MSU Fen Bil. Dergi., 12:1, 105-109 Araştırma Makalesi/ Research Article
MSU J. of Sci., 12:1, 105-109 DOI: 10.18586/msufbd.1367657

Common Fixed Point Results for w-α-Distance

Gülcan ATICI TURAN[1]🖉, Fatma POLAT[2]
[1] Munzur University, Vocational School of Tunceli, Turkey
[2] Dicle University, Diyarbakır, Turkey

✉: gatici23@hotmail.com [1]0000-0002-1009-6072 [2]0009-0007-6100-2252

Received (Geliş): 28.09.2023 Revision (Düzeltme): 09.12.2023 Accepted (Kabul): 30.01.2024

ABSTRACT

In this study, we examined some fixed point theorems in non-full metric spaces. We define the notions of α-lower semi-continuous , w-α-distance, w_0-α– distance, w-α-rational contraction and generalized w-α-rational contraction mapping. We also give related theorem and example. Then, we prove Banach's fixed-point theorem thanks to the concept w-α- distance in metric spaces equipped with an abritrary binary relation. Also, w-α- rational contraction mapping and generalized w-α-rational contraction mapping are defined and by using these definitions, the theorem related fixed point is expressed and proved.

Anahtar Kelimeler: Binary Relation, Fixed Point, α-Complete Metric Space, w-Distance.

w-α-Uzaklık İçin Ortak Sabit Nokta Sonuçları

ÖZ

Bu çalışmada tam metrik olmayan uzaylarda bazı sabit nokta teoremleri incelenmiştir. α-alttan yarı-sürekli, w-α-uzaklık, w_0-α– uzaklık, w-α-rasyonel büzülme ve genelleştirilmiş w-α-rasyonel büzülme dönüşümü kavramları tanımlanmıştır. İlgili teorem ve örneği de verilmiştir. Daha sonra w-α- uzaklık kavramını kullanarak keyfi bir ikili bağıntı ile verilen metrik uzaylarda Banach sabit nokta teoremi ispatlanmıştır. Ayrıca w-α- rasyonel büzülme dönüşümü ve genelleştirilmiş w-α- rasyonel büzülme dönüşümü tanımları yapılmış ve bu tanımlar kullanılarak sabit nokta ile ilgili teorem ifade ve ispat edilmiştir.

Anahtar Kelimeler: İkili Bağıntı, Sabit Nokta, α-Geçişli Dönüşümü, α-Tam Metrik Uzay, w-Uzaklık.

INTRODUCTION

Kada et al [1] presented the idea of w-distance within a metric space. Considering (X,d) as a metric space, a function $\omega: X\times X\to[0,\infty)$ earns the designation of a w-distance on X when it meets these specified conditions for each $x,y,z\in X$,

(w1) $\omega(x,z)\le\omega(x,y)+\omega(y,z)$;

(w2) a function $\omega(x,.): X\to[0,\infty)$ exhibits lower semicontinuous;

(w3) for any $\varepsilon>0$, there exists $\delta>0$ such that $\omega(z,x)\le\delta$ and $\omega(z,y)\le\delta$ imply $d(x,y)\le\varepsilon$ [1].

Later on, they have achieved significant results using this definition in fixed point theory. In 2012, Samet et al [2] defined α-admissible mapping. On the other hand they have expressed and proved the theorems related to fixed point in complete metric spaces.

Hussain et al [3] have obtained fixed point results for rational contraction mapping in α-η-complete metric space. Kutbi and Sintunavarat [4] defined generalized w_α-multivalued contraction mapping and then they have proven fixed point theorems using this mapping in α -complete metric spaces. Many studies have been carried out on fixed points [5, 6, 7, 8].

Definition 1. Consider (X,d) as a metric space, and let $T: X\to Cl(X)$ represent a multivalued mapping. A point $x\in X$ is termed a fixed point of T if $x\in Tx$, and the collection of fixed points of T is symbolized as $F(T)$ [9].

