$$I(0,b)=\int_u^v\sum_{n=0}^\infty\frac{q^{n(n-1)/2}(bx)^n}{(q,\beta q;q)_n}\frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}d_qx=\sum_{n=0}^\infty\frac{q^{n(n-1)/2}b^n}{(q,\beta q;q)_n}\int_u^v\frac{x^n(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}d_qx=\sum_{n=0}^\infty\lambda_nb^n.\tag{5.25}$$

We note that interchange the order of summation and the q-integral in (5.25) is reasonable, since

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}b^n}{(q,\beta q;q)_n} \quad \text{and} \quad \int_u^v \frac{x^n(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty} d_qx$$

can easily infer that they are converges absolutely and uniformly by using the ratio test. Then by q-integral [17, (3.4)]:

$$\int_u^v \frac{x^n(qx/u,qx/v;q)_\infty}{(dx,cx;q)_\infty} d_qx = \frac{(1-q)v(q,u/v,qv/u,dcuv;q)_\infty}{(du,dv,cu,cv;q)_\infty} W_n(d,c,u,v|q).$$

Substituting the above equation into (5.25), we have

$$I(0,b) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}b^n}{(q,\beta q;q)_n} \frac{(1-q)v(q,u/v,qv/u,dcuv;q)_\infty}{(du,dv,cu,cv;q)_\infty} W_n(d,c,u,v|q) = \sum_{n=0}^{\infty} \lambda_n b^n.$$

Equating the coefficients of b^n on both sides of the above equation, we obtain

$$\lambda_n = \frac{(1-q)v(q,u/v,qv/u,dcuv;q)_\infty}{(du,dv,cu,cv;q)_\infty} \frac{q^{n(n-1)/2}W_n(d,c,u,v|q)}{(q,\beta q;q)_n}.$$

Finally, substituting the above equation into (5.24) and Theorem 5.9 follows. □

Remark 5.10. *(1) When $a=b=0$, Theorem 5.9 immediately reduces to the Andrews-Askey integral.*
(2) Setting $d=0$ in Theorem 5.9, we immediately obtain the following corollary.

Corollary 5.11. *For $\max\{|cu|,|cv|,|\alpha q|,|\beta q|\}<1$, we have*

$$\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx;q)_\infty} {}_0\phi_1\left(\begin{matrix} - \\ \alpha q \end{matrix}; q, -\alpha qax\right) {}_2\phi_1\left(\begin{matrix} b/a, - \\ \beta q \end{matrix}; q, -ax\right) d_qx$$
$$= \frac{(1-q)v(q,u/v,qv/u;q)_\infty}{(cu,cv;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(\beta q;q)_n(q;q)_n} \Psi_n^{(cv)}(u,v|q) p_n^{(\alpha,\beta)}(a,b|q).$$

6. Concluding remark

1. This article interprets homogeneous q-Laguerre polynomials and homogeneous little q-Jacobi polynomials mainly from the perspective of q-partial differential equations, providing a new method for studying these two q-orthogonal polynomials. This research method also belongs to Liu's theory of q-partial differential equations.
2. We notice that homogeneous q-Laguerre polynomials and homogeneous Hahn polynomials appear in the Corollary 5.8. To calculate their generating function, we introduce the general double basic hypergeometric series is defined as follows [6, p. 282]

$$\Phi_{D:E;F}^{A:B;C}\left[\begin{matrix} a_A : b_B; c_C \\ d_D : e_E; f_F \end{matrix}; q; x, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_A;q)_{m+n}(b_B;q)_m(c_C;q)_n}{(d_D;q)_{m+n}(q,e_E;q)_m(q,f_F;q)_n}$$
$$\times \left[(-1)^{m+n}q^{\binom{m+n}{2}}\right]^{D-A} \left[(-1)^m q^{\binom{m}{2}}\right]^{1+E-B} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+F-C} x^m y^n, \tag{6.1}$$

where a_A abbreviates the array of A parameters $a_1,a_2,\cdots,a_A$, etc, and $q\neq 0$ when $\min\{D-A,1+E-B,1+F-C\}<0$. The series (6.1) converges absolutely for $|x|,|y|<1$ when $\min\{D-A,1+E-B,1+F-C\}\geq 0$ and $|q|<1$. The series (6.1) is called the q-Kampé de Fériet series when $B=C$ and $E=F$.

