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BASIC BOUNDARY VALUE PROBLEM WITH RETARDED ARGUMENT CONTAINING AN EIGENPARAMETER IN THE TRANSMISSION CONDITION

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Abstract. In this paper basic boundary value problem with retarded argument that has a discontinuity point inside the interval will be studied. At the discontinuity point transmission conditions contain eigenparameter. Existence of eigenvalues and eigenfunctions will be studied. Asmyptotic properties of eigenvalues and eigenfunctions will be obtained.

1. Introduction

Many realistic system depend not only on current state but also the past. These systems can be modeled by using retarded argument equations. In detail these type of equations can be considered in two groups. Equations with constant delay is called equations with time lag and equations with functioanal delay is called equations with after affect.

After the development of control systems in engineering retarded equations become important. Before that scientists were aware of this type of delays in the control systems but there was not enough theory about this subject. Because of that this type of affects were ignored in the models. Delays have an important role to explain complex models mathematically and it also has important affects. Equations with retarded argument is used modeling problems in the fields of biology, chemistry, economics, mechanics, physics, physiology, population change, social networks, heat dissipation, interaction of species, microbiology and engineering. Unlike ordinary differantial equations, equations with retarded argument belong to functional diffrential equation class.

The fundamental study in this subject is made by Norkin in 1956 and 1958 [1, 2]. Şen - Bayramov [3], Yang [4], Akgün-Bayramov-Bayramoğlu [5], Şen - Seo- Arıcı [6], Bayramoğlu - Bayramov - Şen [7], Çetinkaya - Mamedov [8],F. Hira [9] have studied the retarded equation with discontinuity point in the interval.

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Freiling - Yurko [10], Mosazadeh [11], Bondarenko - Yurko [12] have studied the inverse problem.

In this work discontinuous equation on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ with parameter in the transmission condition will be considered.

$$y''(x) + \lambda^2 y(x) + q(x)y(x - \Delta(x)) = 0 \tag{1.1}$$

$$y(0) = y(\pi) = 0 \tag{1.2}$$

$$y\left(\frac{\pi}{2} + 0\right) = \frac{\delta}{\lambda} y\left(\frac{\pi}{2} - 0\right) \tag{1.3}$$

$$y'\left(\frac{\pi}{2} + 0\right) = \frac{\delta}{\lambda} y'\left(\frac{\pi}{2} - 0\right) \tag{1.4}$$

here $q(x)$ and $\Delta(x) \geq 0$ are continuous functions on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ and have finite left right limits at $\frac{\pi}{2}$, if $x \in [0, \frac{\pi}{2})$ then $x - \Delta(x) \geq 0$, if $x \in (\frac{\pi}{2}, \pi]$ then $x - \Delta(x) \geq \frac{\pi}{2}$, λ is a real eigenparameter and $\delta \neq 0$ is arbitrary real number.

Let $\omega_1(x, \lambda)$ be a solution of equation (1.1) on $[0, \frac{\pi}{2})$. After defining this solution, using the transmission conditions (1.3) and (1.4) we can define the solution of equation (1.1) on $(\frac{\pi}{2}, \pi]$ in terms of $\omega_1(x, \lambda)$ as follows:

$$\omega_2\left(\frac{\pi}{2}, \lambda\right) = \frac{\delta}{\lambda}\omega_1\left(\frac{\pi}{2}, \lambda\right), \quad \omega_2'\left(\frac{\pi}{2}, \lambda\right) = \frac{\delta}{\lambda}\omega_1'\left(\frac{\pi}{2}, \lambda\right) \tag{1.5}$$

Consequently, we can define $\omega(x, \lambda)$ on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ as

$$\omega(x, \lambda) = \begin{cases} \omega_1(x, \lambda), & x \in [0, \frac{\pi}{2}) \\ \omega_2(x, \lambda), & x \in (\frac{\pi}{2}, \pi] \end{cases}$$

here $\omega(x, \lambda)$ solves equation (1.1) on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ and satisfies left boundary condition and both transmission conditions (1.3) and (1.4).

