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EXISTENCE OF SOME RICCI-FLAT FINSLER METRICS

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ABSTRACT

This paper shows the existence of some Ricci-flat Finsler metrics defined by a Riemannian metric and 1-form supported by an example.

Keywords: (α,β)-metrics, Einstein Metrics, Ricci Curvature

BAZI RICCI-DÜZ FINSLER ÖLÇÜMLERİNİN VARLIĞI

ÖZET

Bu makalede bir Riemann metrik ve 1-form kullanılarak tanımlanan bazı Ricci-Düz Finsler metriklerin varlığını gösteriyoruz.

Anahtar Kelimeler: *(α, β)*-metrikler, Einstein Metrikleri, Ricci Eğriliği

1. INTRODUCTION

One of the main features that distinguishes Riemannian metrics and Finsler metrics defined on a manifold is that Riemannian metrics are quadratic metrics, whereas Finsler metrics have no restriction on the quadratic property. One can naturally extends the Ricci curvature **Ric** in Riemannian geometry to Finsler metrics. It is a natural problem to study Finsler metrics $F = F(x,y)$ with isotropic Ricci curvature (They are also called *Einstein metrics*.) Ric $=$ Ric(x,y), i.e., $Ric = (n-1)\sigma F^2$, where σ is a scalar function in x on the n-dimensional manifold. It is known that there are Einstein metrics in a certain form that are Ricci-flat. We consider Finsler metrics defined by a Riemannian metric α and a 1-form β in the following form

$$\mathrm{F} = \alpha\phi(s) \quad \text{where} \qquad \mathrm{s} = \frac{\beta}{\alpha}, \tag{1}$$

where ϕ is a positive smooth function. These Finsler metrics in (1) are called (α, β)-*metrics*. Randers metrics defined by $F = \alpha + \beta$ are the simplest (α, β)-metrics. We have more general (α, β)-metrics defined by a polynomial

$$F = \alpha \sum_{i=0}^{k} a_k \left(\frac{\beta}{\alpha}\right)^i, \ \ k \geqq 2 \tag{2}$$

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where $a_0=1$ and a_i's are constants with $a_k \neq 0$. These metrics are called *polynomial metrics*. Bao-Robles, [1], presented equations on α and β characterizing constant Ricci curvature Randers metrics. There are a lot of constant (zero or non-zero) Ricci curvature Randers metrics. On the other hand, if a polynomial metric of non-Randers type in (2) is of constant Ricci curvature, then it is Ricci-flat ([2]). Equations on α, β and ϕ characterizing Douglas type Ricci-flat $(\alpha,\beta)-$metrics were obtained by the authors in [3, 5], independently. Next question arises naturally: Are there any non-Douglas type Ricci-flat $(\alpha,\beta)-$metrics? Cheng-Shen discovered some new Einstein metrics after studying $(\alpha,\beta)-$metrics where β is a Killing form with constant length. They also found singular Einstein metrics on S^3 with $Ric = \pm 2F^2$, and $Ric = 0$, respectively, [4].

In this paper, we set new assumptions on α, β for an $(\alpha,\beta)-$metric F defined in (1) with a goal of characterizing Einstein metrics. These assumptions came as an inspiration through our published papers with Zhongmin Shen, [5,6,7]. Indeed, these papers contain examples for the metrics and similar conditions to form the corresponding Einstein Finsler spaces. (We finalized the assumptions below after a private conversation with Zhongmin Shen). We compute the Riemann curvature and the Ricci curvature for $(\alpha,\beta)-$metrics to characterize Einstein metrics, [2].

Let M^n be an n-dimensional manifold and F an $(\alpha,\beta)-$metric as defined in (1) where $\alpha = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i \neq 0$ is a 1-form on M^n.

$$r_{ij} = \frac{1}{2}\left(b_{i|j} + b_{j|i}\right), \qquad s_{ij} = \frac{1}{2}\left(b_{i|j} - b_{j|i}\right)$$

where "|"denotes the covariant derivative with respect to the Levi-Civita connection of α. We also let

$$r_j := b^m r_{mj}, \quad s_j = b^m s_{mj}$$

where $a^{ij} := (a_{ij})^{-1}$ and $b_j := a^{ij}b_j$. We denote $r_{i0} = r_{ij}y^j$, $s_{i0} = s_{ij}y^j$, and $r_{00} = r_{ij}y^iy^j$, $r_0 = r_iy^j$, $s_0 = s_iy^j$, etc. Here we have $r_i + s_i = 0$ if and only if β has constant length with respect to α. We have the following Assumption I:

Assumption I:

(a) ${}^{\alpha}Ric = (n-1)\tau\,(K_1 + K_2\,(b^2 - s^2))\alpha^2$
(b) $s_{ij} = 0$
(c) $r_{ij} = \epsilon(b^2 a_{ij} - b_ib_j)$

where τ and ϵ are scalar functions in x on M^n with $(\tau,\epsilon) \neq (0,0)$. Then ϵ must satisfy

$$\epsilon_{x^i} = \left(\tilde{\epsilon}\, b^{-2} - \frac{\epsilon^2}{n-2}\right) b_i, \tag{3}$$

where $\tilde{\epsilon} := -\tau K_1 - \frac{\epsilon^2(n-2)}{(n-1)}\, b^2$ (refer to Lemma 2 below). By Assumption I, (a) − (c) $b := ||\beta||_x$ must be constant, here K_1 and K_2 are constants and there is no relationship between τ and ϵ. In particular, if both τ and ϵ are zero, then ${}^{\alpha}Ric = 0$ and $r_{ij} = 0$, $s_{ij} = 0$ (β is parallel with respect to α) by Assumption I. Then, this is the trivial case and $F = \alpha\phi(\beta/\alpha)$ is Ricci-flat for any ϕ. Hence, we assume that $(\tau,\epsilon) \neq (0,0)$ throughout this paper. Next we state the main theorem of the paper.

Theorem 1 Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be an (α,β) - metric on an n-dimensional manifold M where α,β satisfy Assumption I with $(\tau,\epsilon) \neq (0,0)$. We have $Ric = 0$ if and only if

$$\epsilon^2 A(s) + (n-1)\tau b - 2B(s) = 0, \tag{4}$$

where

$$b := \sqrt{a^{ij} b_i b_j},\ \epsilon := \epsilon(x),\ \tau := \tau(x) \text{ and } \tau = K_3 \epsilon^2 \text{ when } B(s) \neq 0$$

and

$$A(s) := (n-1)\{(b^2 - s^2)[3s\Xi + 2(n-2)b^2\Psi] + (b^2 - s^2)^2[-\Xi' + 2(b^2 - s^2)\Psi\Xi' - 2s\Psi\Xi - (b^2 - s^2)^2(\Psi')^2]\} + (b^2 - s^2)^2[\Xi - (b^2 - s^2)\Psi'], \tag{5}$$

$$B(s) := (n-1)[K_1 + K_2(b^2 - s^2)]b^2 - (b^2 - s^2)(K_1 + b^{-2})(2b^2\Psi - s\Xi) \tag{6}$$

and K_1, K_2, **b** are constants, τ and ϵ are scalar functions, and

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q', \quad \Theta := \frac{Q - sQ'}{2\Delta},$$

$$\Psi := \frac{Q'}{2\Delta}, \quad \Xi := (n-1)\Theta + (b^2 - s^2)\Psi'.$$

Note that since the equation (4) also depends on $x \in M^n$, it is not an ODE in s. We divide (4) into some cases. Firstly, if

$$A(s) = 0, \quad \text{and} \quad B(s) = 0, \tag{7}$$

then regardless of the values of ϵ and τ the equation (4) holds. We assume that $(A, B) \neq (0, 0)$.

(a) If $\epsilon \neq 0$, then $\tau = K_3 \epsilon^2$ for some constant K_3, then the equation (4) is reduced to

$$A(s) + (n-1)K_3 B(s) = 0. \tag{8}$$

(b) If $\tau \neq 0$, then $\epsilon^2 = K_4 \tau$ for some constant K_4, then the equation (4) is reduced to

$$K_4 A(s) + (n-1)B(s) = 0. \tag{9}$$

By Theorem 1 (a), $(b^2)_{|k} = 2a^{ij} b_i b_{j|k} = 0$. Thus $b := \sqrt{a^{ij} b_i b_j}$ is a constant.
The equation (8) (or (9)) is a third order ordinary differential equation in ϕ. By ODE theory, for any given initial conditions the local solution of the equation (4) exists nearby 0. It is not possible to express the solution by using simple functions defined on an interval containing $[-b, b]$. This indicates that we might have a singular (α, β) Finsler metric $F = \alpha\phi(\beta/\alpha)$ defined by ϕ.

