Anadolu Üniversitesi Bilim ve Teknoloji Dergisi B- Teorik Bilimler
Anadolu University Journal of Science and Technology B- Theoritical Sciences

2017 - Volume: 5 Number: 2
Page: 111 - 119
DOI: 10.20290/aubtdb.289203
Received: 01 February 2017 Revised: 02 June 2017 Accepted: 07 June 2017

QUINTIC B-SPLINE GALERKIN METHOD FOR THE KdV EQUATION

Dursun IRK *

Department of Mathematics-Computer, Faculty of Art and Science, Eskişehir Osmangazi University, Eskişehir, Türkiye

ABSTRACT

The one-dimensional nonlinear KdV equation is solved numerically based on Crank Nicolson discretization and Galerkin finite element method using quintic B-spline basis function. Two classical test problems, including a single soliton propagation and interaction of two solitons, are used to validate the proposed method. Finally, we can see that the proposed numerical method is a useful approach for numerical solution of KdV equation.

Keywords: Soliton, B-spline, Galerkin Finite Element Method

KdV DENKLEMİ İÇİN KUİNTİK B-SPLİNE GALERKİN METODU

ÖZET

Bir boyutlu lineer olmayan KdV denklemi, Crank Nicolson parçalanması ile birlikte kuintik B-spline taban fonksiyonlarının kullanıldığı Galerkin sonlu elemanlar metoduyla yaklaşık olarak çözülmüştür. Bir solitonun yayılması ve iki solitonun çarpışmasını içeren iki klasik test problemi kullanılarak önerilen yöntemin doğruluğu kontrol edilmiştir. Sonuç olarak önerilen yaklaşık yöntemin KdV denkleminin sayısal çözümü için faydalı bir yöntem olduğu görülmüştür.

Anahtar Kelimeler: Soliton, B-spline, Galerkin Sonlu Elemanlar Metodu

1. INTRODUCTION

We consider the following nonlinear PDE known as the KdV equation given by

$$u_t + \varepsilon u u_x + \mu u_{xxx} = 0 \tag{1}$$

where ε and μ are convection and dispersion parameters. Appropriate boundary conditions over the space interval $[a,b]$ will be taken from

$$\begin{aligned} &u(a,t)=0, \quad u(b,t)=0, \\ &u_x(a,t)=0, \quad u_x(b,t)=0, \quad t\geq 0 \\ &u_{xx}(a,t)=0, \quad u_{xx}(b,t)=0 \end{aligned} \tag{2}$$

to model the analytical conditions that $u\to\infty$ as $x\to\pm\infty$ and we will consider the initial condition as

$$u(x,0)=f(x). \tag{3}$$

Eq. (1) is a fundamental mathematical model for describing the theory of water waves in shallow channels [1]. These water waves are known as solitons, which are stable and do not disperse with time

*Corresponding Author: dirk@ogu.edu.tr

and they are not deformed after collision with other solitons. The terms uu_x and u_{xxx} in the KdV Eq. (1) represent the nonlinear convection and the dispersion effects, respectively and soliton waves are generated as a result of the balance between these terms.

The KdV equation was first studied numerically by Zabusky and Kruskal using leapfrog finite difference method [2]. Various numerical methods, including finite element, differential quadrature, spline approximation, have also been used for the KdV equation by many researcher [3-8] so far. In this study, a numerical solution of the one-dimensional nonlinear KdV equation is presented using Galerkin B-spline approach based on quintic B-spline approximation. Galerkin method based on different degrees B-spline basis functions have been widely proposed to obtain numerical solutions of the nonlinear partial differential equations so far. The RLW and time split RLW equation were solved numerically by using the quintic B-spline Galerkin finite element method [9]. A numerical solution of one dimensional heat equation was presented using quadratic B-spline Galerkin method [10]. Kutluay and Uçar proposed a quadratic B-spline Galerkin finite element approach for numerical solution of the one-dimensional coupled KdV equation [11]. The Galerkin method for MRLW equation was studied using linear finite elements for space discretization and the Crank-Nicolson and the Runge-Kutta scheme for time discretization [12]. The quadratic B-spline Galerkin method was proposed to obtain the numerical solutions of the Improved Boussinesq type equation by Uçar et. al. [13]. Karakoç and Zeybek obtained the numerical solution of GEW equation by using cubic B-spline Galerkin approach [14].

