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Analysis of Inverse Coefficient Problem for Euler-Bernoulli Equation with Periodic and Integral Conditions

Research Article

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Abstract

The research investigates the solution of the inverse problem of a linear Euler-Bernoulli equation. While finding the solution, Volterra integral equation theory was used. For this purpose, the existence of this problem, its uniqueness, and its constant dependence on the data are demonstrated using the Fourier methods.
Keywords: Fourier method, Periodic conditions, Volterra theorem, Euler-Bernoulli inverse problem.

1. INTRODUCTION

The Euler-Bernoulli problem is common. This problem has been used for many physics and engineering problems. The investigation of various problems concerning 4^{th} order homogeneous, linear, and quasi-linear equations has been one of the most attractive areas for mathematicians and engineers due to their importance in the solution of several engineering problems. Examples of scientists working on this subject can be given [1-4].

The Euler-Bernoulli problem was first developed by Daniel Bernoulli and Leonard Euler. T(t,x) is the displacement at time t and at position x, o(x) is the bending stiffness, and k(x)>0 is the linear mass. The transverse motion of an unloaded thin beam is represented by the following fourth-order partial differential equation (PDE):

$$k(x)(\partial^2 T)/(\partial t^2)+o(x)(\partial^4 T)/(\partial x^4)=0, t>0, 0<x<L.$$

The vibration, buckling, and dynamic behavior of various building elements widely used in nanotechnology (nanotube, nanofillers for nanomotors, nanobearings, and nanosprings) and population dynamics, thermoelasticity, medical science, electrochemistry, engineering, wide scope, chemical engineering are represented by the Euler-Bernoulli equations [5-6].

In this problem, the periodic [7] and integral conditions were used [8]. The periodic boundary conditions are highly challenging. The periodic boundary conditions arise from many important applications in heat transfer, and life sciences [8].
The paper is organized as follows. In Section 2, the existence, and the uniqueness of the solution of the problem are proved by using the Fourier method and iteration method. In Section 3, the stability of the method for the solution is shown.

2. MATERIAL AND METHODS

The Fourier Method is a successive approximation method to solve the problem. The Fourier method is one of the very common but highly difficult methods to apply. It is used in all partial type differential equations. Volterra theorem is very difficult to satisfy theory. However, they are very successful methods in analytical solutions. Many scientists have used these methods [3-4].

3. STATEMENT OF SOLUTIONS

The inverse coefficient problem

$$\frac{\partial^2 T}{\partial t^2}+\frac{\partial^4 T}{\partial \chi^4}=j(t)f\left(\chi,t\right), \tag{1}$$

the initial condition

$$\begin{aligned} &T(\chi,0)=\varphi(\chi), \chi\in\left[0,\pi\right] \\ &T_t(\chi,0)=\psi(\chi), \chi\in\left[0,\pi\right] \end{aligned} \tag{2}$$

the periodic boundary conditions

$$\begin{aligned} &T(0,t)=T(0,\pi), t\in\left[0,T\right] \\ &T_\chi(0,t)=T_\chi(0,\pi), t\in\left[0,T\right] \\ &T_{\chi\chi}(0,t)=T_{\chi\chi}(0,\pi), t\in\left[0,T\right] \\ &T_{\chi\chi\chi}(0,t)=T_{\chi\chi\chi}(0,\pi), t\in\left[0,T\right] \end{aligned} \tag{3}$$

the integral overdetermination data

$$\delta(t)=\int_0^\pi \chi T(\chi,t)d\chi, t\in\left[0,T\right]. \tag{4}$$

By applying the Fourier Method, the ensuing model is as follows

$$\begin{aligned} T(\chi,t)=&\frac{1}{2}\left[\varphi_0+\psi_0 t+\frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)j(\tau)f(\chi x,\tau)d\chi d\tau\right] \\ &+\sum_{m=1}^{\infty}\left[\varphi_{cm}\cos\left(2m\right)^2 t+\frac{\psi_{cm}}{\pi(2m)^2}\sin\left(2m\right)^2 t\right]\cos 2m\chi \end{aligned}$$

