Modification of Hardening Parameter for Computational Plasticity

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ABSTRACT

Hardening can be defined as increase in the strength of a material due to plastic deformation. A type of hardening, which is work hardening, is performed under the cold working conditions. In metallic solids permanent change of shape is generally carried out on a microscopic scale by defects called dislocations which are created by stress. In addition, Hardening parameter is so critic for the computational plasticity. In this study, the hardening parameter, which has emerged from the variation of yield surface equation, has been considered. It has been isolated from hardening rule and investigated that the parameter must have a unique value for any hardening rule.

Keywords: Hardening parameter, hardening rule, elasto-plasticity.

Hesaplamalı Plastizite İçin Sertleştirme Parametresinin Modifikasyonu

ÖZ

Sertleştirme plastik deformasyondan dolayı malzemenin mukavemetindeki artış olarak tanımlanmaktadır. Sertleştirme türlerinden biri olan pekleşme soğuk şekil değiştirme sonucunda oluşmaktadır. Metallerdeki kalıcı şekil değişimi, genellikle malzemenin içyapısındaki gerilmelerin dislokasyon adı verilen mikroskobik ölçüdeki kusurlara neden olmasından kaynaklanmaktadır. Buna ek olarak sertleştirme parametresi hesaplamalı plastizite için oldukça önemlidir. Bu çalışmada akma yüzey denkleminden ortaya çıkan sertleştirme parametresi dikkate alınmıştır. Bu parametrenin her sertleştirme kuralı için özgün bir değere sahip olduğu sertleştirme kuralından izole edilmiş ve incelenmiştir.

Anahtar Kelimeler: Sertleştirme parametresi, sertleştirme kuralı, elasto-plastizite.

1. INTRODUCTION

Computational plasticity requires to describe plastic strain and stress increments. This can be proceeded by means of the following conditions in the associated plasticity; the instantaneous yield surface must be convex, the plastic strain increment vector must be on the outward normal to the instantaneous yield surface and the rate of change of plastic strain must be a linear function of the rate of change of the stress [1]. These conditions can be satisfied under the assumption of the elasto-plastic behaviour of a given material under multiaxial stresses obtained in terms of its uniaxial behaviour.

Due to the classical theory of plasticity, elasto-plastic equations are derived based upon the yield criteria, flow rules and hardening rules. Yield Criteria can be defined by the yield surfaces. Yield surface can be generally expressed as follows:

$$F(\underline{\sigma},\underline{k})=0 \quad (1)$$

Where $\underline{k}$ is the hardening vector, which is generally a function of the plastic strain $\underline{\varepsilon}_p$ and a scalar hardening parameter κ, i.e.

$$F(\underline{\sigma},\underline{\varepsilon}_p,\kappa)=0 \quad (2)$$

There are several yield criteria in the literature; the most popular ones for metals and alloys are Von Mises and Tresca's criteria. The yield surfaces of Von Mises and Tresca can be defined by referring to Von Mises`s and Tresca's equivalent stresses $\overline{\sigma}$ and uniaxial yield stresses Y in the following equation:

$$F=\overline{\sigma}-Y=0 \quad (3)$$

Flow Rule, which defines the plastic strain increment $d\underline{\varepsilon}_p$, may be expressed by Nayak and Zienkiewicz [2] for associative plasticity as follows:

$$d\underline{\varepsilon}_p=d\lambda\frac{\partial F}{\partial\underline{\sigma}}=d\lambda\,\underline{a} \quad (4)$$

Where $d\lambda$ is proportionality constant, F is the yield surface function and $\underline{a}$ is the variation of F with respect to stresses.