In the proposed method, quintic B-splines as both interpolation and weight functions in space and the Crank-Nicolson approximation in time are set up. Finally, the method is validated by comparing the present results with earlier numerical results.

2. APPLICATION OF THE METHOD

Given an uniform grid with the time step Δt and space step h, the exact solution of the KdV equation at the grid point is denoted by

$$u(x_m,t_n)=u_m^n, \quad m=0,1,\ldots,N \quad n=0,1,2,\ldots$$

and the notation U_m^n is also used to show the numerical value of u_m^n.

Using Crank-Nicolson method [15], time discretization form of the KdV equation is obtained as

$$u^{n+1}+\frac{\Delta t}{2}\varepsilon u^{n+1}u_x^{n+1}+\frac{\Delta t}{2}\mu\left(u_{xxx}\right)^{n+1}=u^n-\frac{\Delta t}{2}\varepsilon u^n u_x^n-\frac{\Delta t}{2}\mu\left(u_{xxx}\right)^n. \tag{4}$$

The space domain $[a,b]$ is discretized into N subintervals of equal length h as

$$a=x_0<x_1<\ldots<x_{N-1}<x_N=b.$$

The quintic B-splines basis functions ϕ_l, $\left(l=-2,\ldots,N+2\right)$ are defined over the above uniform mesh as

$$\phi_l(x)=\frac{1}{h^5}\begin{cases} g^5(x_{l-3}), & x_{l-3}\le x<x_{l-2},\\ g^5(x_{l-3})-6g^5(x_{l-2}), & x_{l-2}\le x<x_{l-1},\\ g^5(x_{l-3})-6g^5(x_{l-2})+15g^5(x_{l-1}), & x_{l-1}\le x<x_l,\\ g^5(x_{l-3})-6g^5(x_{l-2})+15g^5(x_{l-1}) & \\ -20g^5(x_l), & x_l\le x<x_{l+1},\\ g^5(x_{l-3})-6g^5(x_{l-2})+15g^5(x_{l-1}) & \\ -20g^5(x_l)+15g^5(x_{l+1}), & x_{l+1}\le x<x_{l+2},\\ g^5(x_{l-3})-6g^5(x_{l-2})+15g^5(x_{l-1}) & \\ -20g^5(x_l)+15g^5(x_{l+1})-6g^5(x_{l+2}), & x_{l+2}\le x<x_{l+3},\\ 0, & \text{otherwise.}\end{cases} \tag{5}$$

where $g(l)=x-x_l$. The set of quintic B-splines ϕ_l forms a basis for functions defined over the space interval $a\le x\le b$ [16]. Each quintic B-spline ϕ_l covers 6 elements so that each element $[x_l,x_{l+1}]$ is covered by six splines. Then an approximation $U(x,t)$ in terms of quintic B-splines can be written over the element $[x_l,x_{l+1}]$ as

$$U(x,t)=\sum_{j=l-2}^{l+3}\phi_j(x)\delta_j(t), \tag{6}$$

where δ_j are time dependent unknown parameters which will be determined from the quintic B-spline Galerkin form of the KdV equation. Since the quintic B-spline functions (5) and its first four derivatives are continuous, trial solutions (6) have continuity up to fourth order. Then using quintic B-splines (5) in (6), approximation U and its first four derivatives at the knots are calculated as

$$\begin{aligned} U_l &= U(x_l)=\delta_{l-2}+26\delta_{l-1}+66\delta_l+26\delta_{l+1}+\delta_{l+2},\\ U_l' &= U'(x_l)=\tfrac{5}{h}\left(-\delta_{l-2}-10\delta_{l-1}+10\delta_{l+1}+\delta_{l+2}\right),\\ U_l'' &= U''(x_l)=\tfrac{20}{h^2}\left(\delta_{l-2}+2\delta_{l-1}-6\delta_l+2\delta_{l+1}+\delta_{l+2}\right),\\ U_l''' &= U'''(x_l)=\tfrac{60}{h^3}\left(-\delta_{l-2}+2\delta_{l-1}-2\delta_{l+1}+\delta_{l+2}\right),\\ U_l^{(4)} &= U^{(4)}(x_l)=\tfrac{120}{h^4}\left(\delta_{l-2}-4\delta_{l-1}+6\delta_l-4\delta_{l+1}+\delta_{l+2}\right). \end{aligned} \tag{7}$$