Lemma 1.1. *Let $\omega(x, \lambda)$ be a solution of* (1.1) *and $\lambda > 0$. Then $\omega_1(x, \lambda)$ and $\omega_2(x, \lambda)$ are defined as:*

$$\omega_1(x, \lambda) = \sin \lambda x - \frac{1}{\lambda}\int_0^x \sin \lambda(x - \tau) q(\tau)\omega_1(\tau - \Delta(\tau), \lambda) d\tau \tag{1.6}$$

$$\begin{aligned} \omega_2(x, \lambda) = {} & \frac{\delta}{\lambda}\omega_1\left(\frac{\pi}{2}, \lambda\right)\cos \lambda\left(x - \frac{\pi}{2}\right) + \frac{\delta}{\lambda^2}\omega_1'\left(\frac{\pi}{2}, \lambda\right)\sin \lambda\left(x - \frac{\pi}{2}\right) \\ & - \frac{1}{\lambda}\int_{\frac{\pi}{2}}^x \sin \lambda(x - \tau) q(\tau)\omega_2(\tau - \Delta(\tau), \lambda) d\tau \end{aligned} \tag{1.7}$$

Theorem 1.2. *Eigenvalues of the problem* (1.1)-(1.4) *are simple.*

Proof. Let $\tilde{\lambda}$ be an eigenvalue of the problem (1.1)-(1.4) and

$$\tilde{u}(x, \tilde{\lambda}) = \begin{cases} \tilde{u}_1(x, \tilde{\lambda}), & x \in [0, \frac{\pi}{2}) \\ \tilde{u}_2(x, \tilde{\lambda}), & x \in (\frac{\pi}{2}, \pi] \end{cases}$$

be a corresponding eigenfunction. Then from (1.2), the wronskien becomes zero.

$$W[\tilde{u}_1(x, \tilde{\lambda}), \omega(x, \tilde{\lambda})] = \begin{vmatrix} \tilde{u}_1(0, \tilde{\lambda}) & 0 \\ \tilde{u}_1'(0, \tilde{\lambda}) & 1 \end{vmatrix} = 0$$

It means that these two functions corresponding to $\tilde{\lambda}$ are linearly deppendent. Similarly it can be shown that $\tilde{u}_2(x, \tilde{\lambda})$ and $\omega_2(x, \tilde{\lambda})$ are linearly dependent. Therefore eigenvalues are simple. □

Pluging $\omega(x,\lambda)$ into the other boundary condition, characteristic equation is obtained:

$$\begin{aligned} F(\lambda) = & \frac{\delta}{\lambda}\sin\lambda\pi - \frac{\delta}{\lambda^2}\int_0^{\frac{\pi}{2}} \sin\lambda(\pi-\tau)q(\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau \\ & - \frac{1}{\lambda}\int_{\frac{\pi}{2}}^{\pi} \sin\lambda(\pi-\tau)q(\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau = 0 \end{aligned} \tag{1.8}$$

By Theorem 1.2 the the set of eigenvalues of the problem (1.1)-(1.4) and the set of real roots of equation (1.8) are same.

Lemma 1.3. *Let* $q_1 = \int\limits_0^{\frac{\pi}{2}} |q(\tau)|d\tau$ *and* $q_2 = \int\limits_{\frac{\pi}{2}}^{\pi} |q(\tau)|d\tau$

(1) *Let* $\lambda \geq 2q_1$*, then the solution of* (1.6) *satisfies*

$$|\omega_1(x,\lambda)| \leq 2 \tag{1.9}$$

(2) *Let* $\lambda \geq \max\{2q_1, 2q_2\}$*, then the solution of* (1.7) *satisfies*

$$|\omega_2(x,\lambda)| \leq \frac{8\delta}{q_1} \tag{1.10}$$

Proof. Let $B_{1,\lambda} = \max\limits_{x\in[0,\frac{\pi}{2})} |\omega_1(x,\lambda)|$. Then from (1.6),

$$B_{1,\lambda} \leq 1 + \frac{1}{\lambda}q_1 B_{1,\lambda}$$

for $\lambda \geq 2q_1$, it is obvious that $B_{1,\lambda} \leq 2$.