The total strain increment during plastic flow is expressed as follows:

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$$d\underline{\varepsilon} = d\underline{\varepsilon}_e + d\underline{\varepsilon}_p \tag{5}$$

Where $d\underline{\varepsilon}_e = D^{-1} d\sigma$ and $\underline{D}$ is the elastic stress-strain matrix. So, it can be deduced from Equations (4) and (5) as follows:

$$d\underline{\sigma} = \underline{D}\, d\underline{\varepsilon} - d\lambda\, \underline{D}\, \underline{\mathrm{a}} \tag{6}$$

Considering the Equation (2) and by differentiation it can be deduced that

$$dF = (\frac{\partial F}{\partial \underline{\sigma}})^t\, d\underline{\sigma} + (\frac{\partial F}{\partial \underline{\varepsilon}_p})^t\, d\underline{\varepsilon}_p + \frac{\partial F}{\partial \kappa}\, d\kappa = 0 \tag{7}$$

where the last two terms are called as hardening term and it has been re-defined and expressed as follows:

$$A = -[(\frac{\partial F}{\partial \underline{\varepsilon}_p})^t\, d\underline{\varepsilon}_p + (\frac{\partial F}{\partial \kappa})\, d\kappa\,]/\,d\lambda \tag{8}$$

where

$$d\lambda = \frac{\underline{\mathrm{a}}^t \underline{D}}{A + \underline{\mathrm{a}}^t \underline{D}\underline{\mathrm{a}}}\, d\underline{\varepsilon} \tag{9}$$

So, the Equation (7) is simplified to the following equation

$$dF = \underline{\mathrm{a}}^t d\underline{\sigma} - A\, d\lambda = 0 \tag{10}$$

where the A is the slope in the equivalent stress-equivalent plastic strain curve and it may be called as hardening parameter.

2. MODIFIED HARDENING PARAMETER

The definition and the derivation of hardening parameter is quite standard procedure in classical plasticity theory. But hardening parameter of each case should result with a different formulation and value due to different definitions of yield surfaces in different hardening rules. So, it is required a new interpretation and derivation to handle this problem. For this purpose, the well-known derivations of hardening rules may be given to see the difficulty mentioned above.

2.1 Hardening Rules

2.1.1 Isotropic hardening

This rule states that the instantaneous yield surface will deform uniformly during plastic deformation. So, the yield surface can be formulated as follows:

$$F\,(\underline{\sigma}, \mathrm{Y}(k)\,) = 0 \tag{11}$$

where, k is a function of the plastic strain history. If the history of the process is taken into account throughout the effective plastic strain ε_p then this type of hardening is called as strain hardening. If the hardening parameter depends on the total plastic work, this is known as work hardening. In this study, work hardening material will be used in the analysis so k may be defined as the amount of plastic work done during plastic deformation as follows:

$$dk = dW_p = \underline{\sigma}^t d\underline{\varepsilon}_P \tag{12}$$

If the uniaxial loading case is considered, then the plastic work may be found. Thus

$$dk = dW_p = Y\, d\,\bar{\varepsilon}_p \tag{13}$$

where $d\bar{\varepsilon}_p$ is uniaxial plastic strain increment.

Last two equations are accepted as equivalent to each other with an acceptable correlation between the uniaxial and multiaxial cases, i.e.

$$dk = Y\, d\,\bar{\varepsilon}_p = d\underline{\varepsilon}_P^t \underline{\sigma} = d\lambda\;\, \underline{\mathrm{a}}^t \underline{\sigma} \tag{14}$$

From yield surface function, applying the Euler's theorem [2] and using Equation (14), the hardening and flow parameters are defined as follows:

$$dk = -Y \frac{\partial F}{\partial Y} d\lambda \tag{15}$$

and

$$d\lambda = -\frac{\partial\,\bar{\varepsilon}_P}{(\dfrac{\partial F}{\partial Y})} \tag{16}$$

The Equation (11) is the implicit form of the yield surface but it may be expressed in the following explicit form for the uniaxial case:

$$F(\underline{\sigma}, Y) = f(\underline{\sigma}) - Y = 0 \tag{17}$$

Apply Euler's theorem to Equation (17) then the following form is obtained:

$$(\frac{\partial F}{\partial \underline{\sigma}})^{\,t}\, \underline{\sigma} - Y = 0 \tag{18}$$

so

$$d\kappa = f(\underline{\sigma})\, d\lambda \tag{19}$$

and

$$d\lambda = d\bar{\varepsilon}_p \tag{20}$$

The parameter A given by Equation (8) can be represented in the following form for isotropic hardening considering $\dfrac{\partial F}{\partial \underline{\varepsilon}_p} = \underline{0}$:

$$A = \left(\frac{\partial F}{\partial Y}\right)^2 H^{'} \tag{21}$$

where $H^{'} = \dfrac{dY}{d\bar{\varepsilon}_p}$ and $\dfrac{\partial F}{\partial Y} = 1$

So, the parameter A is given for isotropic hardening as follow:

$$A = H^{'} \tag{22}$$

2.1.2 Kinematic hardening

The Bauschinger effect can be represented by the Kinematic Hardening model and for this case, it is assumed that the yield surface translates in the stress space as a rigid body.