The finite element $[x_l,x_{l+1}]$ is mapped onto the interval $[0,h]$ using the transformation $\xi=x-x_l$, $0\le\xi\le h$. Then expressions of quintic B-spline shape functions in terms of a local coordinate ξ are obtained over $[0,h]$ as

$$\begin{aligned} \phi_{l-2} &= 1-5\tfrac{\xi}{h}+10\tfrac{\xi^2}{h^2}-10\tfrac{\xi^3}{h^3}+5\tfrac{\xi^4}{h^4}-\tfrac{\xi^5}{h^5},\\ \phi_{l-1} &= 26-50\tfrac{\xi}{h}+20\tfrac{\xi^2}{h^2}+20\tfrac{\xi^3}{h^3}-20\tfrac{\xi^4}{h^4}+5\tfrac{\xi^5}{h^5},\\ \phi_l &= 66-60\tfrac{\xi^2}{h^2}+30\tfrac{\xi^4}{h^4}-10\tfrac{\xi^5}{h^5},\\ \phi_{l+1} &= 26+50\tfrac{\xi}{h}+20\tfrac{\xi^2}{h^2}-20\tfrac{\xi^3}{h^3}-20\tfrac{\xi^4}{h^4}+10\tfrac{\xi^5}{h^5},\\ \phi_{l+2} &= 1+5\tfrac{\xi}{h}+10\tfrac{\xi^2}{h^2}+10\tfrac{\xi^3}{h^3}+5\tfrac{\xi^4}{h^4}-5\tfrac{\xi^5}{h^5},\\ \phi_{l+3} &= \tfrac{\xi^5}{h^5}. \end{aligned} \tag{8}$$

Applying Galerkin approach with weight function $W(x)$ to Eq. (4) produces

$$\begin{aligned}&\int_a^b W(x)\left(u^{n+1}+\tfrac{\Delta t}{2}\varepsilon u^{n+1}u_x^{n+1}+\tfrac{\Delta t}{2}\mu\left(u_{xxx}\right)^{n+1}\right)dx=\\&\int_a^b W(x)\left(u^{n}-\tfrac{\Delta t}{2}\varepsilon u^{n}u_x^{n}-\tfrac{\Delta t}{2}\mu\left(u_{xxx}\right)^{n}\right)dx.\end{aligned} \tag{9}$$

Using (6) in (9) and identify the weight functions $W(x)$ with quintic B-splines over the element $[0,h]$, we obtain

$$\begin{aligned}&\sum_{j=l-2}^{l+3}\left\{\int_0^h\phi_i\phi_j d\xi+\tfrac{\Delta t}{2}\varepsilon\sum_{k=l-2}^{l+3}\int_0^h\phi_i\left(\phi_k\delta_k^{n+1}\right)\phi_j' d\xi+\tfrac{\Delta t}{2}\mu\int_0^h\phi_i\phi_j''' d\xi\right\}\delta_j^{n+1}\\&-\sum_{j=l-2}^{l+3}\left\{\int_0^h\phi_i\phi_j d\xi-\tfrac{\Delta t}{2}\varepsilon\sum_{k=l-2}^{l+3}\int_0^h\phi_i\left(\phi_k\delta_k^{n}\right)\phi_j' d\xi-\tfrac{\Delta t}{2}\mu\int_0^h\phi_i\phi_j''' d\xi\right\}\delta_j^{n}\end{aligned}\quad i=l-2,\ldots,l+3,\ k=l-2,\ldots,l+3 \tag{10}$$

for $l=0,1,\ldots,N-1$. The previous expression is also written in matrix form as

$$\left[\mathbf{A}^e+\frac{\Delta t}{2}\varepsilon\left(\mathbf{B}^e\left(\boldsymbol{\delta}^e\right)^{n+1}\right)+\frac{\Delta t}{2}\mu\mathbf{C}^e\right]\left(\boldsymbol{\delta}^e\right)^{n+1}-\left[\mathbf{A}^e-\frac{\Delta t}{2}\varepsilon\left(\mathbf{B}^e\left(\boldsymbol{\delta}^e\right)^{n}\right)-\frac{\Delta t}{2}\mu\mathbf{C}^e\right]\left(\boldsymbol{\delta}^e\right)^{n} \tag{11}$$

