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On a General Non-Linear Difference Equation of Third-Order

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Abstract. In this paper, we investigate the following general difference equations

$$x_{n+1} = h^{-1}\left(h(x_n)\frac{Ah(x_{n-1}) + Bh(x_{n-2})}{Ch(x_{n-1}) + Dh(x_{n-2})}\right),\ n \in \mathbb{N}_0,$$

where the parameters A, B, C, D and the initial values $x_{-\Phi}$, for $\Phi = \overline{0,2}$ are real numbers, $A^2 + B^2 \neq 0 \neq C^2 + D^2$, h is a strictly monotone and continuous function, $h(\mathbb{R}) = \mathbb{R}$, $h(0) = 0$. In addition, we obtain closed-form solutions of aforementioned difference equations. Finally, numerical applications are given.

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1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, means the set of natural, non-negative integer, integer, real and complex numbers, respectively. If $\Phi, \Psi \in \mathbb{Z}$, $\Phi \leq \Psi$ the notation $\alpha = \overline{\Phi, \Psi}$ stands for $\{\alpha \in \mathbb{Z} : \Phi \leq \alpha \leq \Psi\}$.
The difference equations are of interest by many authors in these days [2, 11–14, 16, 17, 19, 24–27, 29–35].

Well-known important difference equation is

$$x_{n+2} = \gamma x_{n+1} + \delta x_n,\ n \in \mathbb{N}_0, \tag{1.1}$$

where the parameter γ, δ and the initial conditions x_0, x_1 are real numbers. De Moivre solved the homogeneous linear second-order difference equation (1.1) in [4]. The general solution of the sequence $(x_n)_{n\in\mathbb{N}_0}$, is given by

$$x_n = \frac{(x_1 - \lambda_2 x_0)\lambda_1^n - (x_1 - \lambda_1 x_0)\lambda_2^n}{\lambda_1 - \lambda_2},\ n \in \mathbb{N}_0, \tag{1.2}$$

when $\delta \neq 0$ and $\gamma^2 + 4\delta \neq 0$,

$$x_n = ((x_1 - \lambda_1 x_0)n + \lambda_1 x_0)\lambda_1^{n-1},\ n \in \mathbb{N}_0, \tag{1.3}$$

when $\delta \neq 0$ and $\gamma^2 + 4\delta = 0$, where λ_1 and λ_2 are the roots of the polynomial $P(\lambda) = \lambda^2 - \gamma\lambda - \delta = 0$. Also, the roots of characteristic equation are $\lambda_{1,2} = \frac{\gamma \pm \sqrt{\gamma^2+4\delta}}{2}$.

The difference equation, which transforms into equation (1.1) using the appropriate transformation is

$$x_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \delta},\ n \in \mathbb{N}_0, \tag{1.4}$$

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for $\gamma \neq 0$, $\alpha\delta \neq \beta\gamma$, where the initial value x_0 is real number. Equation (1.4) is called Riccati difference equation.

Similarly, the difference equation, which becomes equation (1.4) using the change of variable is

$$x_{n+1} = \zeta x_n + \frac{\Upsilon x_n x_{n-1}}{\Phi x_{n-1} + \Psi x_{n-2}}, \; n \in \mathbb{N}_0, \tag{1.5}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are positive real numbers and the parameters ζ, Υ, Φ, Ψ are positive constants. The behavior of the solution of equation (1.5) is investigated in [10].

Type of difference equations in (1.5) have been generalized in various ways by lots of authors in [1,3,5–9,20–23,28]. The generalizations are increasing order, adding constant or periodic parameters, etc. The other way to expand is increasing dimensional. There are difference equations systems which are the type of difference equations in (1.5) in literature (see, e.g. [15, 18]).

A natural question is if equation (1.5) generalize by using different way. Here we give a positive answer. Another way to generalize is the form of the following equation:

$$x_{n+1} = h^{-1}\left(h(x_n)\frac{Ah(x_{n-1}) + Bh(x_{n-2})}{Ch(x_{n-1}) + Dh(x_{n-2})}\right), \; n \in \mathbb{N}_0, \tag{1.6}$$

where the initial values $x_{-\Phi}$, for $\Phi = \overline{0,2}$ are real numbers, the parameters $A, B, C, D \in \mathbb{R}$, $A^2 + B^2 \neq 0 \neq C^2 + D^2$, h is a strictly monotone and continuous function, $h(\mathbb{R}) = \mathbb{R}$, $h(0) = 0$.

Our aim to show that equation (1.6) is solvable in closed form according to states of parameters by changing of the variable. Also, we give numerical applications, which indicate some things in [10] are not correct.