Definition 2. Consider (X,d) as a metric space, and let $T: X\to Cl(X)$ represents a multivalued mapping. T is termed a contraction if there exists a constant $\lambda\in(0,1)$ such that, for every x and y in X, $H(Tx,Ty)\le\lambda d(x,y)$ [9].

Definition 3. Suppose (X,d) represents a metric space, and $\alpha: X\times X\to[0,\infty)$ is a specified mapping. The multivalued mapping $T: X\to Cl(X)$ is termed a w_α-contraction if there exists a w_α-distance $\omega: X\times X\to[0,\infty)$ on X and a value $\lambda\in(0,1)$. This condition ensures that for any $x,y\in X$ and $u\in Tx$, there exists $v\in Ty$ such that

$$\alpha(u,v)\omega(u,v)\le\lambda\omega(x,y)\ [4].$$

Definition 4. In the context of (X,d) being a metric space and $\alpha: X\times X\to[0,\infty)$ a specified mapping, the multivalued mapping $T: X\to Cl(X)$ is referred to as a generalized w_α-contraction if there exists a w_0-distance ω on X and a value $\lambda \in (0,1)$. This condition ensures that for any $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that

$$\alpha(u,v)\omega(u,v) \leq \lambda max\{\omega(x,y), \omega(x,Tx), \omega(y,Ty), \frac{1}{2}[\omega(x,Ty) + \omega(y,Tx)]\ [4].$$

MAİN RESULTS

Definition 5. Let (X,d) be a metric space and $\alpha: X \times X \to [0,\infty)$. A function $f: X \to \mathbb{R} \cup \{-\infty,\infty\}$ is said to be α-lower semi-continuous at point x if for all sequence (x_n) which converges to $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \inf f(x_n) \geq f(x).$$

Definition 6. Let (X,d) be a metric space and $\alpha: X \times X \to [0,\infty)$. A function $\omega: X \times X \to [0,\infty)$ is said to be a w-α-distance on X if

(i) $\omega(x,z) \leq \omega(x,y) + \omega(y,z)$ for any $x, y, z \in X$,

(ii) For any $x \in X$, $\omega(x,.): X \to [0,\infty)$ is α-lower semi-continuous,

(iii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\omega(z,x) \leq \delta$ and $\omega(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$.

Definition 7. Let (X,d) be a metric space. The w-α-distance $\omega: X \times X \to [0,\infty)$ on X is said to be a w_0-α–distance if $\omega(x,x) = 0$ for all $x \in X$.

Example 8. Let $X = [0,\infty)$. Define $T: X \to X$ and $\alpha: X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & x,y \in \left[0,\frac{1}{2}\right) \\ 0, & otherwise, \end{cases}$$

$$Tx = \begin{cases} \frac{3}{2}, & x \in \left[0,\frac{1}{2}\right) \\ \frac{4}{5}, & x = \frac{1}{2} \\ x, & x > \frac{1}{2}\ . \end{cases}$$

Clearly, T is neither α- continuous nor lower semi-continuous. However, it is α-lower semi-continuous.
In fact, let (x_n) be a sequence that not fixed convergent at point $x = \frac{1}{2}$. If $x_n \to \frac{1}{2}^-$, then $Tx_n = \frac{3}{2}$ for all $n \in \mathbb{N}$. If $x_n \to \frac{1}{2}^+$, then $Tx_n = x_n$ for all $n \in \mathbb{N}$ and so $\lim_{n\to\infty} Tx_n = \frac{1}{2}$. Therefore, it is not $\lim_{n\to\infty} inf\, Tx_n \geq T\frac{1}{2}$. Hence, T is not lower semi-continuous at point $x = \frac{1}{2}$. Now, let (x_n) be a sequence not fixed such that $\alpha(x_n, x_{n+1}) \geq 1$ and convergent at point $x = \frac{1}{2}$. Then $Tx_n = \frac{3}{2}$ where $(x_n) \subseteq \left[0,\frac{1}{2}\right)$. However, T is not α-continuous at point $\frac{1}{2}$ due to $Tx_n \to \frac{3}{2} \neq T\frac{1}{2} = \frac{4}{5}$. Also, T is α-lower semi-continuous at point $x = \frac{1}{2}$. Thus, $\frac{3}{2} = \lim_{n\to\infty} inf\, Tx_n \geq T\frac{1}{2} = \frac{4}{5}$.