$$+\sum_{m=1}^{\infty}\left[\frac{2}{\pi(2m)^2}\int_0^t\int_0^\pi j(\tau)f(\chi,\tau)\sin(2m)^2(t-\tau)\cos 2m\chi dxd\tau\right]\cos 2m\chi \quad (5)$$
$$+\sum_{m=1}^{\infty}\left[\varphi_{sm}\cos\left(2m\right)^2 t+\frac{\psi_{sm}}{\pi(2m)^2}\sin\left(2m\right)^2 t\right]\sin 2m\chi$$
$$+\sum_{m=1}^{\infty}\left[\frac{2}{\pi(2m)^2}\int_0^t\int_0^\pi j(\tau)f(\chi,\tau)\sin(2m)^2(t-\tau)\sin 2m\chi d\chi d\tau\right]\sin 2m\chi .$$

Definition 3.1. $\{T(\chi,t), j(t)\}$ is called the solution of the inverse problem (1)-(4).
Theorem 3.2. Let below the assumptions be provided

(A1) $\delta(t)\in C^2\left[0,T\right]$
(A2) $\varphi(\chi)\in C^3\left[0,\pi\right],\psi(\chi)\in C^1\left[0,\pi\right]$
(A3) $f(\chi,t)\in C\left(\left[0,\pi\right]x\left[0,t\right]\right)$
(A4) $0\neq\int_0^\pi \chi T(\chi,t)d\chi, t\in\left[0,T\right]$

then the solution of above the problem (1)-(4) has solutions.

Proof. Let the assumptions is verified:
$\varphi(0)=\varphi(\pi),\varphi'(0)=\varphi'(\pi),$
$\psi(0)=\psi(\pi),\psi'(0)=\psi'(\pi),$
$f(0,t)=f(\pi,t).$
Since our series (5) is absolutely convergent, it is also uniformly convergent. Naturally, $T(\chi,t), T_\chi(\chi,t), T_{\chi\chi}(\chi,t), T(\chi,t), T_t(\chi,t), T_{tt}(\chi,t)$ are continuous and absolutely convergent.

According to (5) and (A1) to get:

$$\delta''(t)=\int_0^\pi \chi T_{tt}(\chi,t)dx, t\in\left[0,T\right]. \quad (6)$$

From (5) and (6)

$$j(t)=\frac{\delta''(t)-\pi\sum_{m=1}^{\infty}(2m)\left[\varphi_{cm}\cos\left(2m\right)^2 t+\frac{\psi_{cm}}{\pi(2m)^2}\sin\left(2m\right)^2 t\right]}{\int_0^\pi \chi f(\chi,t)d\chi}$$
$$-\frac{\pi\sum_{m=1}^{\infty}(2m)\int_0^t f_{sm}(\tau)j(\tau)\sin(2m)^2(t-\tau)d\tau}{\int_0^\pi \chi f(\chi,t)d\chi}$$

The equation given below is the second type of Volterra integral equation:

$$F(t)=\frac{\delta''(t)-\pi\sum_{m=1}^{\infty}(2m)^3\varphi_{cm}\cos\left(2m\right)^2 t+\frac{\psi_{cm}}{\pi}2m\sin\left(2m\right)^2 t}{\int_0^{\pi}\chi f(\chi,t)d\chi},$$

$$K(t,\tau)=\frac{-\pi\sum_{m=1}^{\infty}(2m)\int_0^t f_{sm}(\tau)j(\tau)\sin(2m)^2(t-\tau)d\tau}{\int_0^{\pi}\chi f(\chi,t)d\chi} \tag{7}$$