where the element matrices and element parameters are

$$\begin{aligned}&A_{ij}^e=\int_0^h\phi_i\phi_j d\xi, && B_{ij}^e\left(\boldsymbol{\delta}^{n+1}\right)=\int_0^h\phi_i\left(\phi_k\delta_k^{n+1}\right)\phi_j' d\xi,\\&C_{ij}^e=\int_0^h\phi_i\phi_j''' d\xi, && \left(\boldsymbol{\delta}^e\right)^{n+1}=(\delta_{l-2}^{n+1},\ldots,\delta_{l+3}^{n+1})^T.\end{aligned} \tag{12}$$

Note that the element matrices $\mathbf{A}^e$ and $\mathbf{C}^e$, which independent from the parameters δ_k^e, are 6×6 and the matrix $\mathbf{B}^e$, which dependent from the parameters δ_k^e, is $6\times 6\times 6$. Assembling together contributions from all elements lead to the following nonlinear matrix equation

$$\left[2\mathbf{A}+\Delta t\mu\mathbf{C}+\Delta t\varepsilon\mathbf{B}\boldsymbol{\delta}^{n+1}\right]\boldsymbol{\delta}^{n+1}=\left[2\mathbf{A}-\Delta t\mu\mathbf{C}-\Delta t\varepsilon\mathbf{B}\boldsymbol{\delta}^{n}\right]\boldsymbol{\delta}^{n} \tag{13}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are assembled from the element matrices $\mathbf{A}^e$, $\mathbf{B}^e$ and $\mathbf{C}^e$. The nonlinear system (13) is a set of $N+5$ nonlinear equations in $N+5$ unknown parameters

$$\left(\delta_{-2},\delta_{-1},\ldots,\delta_{N+1},\delta_{N+2}\right).$$

After the first and last two equations are deleted in the system, imposition of the boundary conditions

$$U(a,t)=U_x(a,t)=U(b,t)=U_x(b,t)=0,$$

enables us to eliminate the 4 parameters δ_{-2}^{n+1}, δ_{-1}^{n+1} and δ_{N+1}^{n+1}, δ_{N+2}^{n+1} from the above system. After initial vector

$$\mathbf{d}^0=(\delta_{-2}^0,\ldots,\delta_{N+1}^0,\delta_{N+2}^0)$$

is found using the initial and boundary conditions

$$
\begin{aligned}
&U(x_l,0)=\delta_{l-2}+26\delta_{l-1}+66\delta_l+26\delta_{l+1}+\delta_{l+2}=f(x_l),\ l=0,\ldots,N\\
&U'(x_0,0)=\tfrac{5}{h}\left(-\delta_{-2}-10\delta_{-1}+10\delta_1+\delta_2\right)=0,\\
&U'(x_N,0)=\tfrac{5}{h}\left(-\delta_{N-2}-10\delta_{N-1}+10\delta_{N+1}+\delta_{N+2}\right)=0,\\
&U''(x_0,0)=\tfrac{20}{h^2}\left(\delta_{-2}+2\delta_{-1}-6\delta_0+2\delta_1+\delta_2\right)=0,\\
&U''(x_N,0)=\tfrac{20}{h^2}\left(\delta_{N-2}+2\delta_{N-1}-6\delta_N+2\delta_{N+1}+\delta_{N+2}\right)=0,
\end{aligned}
$$

unknown vectors

$$\mathbf{d}^{n+1}=(\delta_{-2}^{n+1},\ldots,\delta_{N+1}^{n+1},\delta_{N+2}^{n+1}),\ n=0,1,\ldots$$

can be found repeatedly by solving the system (13). Since the system (13) is a nonlinear system of equations, the following inner iteration algorithm is used for all time steps:

Step 1: Set $error=1$ and $\delta_l^*=\delta_l^{n+1}$ in $\mathbf{B}\delta^{n+1}$ and taking $\delta_l^*=\delta_l^n$, then compute U_m^*

Step 2: While $error\geq 10^{-10}$ do Steps 3-4,

Step 3: Find U_l^{n+1}

Step 4: $error=\max\limits_l\left|U_l^{n+1}-U_l^*\right|$ and set $\delta_l^*=\delta_l^{n+1}$,

Stop and go to next time step.