Lemma 9. Consider (X,d) as a metric space, where $\alpha: X \times X \to [0,\infty)$ and $\omega: X \times X \to [0,\infty)$ are w-α-distances on X. Suppose (x_n) and (y_n) are sequences in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $(y_n, y_{n+1}) \geq 1$, respectively, with $x, y, z \in X$. Let (u_n) and (v_n) be sequences of positive real numbers approaching 0. Under these conditions, the following statements hold true:

(i) If $\omega(x_n, y) \leq u_n$ and $\omega(x_n, z) \leq v_n$ for all $n \in \mathbb{N}$, then $y = z$. Moreover, if $\omega(x,y) = 0$ and $\omega(x,z) = 0$, then $y = z$.

(ii) If $\omega(x_n, y_n) \leq u_n$ and $\omega(x_n, z) \leq v_n$ for all $n \in \mathbb{N}$, then $y_n \to z$.

(iii) If $\omega(x_n, x_m) \leq u_n$ for all $n, m \in \mathbb{N}$ such that $m > n$, then (x_n) be a Cauchy sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ in X.

(iv) If $\omega(x_n, y) \leq u_n$ for all $n \in \mathbb{N}$, then (x_n) be a Cauchy sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ in X.

Definition 10. Let (X,d) be a metric space and $\alpha: X \times X \to [0,\infty)$ and $T: X \to Cl(X)$ be given two mappings. T is said to be generalized multivalued w-α-rational contraction mapping if there exist $\lambda \in (0,1)$ and a w_0-α-distance $\omega: X \times X \to [0,\infty)$ on X such that for all $x, y \in X$ and $u \in Tx$ there is a $v \in Ty$ with

$$\alpha(u,v)\omega(u,v) \leq \lambda max\left\{\omega(x,y), \frac{\omega(x,Tx)}{1+\omega(x,Tx)}, \frac{\omega(y,Ty)}{1+\omega(y,Ty)}, \frac{1}{2}[\omega(x,Ty) + \omega(y,Tx)]\right\}.$$

Definition 11. Let (X,d) be a metric space, $\omega: X \times X \to [0,\infty)$ be a w_0-α-distance on X and $T: X \to Cl(X)$. Let

$$M(x,y) = max\left\{\omega(x,y), \frac{\omega(x,Tx)}{1+\omega(x,Tx)}, \frac{\omega(y,Ty)}{1+\omega(y,Ty)}, \frac{\omega(x,Ty) + \omega(y,Tx)}{2}\right\}.$$

Then T is said to be a multivalued w-α-rational contraction mapping if

$\alpha(x,y) \geq 1 \Rightarrow \omega(Tx,Ty) \leq \lambda M(x,y)$ for all $x, y \in X$ where $\lambda \in (0,1)$.

Theorem 12. Let (X,d) be a metric space and $\alpha: X \times X \to [0,\infty)$ be a function. Let $T: X \to Cl(X)$ be a generalized multivalued w-α-rational contraction mapping. Suppose that the following statements are indeed accurate:

(i) There exists $Y \subseteq X$ with $T(X) \subseteq Y$such that (Y,d) is α-complete;

(ii) T is a α-admissible mapping;

(iii) There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

(iv) Either T is α- continuous or

(iv') (x_n) sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in X$ for all $n \in \mathbb{N}$ has a (x_{n_k}) subsequence such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$;

Then $F(T) \neq \emptyset$.