2. Closed-Form Solution of Equation (1.6)

Theorem 2.1. *Suppose that $A^2 + B^2 \neq 0 \neq C^2 + D^2$. So, the equation (1.6) is solvable in closed form.*

Proof. If at least one of the initial conditions $x_{-\theta} = 0$, for $\theta \in \{0, 1, 2\}$, then the solution of equation (1.6) is not defined. Moreover, suppose that $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$. In addition, by using (1.6) we get $x_{n_0+1} = 0$. These facts along with (1.6) imply that x_{n_0+3} is not defined. Hence, for every well-defined solution of (1.6) , we have

$$x_n \neq 0, \; n \geq -2. \tag{2.1}$$

From (2.1) we get

$$h(x_n) \neq 0, \; n \geq -2.$$

Now, we investigate the solution of equation (1.6) for two cases.

2.1. Case 1.

First, assume that $AD \neq BC$ and $C \neq 0$. Let

$$y_n = \frac{h(x_n)}{h(x_{n-1})}, \; n \geq -1. \tag{2.2}$$

From (1.6) and monotonicity of h, we obtain

$$h(x_{n+1}) = h(x_n)\frac{Ah(x_{n-1}) + Bh(x_{n-2})}{Ch(x_{n-1}) + Dh(x_{n-2})}, \; n \in \mathbb{N}_0. \tag{2.3}$$

By using the change of variables (2.2) in (2.3) we get

$$y_{n+1} = \frac{Ay_{n-1} + B}{Cy_{n-1} + D}, \; n \in \mathbb{N}_0. \tag{2.4}$$

Let

$$z_m^{(j)} = y_{2m+j}, \; m \in \mathbb{N}_0, \; j \in \{-1, 0\}. \tag{2.5}$$

Then, from (2.4) and (2.5) we obtain

$$z_{m+1}^{(j)} = \frac{Az_m^{(j)} + B}{Cz_m^{(j)} + D}, \tag{2.6}$$

for $m \in \mathbb{N}_0$, $j \in \{-1, 0\}$. The equation (2.6) is named a Riccati type difference equation in literature.

Let

$$z_m^{(j)} = \frac{u_{m+1}^{(j)}}{u_m^{(j)}} + g_j, \; m \in \mathbb{N}_0, \; j \in \{-1, 0\}, \tag{2.7}$$

for some $g_j \in \mathbb{R}$, $j \in \{-1, 0\}$.
From (2.6)-(2.7) we obtain

$$\left(\frac{u^{(j)}_{m+2}}{u^{(j)}_{m+1}} + g_j\right)\left(C\frac{u^{(j)}_{m+1}}{u^{(j)}_{m}} + Cg_j + D\right) - \left(A\frac{u^{(j)}_{m+1}}{u^{(j)}_{m}} + Ag_j + B\right) = 0,$$

for $m \in \mathbb{N}_0$, $j \in \{-1, 0\}$.
Let

$$g_j = -\frac{D}{C},\ j \in \{-1, 0\}.$$

Then, we have

$$C^2 u^{(j)}_{m+2} - C(A + D)\, u^{(j)}_{m+1} + (AD - BC)\, u^{(j)}_m = 0, \tag{2.8}$$

for $m \in \mathbb{N}_0$, $j \in \{-1, 0\}$.

Assume that $\Delta := (A + D)^2 - 4(AD - BC) \neq 0$. Then, by employing equality (1.2) we get

$$u^{(j)}_m = \frac{\left(u^{(j)}_1 - \lambda_2 u^{(j)}_0\right)\lambda_1^m - \left(u^{(j)}_1 - \lambda_1 u^{(j)}_0\right)\lambda_2^m}{\lambda_1 - \lambda_2}, \tag{2.9}$$

for $m \in \mathbb{N}_0$, $j \in \{-1, 0\}$, where $\lambda_{1,2} = \frac{(A+D)\pm\sqrt{\Delta}}{2C}$, is the general solution to (2.8).
By using (2.9) in (2.7), we get

$$\begin{aligned} z^{(j)}_m &= \frac{\left(u^{(j)}_1 - \lambda_2 u^{(j)}_0\right)\lambda_1^{m+1} - \left(u^{(j)}_1 - \lambda_1 u^{(j)}_0\right)\lambda_2^{m+1}}{\left(u^{(j)}_1 - \lambda_2 u^{(j)}_0\right)\lambda_1^{m} - \left(u^{(j)}_1 - \lambda_1 u^{(j)}_0\right)\lambda_2^{m}} - \frac{D}{C} \\ &= \frac{\left(z^{(j)}_0 + \frac{D}{C} - \lambda_2\right)\lambda_1^{m+1} - \left(z^{(j)}_0 + \frac{D}{C} - \lambda_1\right)\lambda_2^{m+1}}{\left(z^{(j)}_0 + \frac{D}{C} - \lambda_2\right)\lambda_1^{m} - \left(z^{(j)}_0 + \frac{D}{C} - \lambda_1\right)\lambda_2^{m}} - \frac{D}{C}, \end{aligned}$$