Example 1. [4] For the Lie group S^3 we let η^1, η^2, η^3 be the standard right invariant 1-form on S^3 such that

$$d\eta^1 = 2d\eta^2 \wedge d\eta^3, \quad d\eta^2 = 2d\eta^3 \wedge d\eta^1, \quad d\eta^3 = 2d\eta^1 \wedge d\eta^2.$$

For any number $\epsilon \geq 0$, let $\theta^1 := (1+\epsilon)\eta^1$, $\theta^2 := \sqrt{1+\epsilon\eta^2}$, $\theta^3 := \sqrt{1+\epsilon\eta^3}$.

$$\alpha_\epsilon := \sqrt{[\theta^1]^2 + [\theta^2]^2 + [\theta^3]^2}, \quad \beta_\epsilon = b\theta^1,$$

where $b = \sqrt{\epsilon/(1+\epsilon)} < 1$. Then $\bar{F} := \alpha_\epsilon + \beta_\epsilon$ is a Randers metric on S^3 with constant-flag curvature $\sigma = 1$. Further, α_ϵ satisfies

$$^{\alpha_\epsilon}Ric = (n-1)\{\alpha^2 - \frac{n+1}{n-1}(b^2\alpha_\epsilon^2 - \beta_\epsilon^2)\},$$

and $\beta_\epsilon = b_i\,\theta^i$ is a Killing form of constant length b and

$$s_{im}s_j^m = -\left(b^2\delta_{ij} - b_i\,b_j\right), \qquad s_{0;m}^m = (n-1)\beta_\epsilon$$

Thus α_ϵ and β_ϵ satisfy Assumption I with

$$\epsilon = 0, \qquad \tau = -b^2, \qquad K_1 = -b^{-2}, \; K_2 = \frac{n+1}{n-1}b^{-2}.$$

If $\phi = \phi(s)$ satisfies B(s) = 0, then $F = \alpha_\epsilon\phi(\beta_\epsilon/\alpha_\epsilon)$ is Ricci-flat.

2. PRELIMINARIES

A nonnegative scalar function $F = F(x,y)$ on the tangent bundle TM^n is a Finsler metric on a manifold M^n where x is a point in M^n and $y \in T_x M^n$ is a tangent vector at x. The characterization of geodesics for a Finsler metric $F = F(x,y)$ in local coordinates are given by

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0$$

where

$$G^i := \tfrac{1}{4}g^{il}(x,y)\{\,[F^2]_{x^k y^l}(x,y)y^k - [F^2]_{x^l}(x,y)\,\}, \tag{10}$$

and $g_{ij}(x,y) := (\frac{1}{2}F^2)_{y^i y^j}$. A vector field **G** – it is called the spray of F- below is defined by using local functions G^i on TM^n

$$G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}.$$

These local functions $G^i = G^i(x,y)$ are called spray coefficients of F for the spray G. For any $x \in M^n$ and $y \in T_x M^n \setminus\{0\}$, the Riemann curvature $R_y : T_x M^n \to T_x M^n$ is defined by $R_y(u) = R^i{}_k(x,y)\,u^k\frac{\partial}{\partial x^i}\Big|_x$, where

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j\partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The Ricci curvature is given by

$$Ric := 2\frac{\partial G^m}{\partial x^m} - y^j\frac{\partial^2 G^m}{\partial x^j\partial y^m} + 2G^j\frac{\partial^2 G^m}{\partial y^j\partial y^m} - \frac{\partial G^m}{\partial y^j}\frac{\partial G^j}{\partial y^m}.$$

Consider an (α,β)-metric on a manifold M^n defined by

$$F := \alpha\phi(s), \qquad s = \beta/\alpha$$

where $\phi = \phi(s) > 0$ is a C^∞ function on $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $b(x) := \|\beta_x\|_\alpha < b_0$. We suppose that ϕ satisfies the following inequality

$$\emptyset(s) - s\emptyset'(s) + (b^2 - s^2)\emptyset''(s) > 0, \qquad |s| \leq \rho < b_0. \tag{11}$$

Then the (α, β)-metric F is a regular positive definite Finsler metric. The spray coefficients G^i of F obtained by (10), are given in the following lemma.