To investigate the stability of the difference scheme (13) by using Von Neumann stability analysis, the quantity U in the nonlinear term UU_x is locally constant for KdV equation. Substituting the Fourier mode $\delta_m^n=G^n e^{i\beta mh},(i=\sqrt{-1})$ into the linearized difference scheme (13) gives the growth factor G of the form

$$G=\frac{\alpha-i\gamma}{\alpha+i\gamma} \tag{14}$$

where

$$
\begin{aligned}
\gamma=&\frac{1310354}{231}h+\frac{1623019}{231}h\cos(\beta h)+\frac{17488}{11}h\cos(2\beta h)+\frac{50879}{462}h\cos(3\beta h)\\
&+\frac{1623019}{231}h\cos(4\beta h)+\frac{1}{1386}h\cos(5\beta h),
\end{aligned}
$$

$$
\begin{aligned}
\beta=&\Delta t\left(-\frac{6787\varepsilon p}{2}+\frac{6645\mu}{h^2}\right)\sin(\beta h)+\Delta t\left(-\frac{33946\varepsilon p}{21}-\frac{8700\mu}{7h^2}\right)\sin(2\beta h)+\Delta t\left(-\frac{5203\varepsilon p}{28}-\frac{17775\mu}{14h^2}\right)\sin(3\beta h)\\
&+\Delta t\left(-\frac{253\varepsilon p}{63}-\frac{610\mu}{7h^2}\right)\sin(4\beta h)+\Delta t\left(-\frac{\varepsilon p}{252}-\frac{5\mu}{14h^2}\right)\sin(5\beta h).
\end{aligned}
$$

Since the modulus of (14) is $|G|=1$, the difference scheme (13) is unconditionally stable.

3. TEST PROBLEMS

For the first test problem, accuracy of the proposed algorithm is worked out by measuring error norms L_∞ and L_2

$$L_\infty = \max|u_m - U_m|, \quad L_2 = \sqrt{h\sum_{m=0}^{N}|u_m - U_m|}. \tag{15}$$

The following lowest three conservation invariants for the KdV equation are given as

$$C_1 = \int_{-\infty}^{\infty} U dx \approx \int_a^b U dx, \;\; C_2 = \int_{-\infty}^{\infty} U^2 dx \approx \int_a^b U^2 dx, \;\; C_3 = \int_{-\infty}^{\infty}\left(U^3 - \frac{3\mu}{\varepsilon}U_x^2\right)dx \approx \int_a^b\left(U^3 - \frac{3\mu}{\varepsilon}U_x^2\right)dx. \tag{16}$$

Composite trapezium method is used to approximate above integrals.

3.1. Single Soliton Waves

In the first test problem, we will study the motion of a single soliton wave. The analytical single soliton solution of the KdV equation is

$$u(x,t) = 3c\,\text{sech}^2(kx - \bar{x}_0 - kvt), \tag{17}$$

where $v = \varepsilon c$ is the wave velocity, $3c$ is amplitude of the soliton wave, $\bar{x}_0 / k$ is peak position of the initially centered wave and $k = \sqrt{\varepsilon c / (4\mu)}$. This solution represents a soliton of magnitude $3c$, initially centered on the position $\bar{x}_0 / k$ propagating towards the right across the interval $[a,b]$ over the time period without change of shape at constant speed v. To allow comparison with other work, we take $\varepsilon = 1,\ \mu = 4.84\times10^{-4},\ \bar{x}_0 = 6,\ c = 0.3,\ h = 0.01,\ \Delta t = 0.005$ and space interval $[a,b] = [0,2]$. The profiles of the soliton at different time levels up to time $t = 3$ are plotted in Fig. 1. One of the properties of soliton waves is that they are non-dispersive, i.e. maintain their initial shapes and size. We observe in Figure 1 that the soliton moves along space interval with a constant velocity and unchanged amplitude and very little changes in amplitude occurs during run of the algorithm varied from initial value of 0.9 to 0.8994512. When we choose smaller time and space steps as $h = 0.001$, $\Delta t = 0.0005$, the soliton wave's amplitude varies less from initial value of 0.9 to 0.8999982. Therefore, soliton's amplitude remains close to initial soliton's amplitude. If we run the program further in time, the soliton propagates with a constant velocity and almost unchanged amplitude.