for $m \in \mathbb{N}_0$, $j \in \{-1, 0\}$, from the last equality with (2.5) we get

$$y_{2m+j} = \frac{\left(y_j + \frac{D}{C} - \lambda_2\right)\lambda_1^{m+1} - \left(y_j + \frac{D}{C} - \lambda_1\right)\lambda_2^{m+1}}{\left(y_j + \frac{D}{C} - \lambda_2\right)\lambda_1^{m} - \left(y_j + \frac{D}{C} - \lambda_1\right)\lambda_2^{m}} - \frac{D}{C}, \tag{2.10}$$

for $m \in \mathbb{N}_0$, $j \in \{-1, 0\}$.
From (2.2) and (2.10), we obtain

$$h\left(x_{2m+j}\right) = \left(\frac{\left(\frac{h(x_j)}{h(x_{j-1})} + \frac{D}{C} - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{h(x_j)}{h(x_{j-1})} + \frac{D}{C} - \lambda_1\right)\lambda_2^{m+1}}{\left(\frac{h(x_j)}{h(x_{j-1})} + \frac{D}{C} - \lambda_2\right)\lambda_1^{m} - \left(\frac{h(x_j)}{h(x_{j-1})} + \frac{D}{C} - \lambda_1\right)\lambda_2^{m}} - \frac{D}{C}\right) h\left(x_{2m+j-1}\right),$$

for $m \in \mathbb{N}_0$, $j \in \{-1, 0\}$.
From (2.2) we easily get

$$h\left(x_{2m+j_1}\right) = y_{2m+j_1} y_{2m+j_1-1} h\left(x_{2m+j_1-2}\right), \tag{2.11}$$

for $m \in \mathbb{N}_0$, $j_1 \in \{0, 1\}$.
Hence,

$$h(x_{2m}) = h(x_{-2}) \prod_{i=0}^{m} y_{2i} y_{2i-1},$$

$$h(x_{2m+1}) = h(x_{-1}) \prod_{i=0}^{m} y_{2i+1} y_{2i},$$

for $m \in \mathbb{N}_0$, and consequently

$$x_{2m} = h^{-1}\left(h(x_{-2}) \prod_{i=0}^{m} y_{2i} y_{2i-1}\right), \tag{2.12}$$

$$x_{2m+1}=h^{-1}\left(h\left(x_{-1}\right)\prod_{i=0}^{m}y_{2i+1}y_{2i}\right), \tag{2.13}$$

for $m\in\mathbb{N}_0$, where

$$\begin{aligned} y_{2m}y_{2m-1}=&\left(\frac{\left(\frac{h(x_0)}{h(x_{-1})}+\frac{D}{C}-\lambda_2\right)\lambda_1^{m+1}-\left(\frac{h(x_0)}{h(x_{-1})}+\frac{D}{C}-\lambda_1\right)\lambda_2^{m+1}}{\left(\frac{h(x_0)}{h(x_{-1})}+\frac{D}{C}-\lambda_2\right)\lambda_1^{m}-\left(\frac{h(x_0)}{h(x_{-1})}+\frac{D}{C}-\lambda_1\right)\lambda_2^{m}}-\frac{D}{C}\right)\\ &\times\left(\frac{\left(\frac{h(x_{-1})}{h(x_{-2})}+\frac{D}{C}-\lambda_2\right)\lambda_1^{m+1}-\left(\frac{h(x_{-1})}{h(x_{-2})}+\frac{D}{C}-\lambda_1\right)\lambda_2^{m+1}}{\left(\frac{h(x_{-1})}{h(x_{-2})}+\frac{D}{C}-\lambda_2\right)\lambda_1^{m}-\left(\frac{h(x_{-1})}{h(x_{-2})}+\frac{D}{C}-\lambda_1\right)\lambda_2^{m}}-\frac{D}{C}\right), \end{aligned} \tag{2.14}$$

$$\begin{aligned} y_{2m+1}y_{2m}=&\left(\frac{\left(\frac{h(x_{-1})}{h(x_{-2})}+\frac{D}{C}-\lambda_2\right)\lambda_1^{m+2}-\left(\frac{h(x_{-1})}{h(x_{-2})}+\frac{D}{C}-\lambda_1\right)\lambda_2^{m+2}}{\left(\frac{h(x_{-1})}{h(x_{-2})}+\frac{D}{C}-\lambda_2\right)\lambda_1^{m+1}-\left(\frac{h(x_{-1})}{h(x_{-2})}+\frac{D}{C}-\lambda_1\right)\lambda_2^{m+1}}-\frac{D}{C}\right)\\ &\times\left(\frac{\left(\frac{h(x_0)}{h(x_{-1})}+\frac{D}{C}-\lambda_2\right)\lambda_1^{m+1}-\left(\frac{h(x_0)}{h(x_{-1})}+\frac{D}{C}-\lambda_1\right)\lambda_2^{m+1}}{\left(\frac{h(x_0)}{h(x_{-1})}+\frac{D}{C}-\lambda_2\right)\lambda_1^{m}-\left(\frac{h(x_0)}{h(x_{-1})}+\frac{D}{C}-\lambda_1\right)\lambda_2^{m}}-\frac{D}{C}\right) \end{aligned} \tag{2.15}$$

for $m\in\mathbb{N}_0$. By using formulas (2.14)-(2.15) in equations (2.12)-(2.13), we acquire the solution to equation (1.6) if $\Delta\neq 0$.