Lemma 1. [8] The spray coefficients of F for an (α, β)-metric $F = \alpha\phi(s), s = \beta/\alpha$, are given by

$$G^i = {}^{\alpha}G^i + \alpha Q s_0^i + \Theta\{r_{00} - 2Q\alpha s_0\}\frac{y^i}{\alpha} + \Psi\{r_{00} - 2Q\alpha s_0\}b^i \tag{12}$$

where ${}^{\alpha}G^i$ are the spray coefficients of α,

$$Q = \frac{\phi'}{\phi - s\phi'}, \quad \Theta = \frac{Q - sQ'}{2\Delta}, \; \psi = \frac{Q'}{2\Delta}, \quad \Delta = 1 + sQ + (b^2 - s^2)Q',$$

and $r_{00} = r_{ij}y^i y^j, \; a^{ik}s_{ij} = s_j^k, \; s_0^k = y^j s_j^k, \; b^i s_{ij} = s_j, \; s_0 = s_j y^j.$

We also have that

$$\Delta = \frac{\phi(\phi - s\phi' + (b^2 - s^2)\phi'')}{(\phi - s\phi')^2}.$$

In the case that (11) holds, we have $\Delta = \Delta(s) > 0$ for s with $|s| \leq b < b_0$.

3. PROOF OF THE MAIN THEOREM 1

Next we prove the main theorem Theorem 1. First we introduce the following Lemma.

Lemma 2. Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an *n*-dimensional manifold M^n with $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$ satisfying Assumption I. Then $\epsilon = \epsilon(x)$ satisfies

$$\epsilon_{x^i} = (\tilde{\epsilon}\, b^{-2} - \frac{\epsilon^2}{n-2})\, b_i, \tag{13}$$

where $\tilde{\epsilon} \coloneqq -\tau K_1 - \epsilon^2\left(\frac{n-2}{n-1}\right)b^2$.

Proof: By using Ricci identities, we get that

$$\begin{aligned} b_{i|j|k} - b_{i|k|j} &= b^m \; {}^{\alpha}R_{imjk}, \\ -b_{k|i|j} + b_{k|j|i} &= -b^m \; {}^{\alpha}R_{kmij}, \\ b_{j|k|i} - b_{j|i|k} &= b^m \; {}^{\alpha}R_{jmki}. \end{aligned} \tag{14}$$

We also have the following equalities,

$$\begin{aligned} b_{i|k|j} + b_{k|i|j} &= 2r_{ik|j}, \\ -b_{k|j|i} - b_{j|k|i} &= -2r_{kj|i}. \end{aligned} \tag{15}$$

We add all the equations in (14) and (15) to get

$$s_{ij|k} \;=\; \frac{1}{2}\left(b_{i|j|k} - b_{j|i|k}\right) \;=\; -b^{m}\ {}^{\alpha}R_{kmij} + r_{ik|j} - r_{kj|i}. \qquad (16)$$

Hence, we have

$$s^{m}_{j|m} \;=\; b^{m}\ {}^{\alpha}Ric_{mj} + r^{m}_{m|j} - r^{m}_{j|m},$$

$$s^{m}_{0|m} \;=\; b^{m}\ {}^{\alpha}Ric_{m0} + r^{m}_{m|0} - r^{m}_{0|m}. \qquad (17)$$

By using (a) and (b) in the Assumption I, we obtain

$$b^{m}\ {}^{\alpha}Ric_{mk} = (n-1)\tau\, K_1\, b_k\ . \qquad (18)$$

By using (c) in the Assumption I, we get

$$r^{i}_{i} = \epsilon(n-1)b^2,$$

$$r^{i}_{0} = \epsilon\left(b^2 y^i - s\alpha b^i\right),$$

and hence,

$$r^{i}_{i|0} = \epsilon_0(n-1)b^2,$$

$$r^{i}_{0} = \epsilon_0 b^2 - \tilde{\epsilon} s\alpha - (n-1)\epsilon^2 s b^2 \alpha\,, \qquad (19)$$

where

$$\epsilon_0 \;=\; \epsilon_{x^k} y^k \qquad and \qquad \tilde{\epsilon} = \epsilon_{x^k} b^k.$$