Proof. There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$ from (ii). Since T is a generalized w-α-

rational contraction mapping, we obtain $x_2 \in Tx_1$ such that

$$\alpha(x_1,x_2)\omega(x_1,x_2) \leq \lambda max\left\{\omega(x_0,x_1), \frac{\omega(x_0,Tx_0)}{1+\omega(x_0,Tx_0)}, \frac{\omega(x_1,Tx_1)}{1+\omega(x_1,Tx_1)}, \frac{1}{2}[\omega(x_0,Tx_1)+\omega(x_1,Tx_0)]\right\} \quad (2.1)$$

Since T is a α-admissible mapping and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$, we have

$$\alpha(x_1,x_2) \geq 1. \quad (2.2)$$

Then by (2.1) and (2.2) we get

$$\omega(x_1,x_2) \leq \alpha(x_1,x_2)\omega(x_1,x_2) \leq \lambda max\left\{\omega(x_0,x_1), \frac{\omega(x_0,Tx_0)}{1+\omega(x_0,Tx_0)}, \frac{\omega(x_1,Tx_1)}{1+\omega(x_1,Tx_1)}, \frac{1}{2}[\omega(x_0,Tx_1)+\omega(x_1,Tx_0)]\right\}.$$

Again, since T is a generalized w-α-rational contraction, there exists $x_3 \in Tx_2$ such that

$$\alpha(x_2,x_3)\omega(x_2,x_3) \leq \lambda max\left\{\omega(x_1,x_2), \frac{\omega(x_1,Tx_1)}{1+\omega(x_1,Tx_1)}, \frac{\omega(x_2,Tx_2)}{1+\omega(x_2,Tx_2)}, \frac{1}{2}[\omega(x_1,Tx_2)+\omega(x_2,Tx_1)]\right\} \quad (2.3)$$

Since $\alpha(x_1,x_2) \geq 1$ and T be a α- admissible mapping, we have

$$\alpha(x_2,x_3) \geq 1. \quad (2.4)$$

Then we get

$$\omega(x_2,x_3) \leq \alpha(x_2,x_3)\omega(x_2,x_3) \leq \lambda max\left\{\omega(x_1,x_2), \frac{\omega(x_1,Tx_1)}{1+\omega(x_1,Tx_1)}, \frac{\omega(x_2,Tx_2)}{1+\omega(x_2,Tx_2)}, \frac{1}{2}[\omega(x_1,Tx_2)+\omega(x_2,Tx_1)]\right\}$$

by (2.3) and (2.4). Continuing this process, we get $x_n \in Tx_{n-1}$,

$$\alpha(x_n,x_{n+1}) \geq 1 \quad (2.5)$$

and

$$\omega(x_n,x_{n+1}) \leq \lambda max\left\{\omega(x_{n-1},x_n), \frac{\omega(x_{n-1},Tx_{n-1})}{1+\omega(x_{n-1},Tx_{n-1})}, \frac{\omega(x_n,Tx_n)}{1+\omega(x_n,Tx_n)}, \frac{1}{2}[\omega(x_{n-1},Tx_n)+\omega(x_n,Tx_{n-1})]\right\}$$

for all $n \in \mathbb{N}$. Now, we obtain

$$\begin{aligned}\omega(x_n,x_{n+1}) &\leq \lambda max\left\{\omega(x_{n-1},x_n), \frac{\omega(x_{n-1},Tx_{n-1})}{1+\omega(x_{n-1},Tx_{n-1})}, \frac{\omega(x_n,Tx_n)}{1+\omega(x_n,Tx_n)}, \frac{1}{2}[\omega(x_{n-1},Tx_n)+\omega(x_n,Tx_{n-1})]\right\}\\
&= \lambda max\left\{\omega(x_{n-1},x_n), \frac{\omega(x_{n-1},x_n)}{1+\omega(x_{n-1},x_n)}, \frac{\omega(x_n,x_{n+1})}{1+\omega(x_n,x_{n+1})}, \frac{1}{2}[\omega(x_{n-1},x_{n+1})+\omega(x_n,x_n)]\right.\\
&\leq \lambda max\left\{\omega(x_{n-1},x_n), \omega(x_n,x_{n+1}), \frac{1}{2}[\omega(x_{n-1},x_{n+1})]\right\}\\
&\leq \lambda max\left\{\omega(x_{n-1},x_n), \omega(x_n,x_{n+1}), \frac{1}{2}[\omega(x_{n-1},x_n)+\omega(x_n,x_{n+1})]\right\}.\\
&\leq \lambda max\{\omega(x_{n-1},x_n), \omega(x_n,x_{n+1})\} \quad (2.6)\end{aligned}$$