Suppose that $\Delta=(A+D)^2-4(AD-BC)=0$. So, by employing equality (1.3) we have

$$u_m^{(j)}=\left(\left(u_1^{(j)}-\lambda_1 u_0^{(j)}\right)m+\lambda_1 u_0^{(j)}\right)\lambda_1^{m-1}, \tag{2.16}$$

for $m\in\mathbb{N}_0$, $j=\overline{-1,0}$ where

$$\lambda_1=\frac{A+D}{2C}\neq 0.$$

Note that equation (2.16) is the solution to the equation (2.8). From (2.7) and (2.16), we obtain

$$\begin{aligned} z_m^{(j)}&=\frac{\left(\left(u_1^{(j)}-\lambda_1 u_0^{(j)}\right)(m+1)+\lambda_1 u_0^{(j)}\right)\lambda_1}{\left(u_1^{(j)}-\lambda_1 u_0^{(j)}\right)m+\lambda_1 u_0^{(j)}}-\frac{D}{C}\\ &=\frac{\left(\left(z_0^{(j)}+\frac{D}{C}-\lambda_1\right)(m+1)+\lambda_1\right)\lambda_1}{\left(z_0^{(j)}+\frac{D}{C}-\lambda_1\right)m+\lambda_1}-\frac{D}{C}, \end{aligned} \tag{2.17}$$

for $m\in\mathbb{N}_0$, $j\in\{-1,0\}$. From (2.5) and (2.17) we obtain

$$y_{2m+j}=\frac{\left(\left(y_j+\frac{D}{C}-\lambda_1\right)(m+1)+\lambda_1\right)\lambda_1}{\left(y_j+\frac{D}{C}-\lambda_1\right)m+\lambda_1}-\frac{D}{C}, \tag{2.18}$$

for $m\in\mathbb{N}_0$, $j\in\{-1,0\}$. From (2.2) and (2.18) we have

$$h\left(x_{2m+j}\right)=\left(\frac{\left(\left(\frac{h(x_j)}{h(x_{j-1})}+\frac{D}{C}-\lambda_1\right)(m+1)+\lambda_1\right)\lambda_1}{\left(\frac{h(x_j)}{h(x_{j-1})}+\frac{D}{C}-\lambda_1\right)m+\lambda_1}-\frac{D}{C}\right)h\left(x_{2m+j-1}\right),$$

for $m\in\mathbb{N}_0$, $j\in\{-1,0\}$.
We also have

$$\begin{aligned} y_{2m}y_{2m-1}=&\left(\frac{\left(\left(\frac{h(x_0)}{h(x_{-1})}+\frac{D}{C}-\lambda_1\right)(m+1)+\lambda_1\right)\lambda_1}{\left(\frac{h(x_0)}{h(x_{-1})}+\frac{D}{C}-\lambda_1\right)m+\lambda_1}-\frac{D}{C}\right)\\ &\times\left(\frac{\left(\left(\frac{h(x_{-1})}{h(x_{-2})}+\frac{D}{C}-\lambda_1\right)(m+1)+\lambda_1\right)\lambda_1}{\left(\frac{h(x_{-1})}{h(x_{-2})}+\frac{D}{C}-\lambda_1\right)m+\lambda_1}-\frac{D}{C}\right), \end{aligned} \tag{2.19}$$

$$y_{2m+1}y_{2m} = \left(\frac{\left(\left(\frac{h(x_{-1})}{h(x_{-2})} + \frac{D}{C} - \lambda_1\right)(m+2) + \lambda_1\right)\lambda_1}{\left(\frac{h(x_{-1})}{h(x_{-2})} + \frac{D}{C} - \lambda_1\right)(m+1) + \lambda_1} - \frac{D}{C}\right) \times \left(\frac{\left(\left(\frac{h(x_0)}{h(x_{-1})} + \frac{D}{C} - \lambda_1\right)(m+1) + \lambda_1\right)\lambda_1}{\left(\frac{h(x_0)}{h(x_{-1})} + \frac{D}{C} - \lambda_1\right)m + \lambda_1} - \frac{D}{C}\right) \tag{2.20}$$

for $m \in \mathbb{N}_0$.