Differentiating (1.6) with respect to x, we obtain

$$\omega_1'(x,\lambda) = \lambda\cos\lambda x - \int\limits_0^x q(\tau)\cos\lambda(x-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau \tag{1.11}$$

From this we obtain

$$|\omega_1'(x,\lambda)| \leq \lambda + 2q_1 \leq 2\lambda \implies \frac{|\omega_1'(x,\lambda)|}{\lambda} \leq 2 \tag{1.12}$$

Let $B_{2,\lambda} = \max\limits_{x\in(\frac{\pi}{2},\pi]} |\omega_2(x,\lambda)|$. Then from (1.7), (1.9) and (1.12)

$$B_{2,\lambda} \leq \frac{4\delta}{\lambda} + \frac{1}{\lambda}q_2 B_{2,\lambda}$$

Therefore for $\lambda \geq \max\{2q_1, 2q_2\}$, (1.10) is obtained. □

Theorem 1.4. *The problem* (1.1)-(1.4) *has an infinite set of positive eigenvalues.*

Proof. Writing (1.6) and(1.11) into (1.8), we obtain:

$$\begin{aligned} & \frac{\delta}{\lambda}\sin\lambda\pi - \frac{\delta}{\lambda^2}\int\limits_0^{\frac{\pi}{2}} q(\tau)\sin\lambda(\pi-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau \\ & - \frac{1}{\lambda}\int\limits_{\frac{\pi}{2}}^{\pi} q(\tau)\sin\lambda(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau = 0 \end{aligned} \tag{1.13}$$

Let λ be sufficiently large, from (1.9) and (1.10), equation (1.13) may be written as

Let λ be sufficiently large, then by (1.9) and (1.10), (1.8) may be written in the form:

$$\lambda \sin \lambda\pi + O(1) = 0 \tag{1.14}$$

Clearly, for large λ, equation (1.14) has infinite roots. □

2. Asymptotic Properties of Eigenvalues and Eigenfunctions

In this section we will investigate the asymptotic expressions of eigenvalues and eigenfunctions. From now on we will assume λ is sufficienly large. On $[0, \frac{\pi}{2})$, from (1.6) and (1.9)

$$\omega_1(x, \lambda) = O(1) \tag{2.1}$$

On $(\frac{\pi}{2}, \pi]$, from (1.7) and (1.10)

$$\omega_2(x, \lambda) = O\left(\frac{1}{\lambda}\right) \tag{2.2}$$

Derivatives of $\omega_1(x, \lambda)$ and $\omega_2(x, \lambda)$ with respect to λ exist and are continuous on $[0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi]$ respectively[Norkin 1972].

Lemma 2.1.

$$\omega'_{1\lambda}(x, \lambda) = O(1), \quad \textit{for } x \in [0, \frac{\pi}{2}) \tag{2.3}$$

$$\omega'_{2\lambda}(x, \lambda) = O\left(\frac{1}{\lambda}\right), \quad \textit{for } x \in (\frac{\pi}{2}, \pi] \tag{2.4}$$

Proof. Differentiating (1.6) with respect to λ and by (2.1)

$$\omega'_{1\lambda}(x, \lambda) = -\frac{1}{\lambda} \int_0^x q(\tau) \sin \lambda(x-\tau)\omega'_{1\lambda}(\tau-\Delta(\tau), \lambda)d\tau + K_1(x, \lambda), \quad |K_1(x, \lambda)| \leq K_1$$

Let $C_{1,\lambda} = \max_{x\in[0,\frac{\pi}{2})} |\omega'_{1\lambda}(x, \lambda)|$. Existence of $C_{1,\lambda}$ follows from continuity of the derivative of $\omega_1(x, \lambda)$. From the equation above we obtain

$$C_{1,\lambda} \leq \frac{1}{\lambda} q_1 C_{1,\lambda} + K_1$$

Therefore for $\lambda \geq 2q_1$, we obtain $C_{1,\lambda} \leq 2K_1$. Hence (2.3) is prooved. Similarly (2.4) can be proved.

□

Theorem 2.2. *Let* $n \in \mathbb{N}$*. For each sufficiently large* n*, there is only one eigenvalue of the problem* (1.1)-(1.4) *in the neighborhood of* n*.*

Proof. First multiply (1.13) with λ^2, then consider the $O(1)$ term

$$-\delta \int_0^{\frac{\pi}{2}} q(\tau) \sin \lambda(\pi - \tau)\omega_1(\tau - \Delta(\tau), \lambda)d\tau - \lambda \int_{\frac{\pi}{2}}^{\pi} q(\tau) \sin \lambda(\pi - \tau)\omega_2(\tau - \Delta(\tau), \lambda)d\tau$$

By (2.1)-(2.4), for large λ this expression has bounded derivative with respect to λ. Clearly (1.14) has infinitely many solutions. We need to show that these solutions are around natural numbers n for suffiently large n. Consider the function

$F(\lambda) = \lambda \sin \lambda\pi + O(1)$. Its derivative $F'(\lambda) = \sin \lambda\pi + \lambda\pi \cos \lambda\pi + O(1) \neq 0$ for λ close to n for sufficiently large n. Hence by Rolle's theorem proof is completed. □