for all $n \in \mathbb{N}$. In that case we get $\omega(x_n,x_{n+1}) \leq \lambda max\{\omega(x_{n-1},x_n), \omega(x_n,x_{n+1})\}$. If $max\{\omega(x_{k-1},x_k), \omega(x_k,x_{k+1})\} = \omega(x_k,x_{k+1})$ for some $k \in \mathbb{N}$, then $\omega(x_k,x_{k+1}) = 0$ and so we have $\omega(x_{k-1},x_k) = 0$. We get

$$\omega(x_{k-1},x_{k+1}) \leq \omega(x_{k-1},x_k) + \omega(x_k,x_{k+1}) = 0$$

from the property of w-α-distance.

Since $\omega(x_{k-1},x_k) = 0$ and $\omega(x_{k-1},x_{k+1}) = 0$, then we get $x_k = x_{k+1}$ using Lemma 9. This is $x_k \in Tx_k$ and so it means that x_k is a fixed point of T. Now, let's consider the assumption that

$$max\{\omega(x_{n-1},x_n), \omega(x_n,x_{n+1})\} = \omega(x_{n-1},x_n)$$

for all $n \in \mathbb{N}$. We get

$$\omega(x_n,x_{n+1}) \leq \lambda\omega(x_{n-1},x_n) \quad (2.7)$$

for all $n \in \mathbb{N}$ from (2.6). By induction, we have

$$\begin{aligned}\omega(x_n,x_{n+1}) &\leq \lambda\omega(x_{n-1},x_n)\\
&\leq \lambda^2\omega(x_{n-2},x_{n-1})\\
&\vdots\\
&\leq \lambda^n\omega(x_0,x_1)\end{aligned}$$

for all $n \in \mathbb{N}$.

Let $m > n$ for all $n,m \in \mathbb{N}$. Then we have

$$\begin{aligned}\omega(x_n,x_m) &\leq \omega(x_n,x_{n+1}) + \omega(x_{n+1},x_{n+2}) + \cdots + \omega(x_{m-1},x_m)\\
&\leq \lambda^n\omega(x_0,x_1) + \lambda^{n+1}\omega(x_0,x_1) + \cdots + \lambda^{m-1}\omega(x_0,x_1)\\
&\leq \frac{\lambda^n}{1-\lambda}\omega(x_0,x_1).\end{aligned}$$

Since $0 < \lambda < 1$, then we get $\frac{\lambda^n}{1-\lambda}\omega(x_0,x_1) \to 0$ as $n \to \infty$. It is found that (x_n) is a Cauchy sequence in Y satisfying $\alpha(x_n,x_{n+1}) \geq 1$ from Lemma 9 We know that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ from (2.5). Since (Y,d) is α-complete, then we obtain $x_n \to z$ as $n \to \infty$ for some $z \in Y$. We now show that z is a fixed point of T. First, we consider that T is α-continuous. Then we obtain

$$d(z,Tz) = \lim_{n\to\infty} d(x_{n+1},Tz) = \lim_{n\to\infty} d(Tx_n,Tz) = d(Tz,Tz) = 0$$

Here, z is a fixed point of T.