We offer the solution to equation (1.6) by using formulas (2.19)-(2.20) in equations (2.12)-(2.13), if $\Delta = 0$.

Now, suppose that $C = 0$. So, $D \neq 0$ and equation (2.4) turns into

$$y_{n+1} = \frac{A}{D}y_{n-1} + \frac{B}{D},\ n \in \mathbb{N}_0.$$

Hence,

$$z_{m+1}^{(j)} = \frac{A}{D}z_m^{(j)} + \frac{B}{D},\ m \in \mathbb{N}_0,\ j \in \{-1, 0\}. \tag{2.21}$$

If $A = D$, then from (2.21) we obtain

$$z_m^{(j)} = \frac{B}{D}m + z_0^{(j)},\ m \in \mathbb{N}_0,\ j \in \{-1, 0\},$$

so

$$y_{2m+j} = \frac{B}{D}m + y_j,\ m \in \mathbb{N}_0,\ j \in \{-1, 0\},$$

from which along with (2.2) and (2.11) it follows that

$$h(x_{2m}) = \left(\frac{B}{D}m + \frac{h(x_0)}{h(x_{-1})}\right)\left(\frac{B}{D}m + \frac{h(x_{-1})}{h(x_{-2})}\right)h(x_{2m-2}),$$

$$h(x_{2m+1}) = \left(\frac{B}{D}(m+1) + \frac{h(x_{-1})}{h(x_{-2})}\right)\left(\frac{B}{D}m + \frac{h(x_0)}{h(x_{-1})}\right)h(x_{2m-1}),$$

where $m \in \mathbb{N}_0$. After some calculations in the last two equations, we get

$$h(x_{2m}) = h(x_{-2})\prod_{j=0}^{m}\left(\frac{B}{D}j + \frac{h(x_0)}{h(x_{-1})}\right)\left(\frac{B}{D}j + \frac{h(x_{-1})}{h(x_{-2})}\right),$$

$$h(x_{2m+1}) = h(x_{-1})\prod_{j=0}^{m}\left(\frac{B}{D}(j+1) + \frac{h(x_{-1})}{h(x_{-2})}\right)\left(\frac{B}{D}j + \frac{h(x_0)}{h(x_{-1})}\right),$$

for $m \in \mathbb{N}_0$, and consequently

$$x_{2m} = h^{-1}\left(h(x_{-2})\prod_{j=0}^{m}\left(\frac{B}{D}j + \frac{h(x_0)}{h(x_{-1})}\right)\left(\frac{B}{D}j + \frac{h(x_{-1})}{h(x_{-2})}\right)\right), \tag{2.22}$$

$$x_{2m+1} = h^{-1}\left(h(x_{-1})\prod_{j=0}^{m}\left(\frac{B}{D}(j+1) + \frac{h(x_{-1})}{h(x_{-2})}\right)\left(\frac{B}{D}j + \frac{h(x_0)}{h(x_{-1})}\right)\right), \tag{2.23}$$

for $m \in \mathbb{N}_0$. Hence, the equalities in (2.22)-(2.23) are solutions of the equation (1.6) in this case.

Suppose that $A \neq D$. By using (2.21), we get

$$z_m^{(j)} = \left(\frac{A}{D}\right)^m z_0^{(j)} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^m - 1\right),\ m \in \mathbb{N}_0,\ j \in \{-1, 0\}.$$

That is,

$$y_{2m+j} = \left(\frac{A}{D}\right)^m y_j + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^m - 1\right), \tag{2.24}$$

for $m \in \mathbb{N}_0$, $j \in \{-1, 0\}$.

From (2.2), (2.11) and (2.24) we have

$$h(x_{2m}) = \left[\left(\frac{A}{D}\right)^m \frac{h(x_0)}{h(x_{-1})} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^m - 1\right)\right] \times \left[\left(\frac{A}{D}\right)^m \frac{h(x_{-1})}{h(x_{-2})} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^m - 1\right)\right] h(x_{2m-2}),$$

$$h(x_{2m+1}) = \left[\left(\frac{A}{D}\right)^{m+1} \frac{h(x_{-1})}{h(x_{-2})} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^{m+1} - 1\right)\right] \times \left[\left(\frac{A}{D}\right)^m \frac{h(x_0)}{h(x_{-1})} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^m - 1\right)\right] h(x_{2m-1})$$

for $m \in \mathbb{N}_0$.