$$j(t)=F(t)+\int_0^t K(t,\tau)j(\tau)d\tau, t\in\left[0,T\right] \tag{8}$$

Let show F(t) and the kernel K(t,τ) are continuous in [0,T] and [0,T]x[0,T] respectively,

$$F(t)=\frac{\delta''(t)-\pi\sum_{m=1}^{\infty}(2m)^3\left(\int_0^{\pi}\varphi(\chi)\sin 2m\chi d\chi\right)\cos\left(2m\right)^2 t}{\int_0^{\pi}\chi f(\chi,t)d\chi}$$

$$+\frac{-\pi\sum_{m=1}^{\infty}\left(\int_0^{\pi}\psi(\chi)\sin 2m\chi d\chi\right)\frac{2m}{\pi}\sin\left(2m\right)^2 t}{\int_0^{\pi}\chi f(\chi,t)d\chi},$$

$$F(t)=\frac{\delta''(t)-\pi\sum_{m=1}^{\infty}(2m)^3\varphi''_{cm}\cos\left(2m\right)^2 t}{\int_0^{\pi}\chi f(\chi,t)d\chi}$$

$$+\frac{-\pi\sum_{m=1}^{\infty}\psi'_{cm}\frac{2m}{\pi}\sin\left(2m\right)^2 t}{\int_0^{\pi}\chi f(\chi,t)d\chi},$$

Taking maximum,

$$\left\| F(t) \right\| \le \frac{2\left(\left\| \delta''(t) \right\| + \pi \sum_{m=1}^{\infty} \left(\left\| \varphi''_{cm} \right\| + \left\| \psi'_{cm} \right\| \right) \right)}{M\pi^2}$$

$$K(t,\tau) = \frac{-2\sum_{m=1}^{\infty}(2m)\int_0^t\int_0^\pi f(\chi,\tau)j(\tau)\sin(2m)^2(t-\tau)d\chi d\tau}{\int_0^\pi \chi f(\chi,t)d\chi}.$$

Taking maximum,

$$\left\| K(t,\tau) \right\| \le \frac{2\left(\sum_{m=1}^{\infty} \left(\left\| (f_{cm})_x \right\| \left\| T \right\| \right) \right)}{M\pi}.$$

According to (A1)-(A2) and the Weierstrass M test, the kernels F(t) and K(t,τ) are continuous at [0,T] and [0,T]x[0T]. According to Volterra's Theorem, the inverse (1)-(4) problem has only unique solution on [0,T].

4. STABILITY FOR THE PROBLEM

Theorem 4.1. If the data (A1)-(A4) is provided, the solution is constantly dependent on the initial data.

Proof. Let us denote

$$\Delta = \{\varphi, \psi, \kappa, f\},$$
$$\overline{\Delta} = \{\overline{\varphi}, \overline{\psi}, \overline{\kappa}, \overline{f}\},$$

$$\left\| \Delta \right\| = \left(\left\| \kappa \right\| + \left\| \varphi \right\| + \left\| \psi \right\| + \left\| f \right\| \right).$$

$$F(t) - \overline{F(t)} = \frac{\left(\delta''(t) - \overline{\delta''(t)} + \pi \sum_{m=1}^{\infty} \left(\varphi''_{cm} - \overline{\varphi''_{cm}} \right) \cos(2m)^2 t \right)}{\int_0^\pi \chi f(\chi,t)d\chi}$$

$$+ \frac{\pi \sum_{m=1}^{\infty} \left(\psi'_{cm} - \overline{\psi'_{cm}} \right) \cos(2m)^2 t}{\int_0^\pi \chi f(\chi,t)d\chi}$$