The yield surface for kinematic hardening is expressed as follows:

$$F\,(\underline{\sigma}-\underline{\alpha}\,,\; Y_o)=0 \tag{23}$$

where $\underline{\alpha}$ is a shift vector for the translation of the initial yield surface, Y_o is the initial yield stress. The shift vector increment has been defined by Prager [3] as follows:

$$d\underline{\alpha}=C\,d\underline{\varepsilon}_P \tag{24}$$

where C is a parameter which characterises the hardening behaviour of material.

The Ziegler's modification [4] of the Prager hardening rule has been given as follows:

$$d\underline{\alpha}\;=d\mu\,(\underline{\sigma}-\underline{\alpha}\,) \tag{25}$$

Parameters $d\lambda$, A, $d\mu$ and C can be defined as follows for kinematic hardening:

$$d\lambda\;=\frac{1}{C}\frac{\underline{\mathrm{a}}^t d\underline{\sigma}}{\underline{\mathrm{a}}^t\underline{\mathrm{a}}} \tag{26}$$

$$d\mu=\frac{C\mathrm{a}^t\;d\underline{\varepsilon}_p}{Y_\mathrm{o}} \tag{27}$$

$$A=C\,\mathrm{a}^t\;\mathrm{a} \tag{28}$$

where $C=\dfrac{2}{3}H^{'}$

It is valid for a uniaxial case and it may be extendable for multiaxial cases [5].

2.1.3 Mixed hardening

Allen, D. H [6] gives a combination of isotropic and kinematic work hardening rules, which has been used by Guzelbey [1] as mixed hardening for the correctness of isotropic hardening with kinematic hardening to predict the Bauschinger effect during cyclic loadings.

Mixed hardening model simulates the yield surface deforms (isotropic hardening) and translates (kinematic hardening) in the space, and the yield surface equation is given by:

$$F\,(\underline{\sigma}-\underline{\alpha}\,,\; Y_r)=0 \tag{29}$$

Where $\underline{\alpha}$ is the translation of the centre of the yield surface. σ-α=σ_r is the reduced stress vector which is measured from the centre of the translated yield surface. Y_r is the current reduced yield stress in simple tension. The plastic strain increment is expressed as follows:

$$d\underline{\varepsilon}_p=d\underline{\varepsilon}_P^{(i)}\;+\;d\underline{\varepsilon}_P^{(k)} \tag{30}$$

Where the superscripts show the isotropic and kinematic models' contribution, and

$$d\underline{\varepsilon}_P^{(i)}\;=\;M\,d\underline{\varepsilon}_p \tag{31}$$

$$d\underline{\varepsilon}_P^{(k)}\;=(1-M)\,d\underline{\varepsilon}_p \tag{32}$$

where M is a material parameter in the range $-1\leq M\leq 1$ which defines the share of isotropic hardening in the total amount of hardening. The yield surface for mixed hardening can be written in following form:

$$F=F\,(\underline{\sigma}-\underline{\alpha}\,,\;(\underline{\varepsilon}\,)_p^{(k)},\;Y_r) \tag{33}$$

and the hardening parameters are given by:

$$d\lambda=\frac{d\bar{\varepsilon}_p^{(i)}}{M\left(\dfrac{\partial F}{\partial Y_r}\right)} \tag{34}$$

and $d\mu$ of Ziegler's model

$$d\mu\;=\frac{C(1-M)\;\underline{\mathrm{a}}^t d\underline{\varepsilon}_p}{-Y_r\dfrac{\partial F}{\partial Y_r}} \tag{35}$$

$$A=(1-M)\;C\;\underline{\mathrm{a}}^t\;\underline{\mathrm{a}}+\;M\left(\frac{\partial F}{\partial Y_r}\right)^2\frac{dY_r}{d\bar{\varepsilon}_p^{(i)}} \tag{36}$$

where $\dfrac{dY_r}{d\bar{\varepsilon}_p^{(i)}}=\mathrm{H}^{'}(\sigma,\varepsilon)$ and

$C=\dfrac{2}{3}\;\dfrac{dY_r}{d\bar{\varepsilon}_p^{(i)}}=\dfrac{2}{3}H^{'}$ for the special case of a material with an idealized stress-strain diagram.