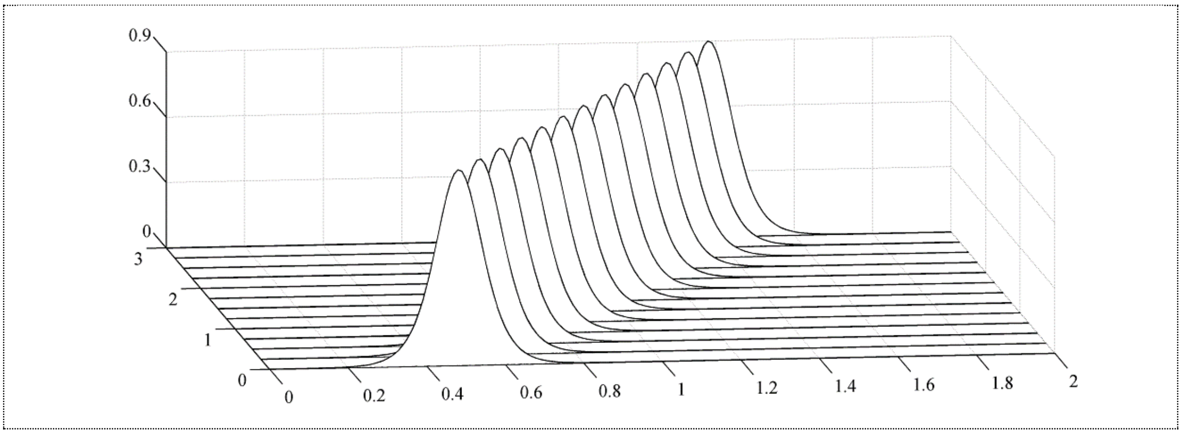

Figure 1. Single soliton at different times

Comparisons are made with several previous works listed in Table 1. As shown in Table 1, the results indicate that the proposed method has a highest accuracy than the previous two works for the first test problem.

Table 1. Invariant and error norms

Method	t	$L_2\times10^3$	$L_\infty\times10^3$	C_1	C_2	C_3
Present	1	0.052	0.141	0.144600	0.086759	0.046850
Present	3	0.125	0.320	0.144597	0.086760	0.046850
[5]	1		0.589	0.144593	0.086759	0.046849
[5]	3		0.740	0.144597	0.086759	0.046849
[6]	1	0.130	0.369	0.144598	0.086759	0.046850
[6]	3	0.387	1.042	0.144600	0.086759	0.046850
Exact		0	0	0.144599	0.086759	0.046850

Absolute error distribution for the proposed method is drawn at time $t=3$ in Figure 2 with space and time steps $h=0.01,\ \Delta t=0.005$, respectively. It can be seen from the figure that the maximum error is taken place around the peak amplitude of the soliton wave.