Now, let's consider the existence of (iv'). So there exist a subsequence (x_{n_k}) of (x_n) such that $\alpha(x_{n_k},z) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$. In this case, we write

$$\omega(x_{n_k+1},z) \leq \lim_{k\to\infty} inf\, \omega(x_{n_k+1},x_{n_k+m}) \leq \lim_{k\to\infty} inf \frac{\lambda^{n_k-1}}{1-\lambda}\omega(x_0,x_1) = 0 \quad (2.8)$$

using w-α-distance lower semi-continuous from inequality $\omega(x_n,x_m) \leq \frac{\lambda^n}{1-\lambda}\omega(x_0,x_1)$. Also, since T be generalized w-α-rational contraction mapping and $\alpha(x_{n_k},z) \geq 1$, we have

$$\begin{aligned}\omega(x_{n_k+1},Tz) &= \omega(Tx_{n_k},Tz)\\
&\leq \lambda max\left\{\omega(x_{n_k},z), \frac{\omega(x_{n_k},x_{n_k+1})}{1+\omega(x_{n_k},x_{n_k+1})}, \frac{\omega(z,Tz)}{1+\omega(z,Tz)}, \frac{1}{2}[\omega(x_{n_k},Tz)+\omega(z,x_{n_k+1})]\right\}\\
&\leq \lambda max\{\omega(x_{n_k},z), \omega(x_{n_k},x_{n_k+1}), \omega(z,Tz), \frac{1}{2}[\omega(x_{n_k},Tz)+\omega(z,x_{n_k+1})]\right\}\end{aligned}$$

$\leq \lambda max\{\omega(x_{n_k}, z), \omega(x_{n_k}, x_{n_k+1}), \omega(z, x_{n_k+1}) + \omega(x_{n_k+1}, Tz)\} \leq \omega(x_{n_k+1}, Tz)\}.$

$$\leq \lambda max\left\{\lim_{k\to\infty} inf \frac{\lambda^{n_k}}{1-\lambda}\omega(x_0, x_1), \lim_{k\to\infty} inf \lambda^{n_k}\omega(x_0, x_1), \lim_{k\to\infty} inf \frac{\lambda^{n_k}}{1-\lambda}\omega(x_0, x_1) + \omega(x_{n_k+1}, Tz)\right\}$$

If $\omega(x_{n_k+1}, Tz) > 0$, then

$$\omega(x_{n_k+1}, Tz) \leq \lambda\omega(x_{n_k+1}, Tz)$$

which is a contradiction. Hence, we have

$$\omega(x_{n_k+1}, Tz) = 0. \quad (2.9)$$

If (2.8) and (2.9) are combined, then we obtain $z = Tz$ from Lemma 9.

Teorem 13. In a metric space (X, d), considering the mapping $\alpha: X \times X \to [0, \infty)$ and $T: X \to Cl(X)$ as a multi-valued w-α-rational contraction mapping, assuming the validity of the following statements:
(i) $Y \subseteq X$ with $T(X) \subseteq Y$ such that (Y, d) is α-complete;
(ii) T is a α-admissible mapping;
(iii) There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(iv) Either T is α- continuous or
(iv') (x_n) sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in X$ for all $n \in \mathbb{N}$ has a (x_{n_k}) subsequence such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$;
Then $F(T) \neq \emptyset$.
Proof. The proof shares resemblance with the one in Theorem 12.

Result 14. Suppose (X, d) represents a metric space equipped with w-$\mathcal{R}$-distance, and $\mathcal{R}$ is any arbitrary binary relation on X. If $T: X \to Cl(X)$ fulfills these conditions, then it implies $F(T)$ is non-empty.
(i) There exists $Y \subseteq X$ with $T(X) \subseteq Y$, such that (Y, d) is $\mathcal{R}$-complete;
(ii) $X(T, \mathcal{R}) \neq \emptyset$ and $\mathcal{R}$ is T-closed;
(iii) Either T is $\mathcal{R}$-continuous
or
(iii') (x_n) such that $(x_n, x_{n+1}) \in \mathcal{R}$ and $x_n \to x \in X$ for all $n \in \mathbb{N}$ has a subsequence (x_{n_k}) such that $(x_{n_k}, x) \in \mathcal{R}$ for all $k \in \mathbb{N} \cup \{0\}$.
(iv) There exists a $\lambda \in [0,1)$ for all $x, y \in X$ such that $x, y \in \mathcal{R}$, then $\omega(Tx, Ty) \leq \lambda M(x, y)$.
There exists

$$M(x, y) = max\left\{\omega(x, y), \frac{\omega(x, Tx)}{1 + \omega(x, Tx)}, \frac{\omega(y, Ty)}{1 + \omega(y, Ty)}, \frac{\omega(x, Ty) + \omega(y, Tx)}{2}\right\}.$$