2.2 Suggested Modification

The effect of hardening on yield surface is represented in Equation (9) by means of the term $-Ad\lambda$, for all hardening models reviewed in this study. The parameter dλ, can be obtained by means of Equation (10). For the special case

$$\bar{\sigma}=\frac{1}{\sqrt{2}}\frac{2\bar{\sigma}}{3d\lambda}\left[\left(d\varepsilon_x^{\,p}-d\varepsilon_y^{\,p}\right)^2+\left(d\varepsilon_y^{\,p}-d\varepsilon_z^{\,p}\right)^2+\left(d\varepsilon_z^{\,p}-d\varepsilon_x^{\,p}\right)^2+\frac{3}{2}\left(d\gamma_{xy}^{\,p}\right)^2\right]^{\frac{1}{2}} \tag{37}$$

i.e.

$$d\lambda=\frac{\sqrt{2}}{3}\left[\left(d\varepsilon_x^{\,p}-d\varepsilon_y^{\,p}\right)^2+\left(d\varepsilon_y^{\,p}-d\varepsilon_z^{\,p}\right)^2+\left(d\varepsilon_z^{\,p}-d\varepsilon_x^{\,p}\right)^2+\frac{3}{2}\left(d\gamma_{xy}^{\,p}\right)^2\right]^{\frac{1}{2}} \tag{38}$$

of Von Mises yield criterion, substituting from Equation (4) into (3), it can be deduced that

which represents a measure of equivalent plastic strain increment, valid for different hardening models. However, the parameter *A* has different definitions depending upon the hardening model used, i.e.

$$A = H^{'} \qquad \textit{(isotropic)}$$

$$A = \frac{2}{3} H^{'} \underline{a}^{t} \underline{a} \qquad \textit{(kinematic)}$$

$$A = MA^{(i)} + (1 - M)A^{(k)} \qquad \textit{(mixed)}$$

(39)

For the case of monotonic increasing load, where different models are supposed to give similar answers, some fluctuation in results has been observed because of the variation in *A* definitions. This suggests that for a more consistent elasto-plastic analysis with different hardening conditions, the parameter *A* should be same definition and independent from hardening rules employed for the analysis. Hence, by considering a uniaxial case,

$$\underline{a}^{t} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad (40)$$

i.e.

$$\underline{a}^{t} \underline{a} = \frac{3}{2} \qquad (41)$$

So the Equation (41) shows that the kinematic and mixed hardening parameter equals to $H^{'}$ which is plastic modulus [7], then the hardening parameter *A* will be the plastic modulus as follows:

$$A = H^{'}$$

for all models of hardening rules.

3. RESULT AND DISCUSSION

In this study, the hardening parameter *A* emerged from the variation of the yield surface equation of elasto-plasticity. The hardening parameter has been improved and expressed as a unique value for different hardening rules. Thus, it may improve the results of elasto-plastic analyses based upon different hardening rules.

NOMENCLAUTURE

k	hardening vector
εp	function of the plastic strain
κ	scalar hardening parameter
$\overline{\sigma}$	Von Mises`s and Tresca's equivalent stresses
Y	uniaxial yield stresses
dλ	proportionality constant
F	yield surface
$\underline{a}$	variation of F
dεp	plastic strain increment
$d\underline{\varepsilon}_e$	elastic strain increment
$d\underline{\varepsilon}$	total strain increment
$\underline{D}$	elastic stress-strain matrix
A	hardening parameter
ε_p	plastic strain
dW_p	plastic work done during plastic deformation
$H^{'}$	isotropic hardening
Yo	initial yield stress
α	shift vector for the translation of the initial yield surface
C	characterises the hardening behaviour of material.
$d\mu$	kinematic hardening
Yr	current reduced yield stress
M	material parameter

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