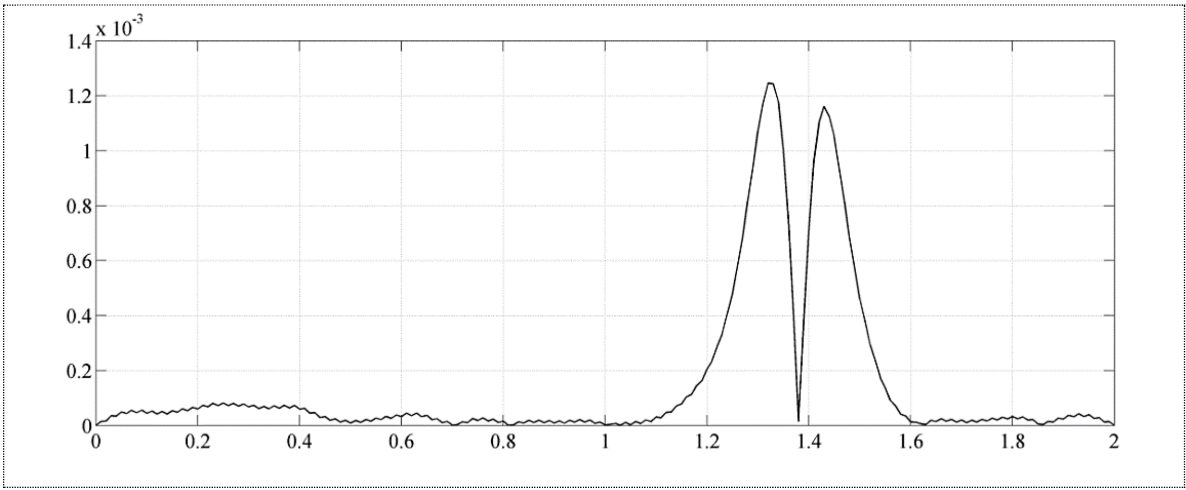

Figure 2. Absolute error at *t* = 3

3.2. Interaction of Two Soliton Waves

We consider interaction of two soliton waves using the following initial condition

$$u(x,0)=3c_1\operatorname{sech}^2(k_1[x-\bar{x}_1])+3c_2\operatorname{sech}^2(k_2[x-\bar{x}_2]), \tag{18}$$

where $k_i=\sqrt{\varepsilon c_i/(4\mu)}, i=1,2.$ All of the computations are done for the parameters $\varepsilon=6,\ \mu=1,$ $c_1=0.3,\ c_2=0.1,\ \bar{x}_1=15$ and $\bar{x}_2=35$ over the region $0\le x\le 90$ for the second test problem. These parameters provide two well separated soliton waves of magnitudes 0.9 and 0.3 sited initially at $x=15$ and 35 respectively. The program is run until $t=30$ with $h=\Delta t=0.1$ and numerical solutions of $u(x,t)$ at several times are drawn for visual views of the solutions in Figure 3. We know that solitons can collide with other solitons, after which both solitons re-emerge with their original form and speed. We can see from the figure that the initial profile consists of two well separated solitons and two soliton propagate to the right at their initial velocities and then they collide after which they separate and resume initial shapes and velocities. After interaction, two solitons propagates with a constant velocity and almost unchanged amplitude.

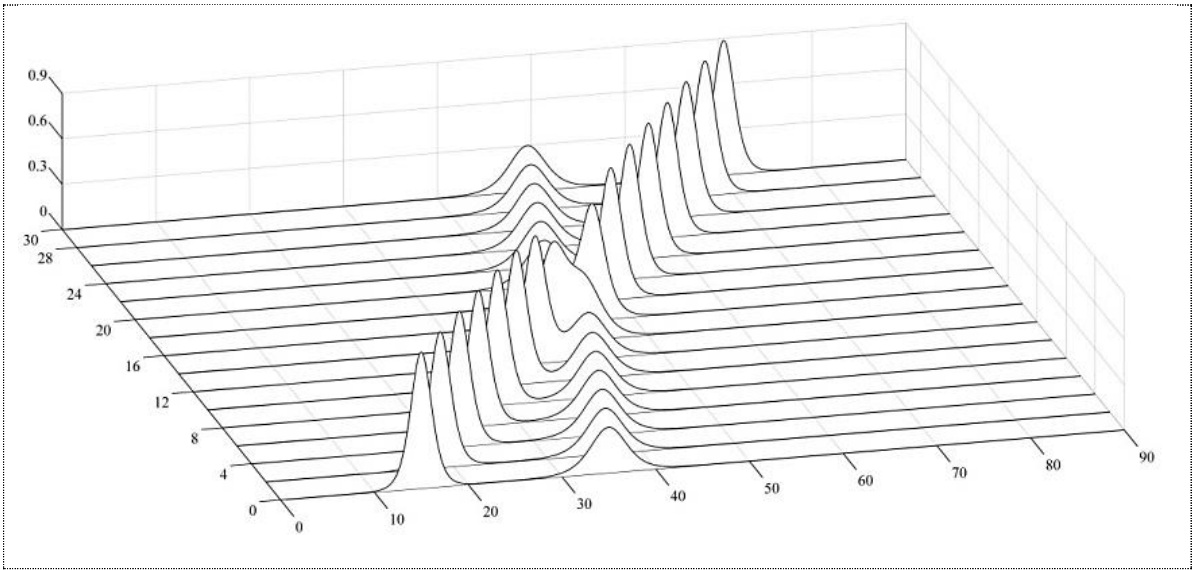

Figure 3. Interaction of two solitons

Table 2 displays numerical values of the invariants and amplitude of wave for various space and time step. According to the Table 2, the agreement between numerical and exact values of invariants together with amplitudes of the waves after the collision is very satisfactory.