Theorem 6.1. *If $\max\{|uyt|,|vyt|\}<1$, then, we have*

$$\sum_{n=0}^{\infty} \frac{\Psi_n^{(\beta)}(u,v|q)L_n^{(\alpha)}(x,y|q)}{(q;q)_n} t^n = \frac{(\beta uyt;q)_\infty}{(uyt,vyt;q)_\infty} \Phi_{2:1;0}^{0:2;1}\left[\begin{matrix} - : \beta, vyt; 0 \\ 0, q^{\alpha+1} : \beta uyt; - \end{matrix}; q; -xutq^{\alpha+1}, -xvtq^{\alpha+1}\right].$$

Proof. Firstly, applying the q-partial derivative operator $\mathcal{D}_{q,t}^k$ to act both sides of the equation (5.3), and then using the formula (1.2), we deduce that

$$\sum_{n=0}^{\infty} \frac{\Psi_{n+k}^{(\beta)}(u,v|q)}{(q;q)_n} t^n = \frac{(\beta ut;q)_\infty}{(ut,vt;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(\beta,vt;q)_j}{(\beta ut;q)_j} u^j v^{k-j}. \tag{6.2}$$

Let LHS to denote the left-hand side of the equation in Theorem 6.1, we have

$$\begin{aligned} \text{LHS} &= \sum_{n=0}^{\infty} \frac{\Psi_n^{(\beta)}(u,v|q)}{(q;q)_n} t^n \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_k} x^k y^{n-k} \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(-1)^k t^n q^{k^2+k\alpha} \Psi_n^{(\beta)}(u,v|q)}{(q;q)_k (q;q)_{n-k} (q^{\alpha+1};q)_k} x^k y^{n-k} \\ &= \sum_{k=0}^{\infty} \frac{(-xt)^k q^{k^2+k\alpha}}{(q,q^{\alpha+1};q)_k} \sum_{n=0}^{\infty} \frac{\Psi_{n+k}^{(\beta)}(u,v|q)}{(q;q)_n} (yt)^n. \end{aligned}$$

Letting $t \to yt$ in (6.2), then substituting it into the above equation yields

$$\begin{aligned} \text{LHS} &= \sum_{k=0}^{\infty} \frac{(-xt)^k q^{k^2+k\alpha}}{(q,q^{\alpha+1};q)_k} \frac{(\beta uyt;q)_\infty}{(uyt,vyt;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(\beta,yvt;q)_j}{(\beta yut;q)_j} u^j v^{k-j} \\ &= \frac{(\beta uyt;q)_\infty}{(uyt,vyt;q)_\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^{(k+j)^2+(k+j)\alpha} (\beta,yvt;q)_j (-xtu)^j (-xtv)^k}{(q^{\alpha+1};q)_{k+j} (q;q)_j (q;q)_k (\beta uyt;q)_j}, \end{aligned}$$

which is equivalent to the right-hand side of the equation in Theorem 6.1. □

Remark 6.2. *(1) Letting $t \to 1$, $x \to a$, $y \to b$ and $\beta \to cv$ in Theorem 6.1, and then substituting that into the equation in Corollary 5.8, we obtain*

$$\begin{aligned} &\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(bx,cx;q)_\infty} {}_0\phi_1 \left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix}; q, -q^{\alpha+1} ax \right) d_q x \\ =\ & \frac{(1-q)v(q,u/v,qv/u,bcuv;q)_\infty}{(bu,bv,cu,cv;q)_\infty} \Phi_{2:1;0}^{0:2;1} \left[\begin{matrix} - : cv, bv; 0 \\ 0, q^{\alpha+1} : bcuv; - \end{matrix}; q; -auq^{\alpha+1}, -avq^{\alpha+1} \right]. \end{aligned}$$

(2) Letting $\beta = 0$ in Theorem 6.1, and we immediately obtain the following corollary.

Corollary 6.3. *If* $\max\{|uyt|,|vyt|\} < 1$*, then, we have*

$$\sum_{n=0}^{\infty} \frac{h_n(u,v|q) L_n^{(\alpha)}(x,y|q)}{(q;q)_n} t^n = \frac{1}{(uyt,vyt;q)_\infty} \Phi_{2:0;0}^{0:1;1} \left[\begin{matrix} - : vyt; 0 \\ 0, q^{\alpha+1} : -; - \end{matrix}; q; -xutq^{\alpha+1}, -xvtq^{\alpha+1} \right].$$

Applying Corollary 6.3 to (5.7), we immediately arrive at the following theorem. The proof will be omitted.

Theorem 6.4. *For $m \in \mathbb{R}$ and $\alpha > -1$, $0 < q = e^{-2k^2} < 1$ and $|yzq| < 1$, we have*

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{e^{-\theta^2+2m\theta}}{(yq^{1/2}e^{2ki\theta};q)_\infty (zq^{1/2}e^{-2ik\theta};q)_\infty} \times \Phi_{2:0;0}^{0:1;1} \left[\begin{matrix} - : yq^{1/2}e^{2ki\theta}; 0 \\ 0, q^{\alpha+1} : -; - \end{matrix}; q; xe^{2mki} q^{\alpha+2}, -xve^{2ki\theta} q^{\alpha+3/2} \right] d\theta \\ =\ & \sqrt{\pi} e^{m^2} \frac{(-yqe^{2mki};q)_\infty (-zqe^{-2mki};q)_\infty}{(yzq;q)_\infty} {}_0\phi_1 \left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix}; q, -q^{\alpha+2} xz \right). \end{aligned}$$

Declarations

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