Taking maximum

$$\left\| F(t) - \overline{F(t)} \right\| \le \frac{2\left(\left\| \delta''(t) - \overline{\delta''(t)} \right\| + \pi \sum_{m=1}^{\infty} \left(\left\| \varphi''_{cm} - \overline{\varphi''_{cm}} \right\| + \left\| \psi'_{cm} - \overline{\psi'_{cm}} \right\| \right) \right)}{M\pi^2} \tag{9}$$

$$K(t,\tau)-\overline{K(t,\tau)}=\frac{\pi\sum_{m=1}^{\infty}\int_0^t\left(f_{cm}\right)_\chi\sin(2m)^2(t-\tau)d\tau}{\int_0^\pi \chi f(\chi,t)d\chi}$$

$$-\frac{\pi\sum_{m=1}^{\infty}\int_0^t\overline{\left(f_{cm}\right)_x}\sin(2m)^2(t-\tau)d\tau}{\int_0^\pi \chi\overline{f(\chi,t)}d\chi}$$

Taking maximum

$$\left\|K(t,\tau)-\overline{K(t,\tau)}\right\|\le\frac{2\left(\left\|f-\overline{f}\right\|\left\|T\right\|+\left\|T\right\|\sum_{m=1}^{\infty}\left(\left\|\left(f_{cm}\right)_\chi-\overline{\left(f_{cm}\right)_\chi}\right\|\right)\right)}{M\pi}\ . \tag{10}$$

Using same estimations,we get

$$\left\|j-\overline{j}\right\|\le\left\|F-\overline{F}\right\|+\left\|T\right\|\left\|K\right\|\left\|j-\overline{j}\right\|+\left\|j\right\|\left\|K-\overline{K}\right\|,$$

From (9)-(10)

$$\left\|j-\overline{j}\right\|\le\frac{2}{\pi^2M(1-\left\|T\right\|\left\|K\right\|)}\left\{\left\|\delta'-\overline{\delta'}\right\|+\pi\sum_{m=1}^{\infty}\left(\left\|\varphi''_{cm}-\overline{\varphi''_{cm}}\right\|+\left\|\psi'_{cm}-\overline{\psi'_{cm}}\right\|\right)\right\}$$

$$+\frac{2\left\|T\right\|\left\|\overline{j}\right\|}{\pi^2M(1-\left\|T\right\|\left\|K\right\|)}\left\|f-\overline{f}\right\|+\frac{2\left\|T\right\|^2\left\|\overline{j}\right\|}{\pi^2M(1-\left\|T\right\|\left\|K\right\|)}\sum_{m=1}^{\infty}\left(\left\|\left(f_{cm}\right)_\chi-\overline{\left(f_{cm}\right)_\chi}\right\|\right)$$

The difference from the (5)

$$T-\overline{T}=\frac{1}{2}\left[\left(\varphi_0-\overline{\varphi_0}\right)+\left(\psi_0-\overline{\psi_0}\right)t+\frac{2}{\pi}\int_0^t(t-\tau)j(\tau)\left(f_0-\overline{f_0}\right)d\tau\right]$$

$$+\sum_{m=1}^{\infty}\left[\left(\varphi_{cm}-\overline{\varphi_{cm}}\right)\cos\left(2m\right)^2t+\frac{\left(\psi_{cm}-\overline{\psi_{cm}}\right)}{\pi(2m)^2}\sin\left(2m\right)^2t\right]\cos 2m\chi$$

$$+\sum_{m=1}^{\infty}\left[\frac{2}{\pi(2m)^2}\int_0^t j(\tau)\left(f_{cm}-\overline{f_{cm}}\right)\sin(2m)^2(t-\tau)d\tau\right]\cos 2m\chi$$

$$+\sum_{m=1}^{\infty}\left[\left(\varphi_{sm}-\overline{\varphi_{sm}}\right)\cos\left(2m\right)^2t+\frac{\left(\psi_{sm}-\overline{\psi_{sm}}\right)}{\pi(2m)^2}\sin\left(2m\right)^2t\right]\sin 2m\chi$$

$$+\sum_{m=1}^{\infty}\left[\frac{2}{\pi(2m)^2}\int_0^t j(\tau)\left(f_{sm}-\overline{f_{sm}}\right)\sin(2m)^2(t-\tau)d\tau\right]\sin 2m\chi$$