From (1.14)

$$\lambda_n = n + O\left(\frac{1}{n}\right) \tag{2.5}$$

is obtained. Writing (2.5) into (1.6) and (1.7), eigenfunctions of the problem (1.1)-(1.4) are obtained.

$$u_{1n}(x) = \omega_1(x, \lambda_n) = \sin nx + O\left(\frac{1}{n}\right)$$

$$u_{2n}(x) = \omega_2(x, \lambda_n) = \frac{\delta}{n} \sin nx + O\left(\frac{1}{n^2}\right)$$

$$u_n(x) = \begin{cases} \sin nx + O\left(\frac{1}{n}\right), & x \in [0, \frac{\pi}{2}) \\ \frac{\delta}{n} \sin nx + O\left(\frac{1}{n^2}\right), & x \in (\frac{\pi}{2}, \pi] \end{cases} \tag{2.6}$$

3. Sharper Estimates for Eigenvalues and Eigenfunctions

Under additional hypotheses about the functions $q(x)$ and $\Delta(x)$, it is possible to improve the expressions given by (2.5), (2.6).

Lemma 3.1. *Suppose the derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, and have finite limits $q'\left(\frac{\pi}{2} \pm 0\right) = \lim_{x \to \frac{\pi}{2} \pm 0} q'(x)$, $\Delta''\left(\frac{\pi}{2} \pm 0\right) = \lim_{x \to \frac{\pi}{2} \pm 0} \Delta''(x)$, $\Delta'(x) \leq h < 2$ and $\Delta(0) = 0$, $\lim_{x \to \frac{\pi}{2} + 0} \Delta(x) = 0$. Then*

$$\int_0^x \cos \lambda(2\tau - \Delta(\tau)) q(\tau) d\tau = O\left(\frac{1}{\lambda}\right) \tag{3.1}$$

and

$$\int_0^x \sin \lambda(2\tau - \Delta(\tau)) q(\tau) d\tau = O\left(\frac{1}{\lambda}\right) \tag{3.2}$$

Proof. See Lemma III.3.3 in [13] □

Theorem 3.2. *Under the hypoteses of Lemma 3.1 eigenvalues of* (1.1)-(1.4) *problem can be improved as*

$$\lambda_n = n - \frac{B(\pi, n, \Delta(\tau))}{n\pi} + O\left(\frac{1}{n^2}\right) \tag{3.3}$$

Proof. From (2.6), we can write

$$\omega_1(\tau - \Delta(\tau), \lambda) = \sin \lambda(\tau - \Delta(\tau)) + O\left(\frac{1}{\lambda}\right) \tag{3.4}$$

$$\omega_2(\tau - \Delta(\tau), \lambda) = \frac{\delta}{\lambda} \sin \lambda(\tau - \Delta(\tau)) + O\left(\frac{1}{\lambda^2}\right) \tag{3.5}$$

Writing these into the characterisitc equation (1.8), and multiplying by λ^2 equation (1.8) turns into

$$\delta\lambda \sin\lambda\pi - \delta \int_0^{\frac{\pi}{2}} q(\tau)\sin\lambda(\pi-\tau)\left[\sin\lambda(\tau-\Delta(\tau)) + O\left(\frac{1}{\lambda}\right)\right]d\tau$$

$$- \lambda \int_{\frac{\pi}{2}}^{\pi} q(\tau)\sin\lambda(\pi-\tau)\left[\frac{\delta}{\lambda}\sin\lambda(\tau-\Delta(\tau)) + O\left(\frac{1}{\lambda^2}\right)\right]d\tau = 0 \tag{3.6}$$

defining

$$A(x,\lambda,\Delta(\tau)) = \frac{1}{2}\int_0^x q(\tau)\sin\lambda\Delta(\tau)d\tau \tag{3.7}$$

and

$$B(x,\lambda,\Delta(\tau)) = \frac{1}{2}\int_0^x q(\tau)\cos\lambda\Delta(\tau)d\tau \tag{3.8}$$

equation (3.6) simplifies as

$$\lambda\sin\lambda\pi + B(\pi,\lambda,\Delta(\tau))\cos\lambda\pi + A(\pi,\lambda,\Delta(\tau))\sin\lambda\pi + O\left(\frac{1}{\lambda}\right) = 0$$

writing $\lambda = \lambda_n = n + \delta_n$ and for large n

$$\tan\delta_n\pi = -\frac{B(\pi,n,\Delta(\tau))}{n} + O\left(\frac{1}{n^2}\right) \implies \delta_n = -\frac{B(\pi,n,\Delta(\tau))}{n\pi} + O\left(\frac{1}{n^2}\right)$$