Proof. Let

$$\alpha(x, y) = \begin{cases} 1, & (x, y) \in \mathcal{R} \\ 0, & otherwise \end{cases}.$$

If there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$, then since $X(T, \mathcal{R}) \neq \emptyset$, there exists a point $x_0 \in X(T, \mathcal{R})$ such that $(x_0, Tx_0) \in \mathcal{R}$. Since $(x_0, x_1) \in \mathcal{R}$ and $\mathcal{R}$ is T-closed, there exists a $x_2 \in Tx_1$ such that $(x_1, x_2) \in \mathcal{R}$. $\alpha(x_1, x_2) \geq 1$ due to the definition of α.

Continuing this process, we get $\alpha(x_n, x_{n+1}) \geq 1$ such that $x_n = Tx_{n-1}$. That is, T is a α-admissible. Since the definition of α and (Y, d) is $\mathcal{R}$-complete, then (Y, d) is α-complete. (iii) and (iii') conditions requires (iv) and (iv') hypotheses of Theorem 12. Now let $\alpha(x, y) \geq 1$. Then $(x, y) \in \mathcal{R}$. Because of the hypothesis (iv) there exists a $\lambda \in [0,1)$ such that
$\omega(Tx, Ty) \leq \lambda M(x, y)$.
Therefore, since it is provide all conditions of Theorem 12, then T has a fixed point. Also, w-$\mathcal{R}$-distance requires w-α-distance.

Result 15. Suppose (X, d) represents a metric space, $\alpha: X \times X \to [0, \infty)$ is a mapping, and $T: X \to Cl(X)$ is a multi-valued w-α -rational contraction mapping, given that the following conditions are satisfied:
(i) T is a α-contraction mapping;
(ii) There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(iii) Either T is α-continuous or (x_n) sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ ve $x_n \to x \in X$ for all $n \in \mathbb{N}$ has a (x_{n_k}) subsequence such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$;
Then $F(T) \neq \emptyset$.
Proof. As (X, d) constitutes a complete metric space, ensuring α-complete, the intended outcome is achieved by employing the proof outlined in Theorem 12.

Example 16. Let $X = (-1, \infty)$ and $d: X \times X \to [0, \infty)$ with the metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $\alpha: X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} x^2 + y^2, & x, y \in [0,1] \\ 0, & otherwise. \end{cases}$$

$T: X \to Cl(X)$ multivalued mapping define by

$$Tx = \begin{cases} \left\{\frac{1}{4}x^2\right\}, & x \in [0,1] \\ \{|x|, |x + 2|\}, & otherwise. \end{cases}$$

Now, we show that this is T a multivalued w-α-rational contraction mapping with $\lambda = \frac{1}{2}$ and w-α-distance $\omega: X \times X \to [0, \infty)$, defined as $\omega(x, y) = \max\{|x|, |y|\}$ for all $x, y \in X$. Let $u \in Tx = \left\{\frac{1}{4}x^2\right\}$ for $x, y \in [0,1]$. That is, we can found in a $v = \frac{1}{4}y^2 \in Ty$ such that $u = \frac{1}{4}x^2$ and

$\alpha(u, v)\omega(u, v) = \alpha\left(\frac{x^2}{4}, \frac{y^2}{4}\right)\omega\left(\frac{x^2}{4}, \frac{y^2}{4}\right)$
$= \left(\frac{x^4}{16} + \frac{y^4}{16}\right)\left(\frac{1}{4}\max\{x^2, y^2\}\right)$
$\leq (1 + 1)\frac{1}{4}\max\{x^2, y^2\}$
$\leq \frac{1}{2}\max\{|x|, |y|\}$
$= \lambda\omega(x, y)$
$\leq \lambda M(x, y)$.