Hence

$$h(x_{2m}) = h(x_{-2}) \prod_{s=0}^{m} \left[\left(\frac{A}{D}\right)^s \frac{h(x_0)}{h(x_{-1})} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^s - 1\right)\right] \times \left[\left(\frac{A}{D}\right)^s \frac{h(x_{-1})}{h(x_{-2})} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^s - 1\right)\right],$$

$$h(x_{2m+1}) = h(x_{-1}) \prod_{s=0}^{m} \left[\left(\frac{A}{D}\right)^{s+1} \frac{h(x_{-1})}{h(x_{-2})} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^{s+1} - 1\right)\right] \times \left[\left(\frac{A}{D}\right)^s \frac{h(x_0)}{h(x_{-1})} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^s - 1\right)\right],$$

for $m \in \mathbb{N}_0$, and consequently

$$x_{2m} = h^{-1}\Bigg[h(x_{-2}) \prod_{s=0}^{m} \left[\left(\frac{A}{D}\right)^s \frac{h(x_0)}{h(x_{-1})} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^s - 1\right)\right] \times \left[\left(\frac{A}{D}\right)^s \frac{h(x_{-1})}{h(x_{-2})} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^s - 1\right)\right]\Bigg], \tag{2.25}$$

$$x_{2m+1} = h^{-1}\Bigg[h(x_{-1}) \prod_{s=0}^{m} \left[\left(\frac{A}{D}\right)^{s+1} \frac{h(x_{-1})}{h(x_{-2})} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^{s+1} - 1\right)\right] \times \left[\left(\frac{A}{D}\right)^s \frac{h(x_0)}{h(x_{-1})} + \frac{B}{A-D}\left(\left(\frac{A}{D}\right)^s - 1\right)\right]\Bigg], \tag{2.26}$$

for $m \in \mathbb{N}_0$. Then, the solution of equation (1.6) is given by equations (2.25)-(2.26) in this case.

2.2. **Case 2.** Suppose that $AD = BC$. If $A = 0$, then $B \neq 0$. This means $C = 0$ and $D \neq 0$. In this case, from equation (1.6), we obtain

$$x_{n+1} = h^{-1}\left(\frac{B}{D} h(x_n)\right), \ n \in \mathbb{N}_0. \tag{2.27}$$

From (2.27) we easily get

$$x_n = h^{-1}\left(\left(\frac{B}{D}\right)^n h(x_0)\right), \tag{2.28}$$

for $n \in \mathbb{N}_0$.

If $A \neq 0$ and $B = 0$, then $D = 0$, from which it follows that $C \neq 0$. Thus,

$$x_{n+1} = h^{-1}\left(\frac{A}{C} h(x_n)\right), \ n \in \mathbb{N}_0. \tag{2.29}$$

From (2.29) we have

$$x_n = h^{-1}\left(\left(\frac{A}{C}\right)^n h(x_0)\right), \tag{2.30}$$

for $n \in \mathbb{N}_0$.

If $D = 0$, so $C \neq 0$. It means $A \neq 0$, $B = 0$. Then, we have equation (2.29). Moreover, equation (2.30) is a solution of equation (2.29). Suppose that $C = 0$, so $D \neq 0$. It means $A = 0$, $B \neq 0$. So, we obtain equation (2.27). In addition, (2.28) is a solution of equation (2.27).

Assume that $ABCD \neq 0$. It means $A = \frac{BC}{D}$. Then, we have equation (2.27). Similarly, it means $B = \frac{AD}{C}$, then we get equation (2.29). □

3. Numerical Applications

Behaviour of solutions to equation (1.5) is mentioned in [10]. But we notice some wrong arguments in [10].

Equation (1.5) can be expressed as

$$x_{n+1} = x_n \frac{\zeta\Psi x_{n-2} + (\zeta\Phi + \Upsilon)\, x_{n-1}}{\Phi x_{n-1} + \Psi x_{n-2}}, \; n \in \mathbb{N}_0. \tag{3.1}$$

Firstly, the authors of [10] studied to obtain the equilibrium point of the equation. Then, using a great deal calculations, they found $\overline{x} = 0$. If

$$(1 - \zeta)(\Phi + \Psi) \neq \Upsilon,$$

an unique equilibrium point of equation (1.5) is $\overline{x} = 0$.

Suppose that an equilibrium point of equation (1.5) is $\overline{x}$. So, we get the following equation

$$\overline{x} = \zeta\overline{x} + \frac{\Upsilon\overline{x}^2}{(\Phi + \Psi)\,\overline{x}}. \tag{3.2}$$

From (3.2), we see that it must be

$$(\Phi + \Psi) \neq 0 \text{ and } \overline{x} \neq 0.$$

This exterminates the probability $\overline{x} = 0$.

Suppose that $\overline{x} \neq 0$. Moreover, equation (3.2) means

$$\overline{x}\left(1 - \zeta - \frac{\Upsilon}{\Phi + \Psi}\right) = 0,$$

so we have

$$1 - \zeta - \frac{\Upsilon}{\Phi + \Psi} = 0. \tag{3.3}$$

From equation (3.3), the equilibrium point of the difference equation is $\overline{x} \neq 0$. It implies that the idea in [10] Theorem 2.1, under the condition, the zero equilibrium point of equation (1.5) is local asymptotic stable is not correct, because it is not an equilibrium point at all.