Therefore the proof is complete. □

Theorem 3.3. *Under the hypoteses of Lemma 3.1 eigenfunctions u_{1n} and u_{2n} of* (1.1)-(1.4) *can be improved as*

$$u_{1n}(x) = \left(1 - \frac{A(x,n,\Delta(\tau))}{n}\right)\sin nx + \\ + \frac{xB(\pi,n,\Delta(\tau)) - \pi B(x,n,\Delta(\tau))}{n\pi}\cos nx + O\left(\frac{1}{n^2}\right) \tag{3.9}$$

$$u_{2n}(x) = \frac{\delta}{n}\left(1 - \frac{A(x,n,\Delta(\tau))}{n}\right)\sin nx + \\ + \frac{\delta(xB(\pi,n,\Delta(\tau)) - \pi B(x,n,\Delta(\tau)))}{n^2\pi}\cos nx + O\left(\frac{1}{n^3}\right) \tag{3.10}$$

Proof. First we write (3.4) into (1.6) and obtain

$$\omega_1(x,\lambda) = \sin\lambda x + \frac{1}{\lambda}\int_0^x q(\tau)\sin\lambda(x-\tau)\left[\sin\lambda(\tau-\Delta(\tau)) + O\left(\frac{1}{\lambda}\right)\right]d\tau$$

Then using (3.7) and (3.8), this expression becomes

$$\omega_1(x,\lambda) = \sin\lambda x + \frac{1}{\lambda}A(x,\lambda,\Delta(\tau))\sin\lambda x - \frac{1}{\lambda}B(x,\lambda,\Delta(\tau))\cos\lambda x + O\left(\frac{1}{\lambda^2}\right)$$

Now writing (3.3), we obtain the eigenfunction on $[0, \frac{\pi}{2})$ as

$$u_{1n}(x) = \omega_1(x, \lambda_n) = \left(1 - \frac{A(x, n, \Delta(\tau))}{n}\right) \sin nx +$$
$$+ \frac{xB(\pi, n, \Delta(\tau)) - \pi B(x, n, \Delta(\tau))}{n\pi} \cos nx + O\left(\frac{1}{n^2}\right)$$

Now we will improve the eigenfunction on $(\frac{\pi}{2}, \pi]$. In order to do that first we will write (1.9) and (1.12) into (1.10) and then we will use (3.4) and (3.5) together with (3.7) and (3.8) to obtain

$$\omega_2(x, \lambda) = \frac{\delta}{\lambda} \sin \lambda x + \frac{\delta}{\lambda^2} A(x, \lambda, \Delta(\tau)) \sin \lambda x - \frac{\delta}{\lambda^2} B(x, \lambda, \Delta(\tau)) \cos \lambda x + O\left(\frac{1}{\lambda^3}\right)$$

Now writing, (3.3) into this expression we obtain the eigenfunction on $(\frac{\pi}{2}, \pi]$.

$$u_{2n}(x) = \omega_2(x, \lambda_n) = \frac{\delta}{n}\left(1 - \frac{A(x, n, \Delta(\tau))}{n}\right) \sin nx +$$
$$+ \frac{\delta(xB(\pi, n, \Delta(\tau)) - \pi B(x, n, \Delta(\tau)))}{n^2\pi} \cos nx + O\left(\frac{1}{n^3}\right)$$

This completes the proof. □

4. Conclusion

In this paper discontinuous differential equation with retarded argument is studied. In the case of transmission condition that contains eigenparameter, eigenvalues and the corresponding eigenfunctions calculated asymptoticaly as follows:

$$\lambda_n = n - \frac{B(\pi, n, \Delta(\tau))}{n\pi} + O\left(\frac{1}{n^2}\right)$$

$$u_n(x) = \begin{cases} u_{1n}(x), & x \in [0, \frac{\pi}{2}) \\ u_{2n}(x), & x \in (\frac{\pi}{2}, \pi] \end{cases} \tag{4.1}$$

where $u_{1n}(x)$ and $u_{2n}(x)$ are defined by (3.9) and (3.10) respectively.

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