We substitute the results (18) and (19) into the equation (17), then we use (b) to get

$$0 \;=\; (n-1)\tau\, K_1\, s\,\alpha + \epsilon_0\,(n-1)b^2 - \epsilon_0\, b^2 + \tilde{\epsilon}\, s\,\alpha + (n-1)\epsilon^2\, s\, b^2\,\alpha \qquad (20)$$

We further use the result (16) and get

$$s^{m}_{|m} \;=\; \left(b^l s^{m}_{l}\right)_{|m} \;=\; b^{l}_{|m} s^{m}_{l} + b^l s^{m}_{l|m}$$

$$s^{m}_{|m} \;=\; -t^{m}_{m} - r^{l}_{m} r^{m}_{l} - b^m b^l\ {}^{\alpha}Ric_{lm} + r^{m}_{|m} - b^l r^{m}_{m|l} \qquad (21)$$

where $t^{m}_{m} = s^{i}_{m} s^{m}_{i}$. By using (b), the equation (21) becomes

$$0 = -r^{l}_{m} r^{m}_{l} - b^m b^l\ {}^{\alpha}Ric_{ml} + r^{m}_{|m} - b^m r^{m}_{m|l}, \qquad (22)$$

and therefore, we obtain

$$\tilde{\epsilon} = -\,\tau K_1 - \frac{\epsilon^2(n-2)}{n-1} b^2. \qquad (23)$$

Thus, by substituting (23) into (22), we get the relationship between ϵ_0 and $\tilde{\epsilon}$ expressed as

$$\epsilon_0 = \tilde{\epsilon} b^{-2} - \frac{\epsilon^2}{n-2}\beta. \qquad (24)$$

Lemma 3. Let $F = \alpha\,\phi(\beta/\alpha)$ be an (α,β)-metric on an n-dimensional manifold M with $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$ satisfying the conditions in Assumption I above. Then the Ricci curvature of F is given by

$$Ric = {}^{\alpha}Ric - (n-1)\tau\,[K_1 + K_2(b^2 - s^2)]\alpha^2 + \frac{1}{n-1}[\epsilon^2 A(s) + (n-1)\tau\, b^{-2}B(s)], \quad (25)$$

where $A(s)$ and $B(s)$ are given in equations (5) and (6).

Proof: Here, we have $b_{i|j}$ as follows:

$$b_{i|j} = \epsilon(b^2 a_{ij} - b_i b_j).$$

We note that $(b^2)_{|i} = 2b^j b_{i|j} = 2b^j r_{ji} = 0$, and we get that b is a constant. Lemma 1 lets us write the spray coefficients of F as follows

$$G^i := {}^{\alpha}G^i + T^i,$$

where

$$T^i = r_{00}\left(\frac{y^i}{\alpha}\Theta + \Psi b^i\right)$$

The flag curvature tensor is written as

$$R^i_k := {}^{\alpha}R^i_k + H^i_k,$$

$$H^i_k := 2T^i_{|k} - T^i_{|j.k}\,y^j + 2\,T^j\,T^i_{.j.k} - T^i_{.j}\,T^j_{.k}$$

Then

$$\boldsymbol{Ric} = {}^{\alpha}\boldsymbol{Ric} + H^i_i \quad (26)$$

where

$$H^i_i := 2T^i_{|i} - T^i_{|j.i}\,y^j + 2T^jT^i_{.j.i} - T^i_{.j}\,T^j_{.i} \quad (27)$$

By using the assumptions (b) and (c) in Assumption I, and the following identities, we compute the Ricci curvature.

$$s_{|i}\,y^i = \epsilon(b^2 - s^2)\alpha, \quad s_{|i}\,b^i = 0, \quad s_{.i}\,b^i = \frac{1}{\alpha}(b^2 - s^2), \quad s_{.i}\,y^i = 0,$$

$$s_{|j.i}b^iy^j = -\epsilon(b^2 - s^2)s, \quad s_{|j.i}y^iy^j = 0, \quad s_{.j.i}y^iy^j = 0,$$

$$s_{.j.i}\,b^i\,y^j = -\frac{1}{\alpha}(b^2 - s^2), \quad s_{.j.i}\,b^i\,b^j = -\frac{3s}{\alpha^2}(b^2 - s^2). \quad (28)$$