Moreover, Theorem 3.1 in [10] is expressed as:

Theorem 3.1. *The equilibrium point $\overline{x}$ of equation (1.5) is global attractor if $\Phi(1 - \zeta) \neq \Upsilon$.*

The particular case of equation (1.6) is equation (3.1) with

$$h(x) = x, \; A = \zeta\Phi + \Upsilon, \; B = \zeta\Psi, \; C = \Phi, \; D = \Psi.$$

Example 3.2. Keep in mind the equation (1.5) with

$$\zeta = 2, \; \Upsilon = -3, \; \Phi = 1, \; \Psi = 4,$$

and then, we get the following equation

$$x_{n+1} = x_n \frac{8x_{n-2} - x_{n-1}}{x_{n-1} + 4x_{n-2}}, \; n \in \mathbb{N}_0. \tag{3.4}$$

Equation (3.4) is derived from equation (1.6) with $h(x) = x$ and $x \in \mathbb{R}$,

$$A = -1, \; B = 8, \; C = 1, \; D = 4. \tag{3.5}$$

By using (3.5) in equation (2.8), we get the following characteristic polynomial to the corresponding linear equation in (2.8)

$$p_1(\lambda) = \lambda^2 - 3\lambda - 12,$$

and its roots are

$$\lambda_1 = \frac{3+\sqrt{57}}{2} \text{ and } \lambda_2 = \frac{3-\sqrt{57}}{2}.$$

Then, we obtain

$$\Phi(1-\zeta) - \Upsilon = 2 \neq 0,$$

the restriction $\Phi(1-\zeta) \neq \Upsilon$ in Theorem 3.1 is valid.

By using the parameters A, B, C, D are as in (3.5) and (2.12)-(2.15), where $h(x) = x$ and $x \in \mathbb{R}$, we get

$$x_{2m} = x_{-2} \prod_{i=0}^{m} y_{2i} y_{2i-1}, \tag{3.6}$$

$$x_{2m+1} = x_{-1} \prod_{i=0}^{m} y_{2i+1} y_{2i}, \tag{3.7}$$

for $m \in \mathbb{N}_0$, where

$$\begin{aligned} y_{2m}y_{2m-1} = &\left(\frac{\left(\frac{x_0}{x_{-1}} + 4 - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} + 4 - \lambda_1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} + 4 - \lambda_2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} + 4 - \lambda_1\right)\lambda_2^{m}} - 4 \right) \\ &\times \left(\frac{\left(\frac{x_{-1}}{x_{-2}} + 4 - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} + 4 - \lambda_1\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} + 4 - \lambda_2\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} + 4 - \lambda_1\right)\lambda_2^{m}} - 4 \right), \end{aligned} \tag{3.8}$$

$$\begin{aligned} y_{2m+1}y_{2m} = &\left(\frac{\left(\frac{x_{-1}}{x_{-2}} + 4 - \lambda_2\right)\lambda_1^{m+2} - \left(\frac{x_{-1}}{x_{-2}} + 4 - \lambda_1\right)\lambda_2^{m+2}}{\left(\frac{x_{-1}}{x_{-2}} + 4 - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} + 4 - \lambda_1\right)\lambda_2^{m+1}} - 4 \right) \\ &\times \left(\frac{\left(\frac{x_0}{x_{-1}} + 4 - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} + 4 - \lambda_1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} + 4 - \lambda_2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} + 4 - \lambda_1\right)\lambda_2^{m}} - 4 \right) \end{aligned} \tag{3.9}$$

for $m \in \mathbb{N}_0$.

Note that

$$\begin{aligned} &\lim_{m\to\infty} \left(\frac{\left(\frac{x_0}{x_{-1}} + 4 - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} + 4 - \lambda_1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} + 4 - \lambda_2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} + 4 - \lambda_1\right)\lambda_2^{m}} - 4 \right) \\ &= \lim_{m\to\infty} \left(\frac{\left(\frac{x_{-1}}{x_{-2}} + 4 - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} + 4 - \lambda_1\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} + 4 - \lambda_2\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} + 4 - \lambda_1\right)\lambda_2^{m}} - 4 \right) \\ &= \lambda_1 - 4 = \frac{-5+\sqrt{57}}{2} > 1, \end{aligned}$$

when

$$\frac{x_{-p}}{x_{-(p+1)}} \neq \lambda_2 - 4 = \frac{-5-\sqrt{57}}{2}, \ p = \overline{0,1}. \tag{3.10}$$

By selecting positive initial conditions providing (3.10) and using equations in (3.6)-(3.9), we obtain

$$\lim_{m\to\infty} x_m = \infty.$$

Then, the solutions are not convergent. It is a counterexample to the claim in Theorem 3.1.