Table 2. Invariants and amplitude of soliton waves at t =30

t	$h = \Delta t$	C_1	C_2	C_3	Small wave amplitude	Large wave amplitude
30	0.1	4.2246681	1.8914716	0.8998216	0.299693	0.886526
30	0.05	4.2328155	1.9202935	0.9255896	0.299938	0.899726
30	0.02	4.2324872	1.9198685	0.9252089	0.299990	0.899956
30	0.01	4.2324740	1.9198130	0.9251590	0.299998	0.899983
Exact		4.2324749	1.9198099	0.9251562		

4. CONCLUSIONS

Galerkin method based on quintic B-spline shape functions was presented for a numerical solution of the KdV equation. The propagation of a single soliton wave and interaction of two soliton waves were used to examine the performance of the method. From the comparison between the previous numerical methods and present method we conclude that our scheme is more accurate than other scheme. Therefore the proposed Galerkin finite element method using quintic B-spline provides accurate method for solving KdV equation.

REFERENCES

[1] Korteweg DJ and de Vries G. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philos Mag. 1895; 39 (240): 422–443.

[2] Zabusky NJ and Kruskal M D. Interaction of "solitons" in a collisionless plasma and the recurrence of initial states. Phys Rev Lett 1965; 15 (6): 240–243.

[3] Soliman AA. Collocation solution of the Korteweg-De Vries equation using septic splines. Int J Comput Math 2004; 81: 325-331.

[4] Canıvar A, Sarı M and Dağ I. A Taylor-Galerkin finite element method for the KdV equation using cubic B-splines. Physica B 2010; 405: 3376-3383.

[5] Ersoy Ö and Dağ İ. The Exponential Cubic B-Spline Algorithm for Korteweg-de Vries Equation. Advances in Numerical Analysis 2015; 2015: 1-8.

[6] Saka B. Cosine expansion-based differential quadrature method for numerical solution of the KdV equation, Chaos Soliton Fract 2009; 40: 2181-2190.

[7] Korkmaz A. Numerical algorithms for solutions of Korteweg-de Vries Equation, Numer Meth Part D E 2010; 26(6): 1504–1521.

[8] Irk D, Dağ İ and Saka B. A small time solutions for the Korteweg–de Vries equation using spline approximation. Appl Math Comput 2006; 173 (2): 834-846.

[9] Dağ İ, Saka B and Irk D. Galerkin method for the numerical solution of the RLW equation using quintic B-splines. J Comput Appl Math 2008; 190: 532–547.

[10 Dhawan S and Kumar S. A galerkin B-spline approach to one dimensional heat equation. International Journal of Research and Reviews in Applied Sciences 2009; 1(1): 47-57.

[11] Kutluy S and Uçar Y. A quadratic B-spline Galerkin approach for solving a coupled KdV equation. Math Model Anal 2013; 18(1): 103-121.

[12] Mei L, Gao Y and Chen Z. A Galerkin finite element method for numerical solutions of the Modified Regularized Long Wave Equation. Abstr Appl Anal 2014; 2014: 1-11.

[13] Uçar Y, Karaağaç B and Esen A. A new approach on numerical solutions of the Improved Boussinesq type equation using quadratic B-spline Galerkin finite element method. Appl Math Comput 2015; 270: 148-155.

[14] Karakoç SBG and Zeybek H. A cubic B-spline Galerkin approach for the numerical simulation of the GEW equation. Statistics, Optimization & Information Computing 2016; 4: 30-41.

[15] Crank J and Nicolson P. A practical method for numerical evaluation of solutions of partial differential equations of the heat conduction type. Proc. Camb. Phil. Soc. 1947; 43: 50–67.

[16] Prenter PM. Splines and Variational Methods, Wiley 1975.