We also easily get

$$r_{00|i}y^i = \epsilon_0(b^2 - s^2)\alpha^2 - 2\epsilon^2(b^2 - s^2)\alpha^3,$$

$$r_{00|i}b^i = \tilde{\epsilon}(b^2 - s^2)\alpha^2,$$

$$r_{00.i}y^i = 2\epsilon(b^2 - s^2)\alpha^2 = 2r_{00}, \qquad r_{00.i}b^i = 0,$$

$$r_{00|j.i}y^iy^j = 2\epsilon_0(b^2 - s^2)\alpha^2 - 4\epsilon^2 s(b^2 - s^2)\alpha^3 = r_{00|i}y^i,$$

$$r_{00|j.i}b^iy^j = -2\epsilon^2 b^2(b^2 - s^2)\alpha^2, \qquad r_{00|j.i}b^jy^i = 2\tilde{\epsilon}(b^2 - s^2)\alpha^2,$$

$$r_{00.i.j}y^jy^i = 2r_{00}, \qquad r_{00.i.j}b^jy^i = 0, \qquad r_{00.i.j}b^jb^i = 0 \tag{29}$$

where $\epsilon_0 = \epsilon_{x^k}y^k$ *and* $\tilde{\epsilon} = \epsilon_{x^k}b^k$.

By using the identities in (28) and (29), we obtain

$$T^i_{|i} = [\epsilon_0\,(b^2 - s^2)\alpha - 2\epsilon r_{00}\,s]\Theta + \epsilon r_{00}\,(b^2 - s^2)\Theta' + [\,\epsilon(b^2 - s^2)\alpha^2 + (n-1)\epsilon r_{00}\,b^2\,]\Psi,$$

$$T^i_{|j.i}y^j = -(n+1)(b^2 - s^2)(2\epsilon^2 s\alpha - \epsilon_0)\alpha\Theta + (n+1)\epsilon(b^2 - s^2)\alpha^2 r_{00}\Theta'$$

$$-(b^2 - s^2)^2(4\epsilon^2 s\alpha - \epsilon_0)\alpha\Psi' + \epsilon(b^2 - s^2)^2 r_{00}\Psi'',$$

$$T^jT^i_{.j.i} = \alpha^{-2}\,r_{00}^2(b^2 - s^2)^2\Psi''\,\Psi - 3\alpha^{-2}s\,r_{00}^2(b^2 - s^2)\,\Psi\Psi' + (n+1)\alpha^{-2}\,r_{00}^2(b^2 - s^2)\Theta'\Psi$$

$$+ \alpha^{-2}\,r_{00}^2(b^2 - s^2)\Theta\Psi' + (n+1)\alpha^{-2}\,r_{00}^2\Theta^2 - (n+1)s\alpha^{-2}\,r_{00}^2\Theta\,\Psi$$

$$T^i_{.j}T^j_{.i} = 4\alpha^{-2}\,r_{00}^2(b^2 - s^2)\Theta'\Psi + 2\alpha^{-2}\,r_{00}^2(b^2 - s^2)\Theta\Psi' + \alpha^{-2}\,r_{00}^2(b^2 - s^2)^2\Psi'^2$$

$$-4\alpha^{-2}\,r_{00}^2 s\,\Psi\Theta + (n+3)\alpha^{-2}\,r_{00}^2\Theta^2,$$

After substituting the equations obtained above in the equations (26), we obtain (25).

We now prove Theorem 1. By assumption on ${}^{\alpha}\boldsymbol{Ric}$, we have

$${}^{\alpha}\boldsymbol{Ric} = (n-1)\tau\big(K_1 + K_2(b^2 - s^2)\big)\alpha^2,$$

and Φ satisfies

$$\epsilon^2 A(s) + (n-1)\tau b^{-2}B(s) = 0.$$

Then by Lemma 3, the Ricci curvature $\boldsymbol{Ric} = 0$ and hence the $(\alpha,\beta)-metric$ is an Einstein space.

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