That is, $\alpha(u, v)\omega(u, v) \leq \lambda M(x, y)$. Therefore T multivalued w-α-rational contraction mapping.
While (Y, d) may not qualify as a complete metric space, it does fulfill the criteria for being an α-complete metric space. Consider (x_n) as a Cauchy sequence within Y, with $\alpha(x_n, x_{n+1}) \geq 1$ for all n in the natural numbers.

Consequently, $x_n \in [0,1]$ for all $n \in \mathbb{N}$. Given that $([0,1], d)$ stands as a complete metric space, there exists $z \in [0,1]$ such that $x_n \to z$ as $n \to \infty$. Therefore, (Y, d) qualifies as an α-complete metric space.
If $\alpha(x,y) \geq 1$, it implies that $x, y \in [0,1]$. Concurrently, $Tc \in [0,1]$ for all $c \in [0,1]$. Consequently, $\alpha(Tx,Ty) \geq 1$, signifying that T qualifies as an α-admissible mapping. There exists $x_0 = 1$ such that $x_1 = \frac{1}{4} \in T1$ and $\alpha(x_0, x_1) = \alpha\left(1, \frac{1}{4}\right) \geq 1$.
$x_n \to x$ as $n \to \infty$ and (x_n) sequence provide $\alpha(x_n, x_{n+1}) \geq 1$ inequality for all $n \in \mathbb{N}$. Hence, $(x_n) \subseteq [0,1]$ for all $n \in \mathbb{N}$ and so $(Tx_n) \subseteq [0,1]$. Since T is continuous on $[0,1]$, then $Tx_n \to Tx$ as $n \to \infty$.
This implies that T is a mapping that maintains α-continuity.
Alternatively, let $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to z \in X$. In this case, there exists a subset (x_{n_k}) such that $x_n \in [0,1]$ and $x_{n_k} \to z$. Thus, $\alpha(x_{n_k}, z) \geq 1$.

REFERENCES

[1] Kada, O., Suzuki, T., Takahashi, W. Nonconvex minimization theorems and fixed point theorems in complete metriz spaces, Mathematica Japonica. 44:2 381-391, 1996.
[2] Samet, B., Vetro, C., Vetro, P. Fixed point theorems α-ψ -conractive type mappings, Nonlinear Analysis: Theory, Methods and Applications. 75:4 2154- 2165, 2012.
[3] Hussain, N., Kutbi, M. A., Salimi, P. Fixed point theory in α-complete metric spaces with applications, Abstract and Applied Analysis. 1:2 1-11, 2014.
[4] Kutbi, M. A., Sintunavarat, W. The existence of fixed point theorems via -distance and α-admissible mappings and applications, Abstract and Applied Analysis. 141 1-8, 2013.
[5] Vetro, C. A Fixed-Point Problem with Mixed-Type Contractive Condition, Constructıve Mathematıcal Analysıs. 3:1, 45-52, 2020.
[6] Karapinar, E. A Short Survey on the Recent Fixed Point Results on b-Metric Spaces, Constructıve Mathematıcal Analysıs. 1:1 15-44, 2018.
[7] Nazam, M., Acar, Ö. Fixed points of (α,ψ)-contractions in Hausdorff partial metric spaces, Mathematical Methods in the Applied Sciences. 42:16 5159-5173, 2019.
[8] Mınak, G., Acar, Ö.,Altun, İ.. Multivalued Pseudo-Picard Operators and Fixed Point Results, Journal of Function spaces and applications. 2013, 2013.
[9] Kutbi, M. A., Sintunavarat, W. Fixed point theorems for generalized w_α- contraction multivalued mappings in α-complete metric spaces, Fixed Point Theory and Applications. 139 1-9, 2014.