Taking maximum

$$\left\|T-\overline{T}\right\| \le \frac{1}{2}\left\|\varphi_0-\overline{\varphi_0}\right\|+\frac{1}{2}\left\|\psi_0-\overline{\psi_0}\right\|\left\|T\right\|$$
$$+\sum_{m=1}^{\infty}\frac{1}{(2m)^2}\left(\left\|\varphi_{cm}-\overline{\varphi_{cm}}\right\|+\left\|\psi_{cm}-\overline{\psi_{cm}}\right\|\right)$$
$$+\sum_{m=1}^{\infty}\frac{1}{(2m)^2}\left(\left\|f_{cm}-\overline{f_{cm}}\right\|\left\|T\right\|\left\|j\right\|+\left\|f_{sm}-\overline{f_{sm}}\right\|\left\|T\right\|\left\|j\right\|\right)$$
$$+\sum_{m=1}^{\infty}\frac{1}{(2m)^2}\left(\left\|\varphi_{sm}-\overline{\varphi_{sm}}\right\|+\left\|\psi_{sm}-\overline{\psi_{sm}}\right\|\right)$$
$$+\sum_{m=1}^{\infty}\frac{1}{(2m)^2}\left(\left\|\overline{f_{cm}}\right\|\left\|T\right\|\left\|j-\overline{j}\right\|+\left\|\overline{f_{sm}}\right\|\left\|T\right\|\left\|j-\overline{j}\right\|\right)$$

Applying Hölder inequality,

$$\left\|T-\overline{T}\right\| \le \frac{1}{2}\left\|\varphi_0-\overline{\varphi_0}\right\|+\frac{1}{2}\left\|\psi_0-\overline{\psi_0}\right\|\left\|T\right\|$$
$$+\frac{\pi^2}{24}\sum_{m=1}^{\infty}\left(\left\|\varphi_{cm}-\overline{\varphi_{cm}}\right\|+\left\|\psi_{cm}-\overline{\psi_{cm}}\right\|\right)$$
$$+\frac{\pi^2}{24}\sum_{m=1}^{\infty}\left\|f_{cm}-\overline{f_{cm}}\right\|\left\|T\right\|\left\|j\right\|+\left\|f_{sm}-\overline{f_{sm}}\right\|\left\|T\right\|\left\|j\right\|$$
$$+\frac{\pi^2}{24}\sum_{m=1}^{\infty}\left(\left\|\varphi_{sm}-\overline{\varphi_{sm}}\right\|+\left\|\psi_{sm}-\overline{\psi_{sm}}\right\|\right)$$
$$+\frac{\pi^2}{24}\sum_{m=1}^{\infty}\left\|\overline{f_{cm}}\right\|\left\|T\right\|\left\|j-\overline{j}\right\|+\left\|\overline{f_{sm}}\right\|\left\|T\right\|\left\|j-\overline{j}\right\|$$
$$\left\|T-\overline{T}\right\| \le M_1\left\|\varphi-\overline{\varphi}\right\|+M_2\left\|\psi-\overline{\psi}\right\|$$
$$+M_3\left\|f-\overline{f}\right\|+M_4\left\|\kappa-\overline{\kappa}\right\|$$
$$\left\|T-\overline{T}\right\| \le M_5\left\|\Delta-\overline{\Delta}\right\|$$

for $\Delta \to \overline{\Delta}$ then $T \to \overline{T}$.

5. CONCLUSION

The solution of the inverse coefficient for the linear Euler equation which involves periodic and integral boundary conditions was examined. Although the problem is ill-posed, the results obtained are quite suitable. This article specifically examines periodic boundary conditions, which are more challenging than local boundary conditions in inverse coefficient problems. In this study, results were obtained using the Fourier method. As a result, the applied methods revealed the analytical solution to this problem.

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