Example 3.3. Keep in mind the equation (1.5) with

$$\zeta = \Upsilon = \Phi = \Psi = 1,$$

and then, we get the following equation

$$x_{n+1} = x_n \frac{x_{n-2} + 2x_{n-1}}{x_{n-1} + x_{n-2}}, \ n \in \mathbb{N}_0. \tag{3.11}$$

Equation (3.11) is derived from equation (1.6) with $h(x) = x$ and $x \in \mathbb{R}$,

$$A = 2,\ B = C = D = 1. \tag{3.12}$$

By using (3.12) in equation (2.8), we get the following characteristic polynomial to the corresponding linear equation in (2.8)

$$p_2(\lambda) = \lambda^2 - 3\lambda + 1,$$

and its roots are

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \text{ and } \lambda_2 = \frac{3-\sqrt{5}}{2}.$$

Then, we obtain

$$\Phi(1-\zeta) - \Upsilon = -1 \neq 0,$$

the restriction $\Phi(1-\zeta) \neq \Upsilon$ in Theorem 3.1 is valid.

By using the parameters A, B, C, D are as in (3.12) and (2.12)-(2.15), where $h(x) = x,\ x \in \mathbb{R}$, we have that the relations in (3.6)-(3.8) valid for $m \in \mathbb{N}_0$, where

$$\begin{aligned} y_{2m}y_{2m-1} = &\left(\frac{\left(\frac{x_0}{x_{-1}} + 1 - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} + 1 - \lambda_1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} + 1 - \lambda_2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} + 1 - \lambda_1\right)\lambda_2^{m}} - 1 \right) \\ &\times \left(\frac{\left(\frac{x_{-1}}{x_{-2}} + 1 - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} + 1 - \lambda_1\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} + 1 - \lambda_2\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} + 1 - \lambda_1\right)\lambda_2^{m}} - 1 \right), \end{aligned} \tag{3.13}$$

$$\begin{aligned} y_{2m+1}y_{2m} = &\left(\frac{\left(\frac{x_{-1}}{x_{-2}} + 1 - \lambda_2\right)\lambda_1^{m+2} - \left(\frac{x_{-1}}{x_{-2}} + 1 - \lambda_1\right)\lambda_2^{m+2}}{\left(\frac{x_{-1}}{x_{-2}} + 1 - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} + 1 - \lambda_1\right)\lambda_2^{m+1}} - 1 \right) \\ &\times \left(\frac{\left(\frac{x_0}{x_{-1}} + 1 - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} + 1 - \lambda_1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} + 1 - \lambda_2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} + 1 - \lambda_1\right)\lambda_2^{m}} - 1 \right) \end{aligned} \tag{3.14}$$

for $m \in \mathbb{N}_0$.

Note that

$$\begin{aligned} &\lim_{m\to\infty} \left(\frac{\left(\frac{x_0}{x_{-1}} + 1 - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} + 1 - \lambda_1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} + 1 - \lambda_2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} + 1 - \lambda_1\right)\lambda_2^{m}} - 1 \right) \\ &= \lim_{m\to\infty} \left(\frac{\left(\frac{x_{-1}}{x_{-2}} + 1 - \lambda_2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} + 1 - \lambda_1\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} + 1 - \lambda_2\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} + 1 - \lambda_1\right)\lambda_2^{m}} - 1 \right) \\ &= \lambda_1 - 1 = \frac{1+\sqrt{5}}{2} > 1, \end{aligned}$$

when

$$\frac{x_{-p}}{x_{-(p+1)}} \neq \lambda_2 - 1 = \frac{1-\sqrt{5}}{2},\ p = \overline{0,1}. \tag{3.15}$$

By selecting positive initial conditions providing (3.15) and using equations in (3.13)-(3.14) we obtain

$$\lim_{m\to\infty} x_m = \infty.$$

Since, the solutions are not convergent, which is a counterexample to the claim in Theorem 3.1 in the case $\min\{\zeta, \Upsilon, \Phi, \Psi\} > 0$.

4. Conclusion

In this study, we have solved general non-linear difference equation of third-order in closed form. The solutions are found according to following states of parameters

(1) if $AD \neq BC$,
  (a) if $C \neq 0$, $(A+D)^2 - 4(AD - BC) \neq 0$,
  (b) if $C \neq 0$, $(A+D)^2 - 4(AD - BC) = 0$,
  (c) if $C = 0$, $A = D$,
  (d) if $C = 0$, $A \neq D$,
(2) if $AD = BC$,
  (a) if $A = 0$,
  (b) if $A \neq 0$,
  (c) if $D = 0$,
  (d) if $D \neq 0$,
  (e) if $ABCD \neq 0$.

Moreover, we have given an application.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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