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RESEARCH ARTICLE

ON THE MAXIMUM CIRCULAR INVERSES OF MAXIMUM CIRCLES

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ABSTRACT

In this study, the inverses of maximum circles under the maximum circular inversion are examined. The maximum circular inversion is observed to transform maximum circles to curves consisting of parabolic arcs, line segments, and sometimes rays. It is seen that the image curve occurs in different shapes depending on the relative positions of the inversion circle and the maximum circle. While the images of maximum circles not passing through the inversion center are closed curves, the images of those passing through the inversion center are not closed. Inverse curves have been studied by considering the radii of the maximum circles and the positions of the lines joining their centers and the center of maximum inversion. Classifications of the inverses of maximum circles are presented.

Keywords: Maximum metric, Inversion in maximum circle, Maximum circular inversion, Inverse of maximum circle

1. INTRODUCTION

Inversion defined on a circle is a geometric transformation such that it maps a point to another point in analytical plane. Apollonius of Perga initially presented the concept of inversion with respect to circle in his treatise titled "Plane Loci". Later, during the 1830s, Steiner researched on the inversion with respect to circle. Since then, inversions with respect to circles have been studied and developed. Also, inversions in some curves and surfaces different from circles are defined and studied, [7, 8, 12, 15, 19-20].

In geometry, the measurement of distances between points in analytical plane can be achieved through various distance functions, each offering a lot of insights. The notable ones among these distance functions are Euclidean, taxicab, maximum, Chinese checker and iso-taxicab distances. When distance functions different from Euclidean distance measures are integrated into analytical plane, various non-Euclidean geometries are formed. There are studies that enrich the literature of these geometries, [1-6, 9-11, 13-14, 16-18, 22-23]. The inversions in circles and spheres obtained by using the taxicab distance, the Chinese Checker distance, the maximum distance are defined and developed, [5, 9-11, 15-19, 21, 23].

The maximum distance between two points $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ is defined by $d_M(A_1, A_2) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. This elementary mathematical formulation coupled with its instinctively coherent explication endow it with utility of significance in the realm of computer science and engineering implementations. The maximum plane is the analytical plane endowed with the maximum distance and symbolized by $\mathbb{R}^2_{\mathbf{M}}$. It is almost the same as the Euclidean plane except the distance function.

In this article, the inverses of maximum circles under the maximum circular inversion have been examined. The maximum circular inversion is observed to transform maximum circles to curves

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consisting of parabolic arcs, line segments, and sometimes rays. It is seen that the image curve occurs in different shapes depending on the relative positions of the inversion circle and the maximum circle. While the images of maximum circles not passing through the inversion center are closed curves, the images of those passing through the inversion center are not closed. Inverse curves have been studied by considering the radii of the maximum circles and the positions of the lines joining their centers and the center of maximum inversion. Classifications of the images are presented.

This study is organized as follows: In section 2, some concepts used in this work are mentioned. In section 3, the inversion map defined in the maximum circle and its some properties are given. In section 4, the images of the maximum circles under the inversion with respect to maximum circle are examined and the results are presented.

2. PRELIMINARIES

It is clear from the distance d_M that the maximum distance between two points $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ is equal to the greatest of the Euclidean lengths of the line segments parallel to the coordinates axes in the right triangle with the hypotenuse A_1A_2.

Krause in [14] gave the classification of lines according to their slopes as follows: Let m be the slope of the line ℓ in plane. The line ℓ is called the steep line, the gradual line and the separator line in the cases of $|m|>1$, $|m|<1$ and $|m|=1$, respectively. In the special cases that the line ℓ is parallel to x-axis or y-axis, ℓ is named as the horizontal line or the vertical line, respectively. This classification is also valid in $\mathbb{R}^2_{\mathrm{M}}$.

The maximum circle centered at $M = (m_1, m_2)$ with the radius r is the set of points (x, y) satisfying the equation

$$\max\{|x - m_1|, |y - m_2|\} = r.$$

Every Euclidean translation preserves the maximum distance. So, it is an isometry in the maximum plane. Reflections in the coordinate axes and the separator lines through the origin and rotations about the origin by integer multiples of $\frac{\pi}{2}$ are isometries in maximum plane, [22].

Maximum distance from the point $A = (x_1, y_1)$ to the line ℓ with the equation $ax + by + c = 0$ is

$$d_M(A, \ell) = \frac{|ax_1 + by_1 + c|}{|a| + |b|},$$

[22].

3. THE INVERSION MAP IN THE MAXIMUM CIRCLE

The inversion in the maximum circle can be defined as an analog of the inversion in the Euclidean circle. While the inversion leaves the points on the inversion circle fixed, it maps points close to the center of the inversion circle to points far from the center of the inversion circle and conversely. However, under the inversion transformation, the center of the inversion circle has no image, and no point is mapped to the center of the inversion circle. Thus, by adding only one point O_∞ called the "ideal point" or "point at infinity" to the maximum plane, the inversion is defined at the center of the inversion circle and becomes a one-to-one map. According to this:

Consider the maximum circle, denoted by $\mathcal{C}$, with the center O and the radius r in $\mathbb{R}^2_M$.The inversion in the maximum circle $\mathcal{C}$ is the mapping $I_{\mathcal{C}}$ of $\mathbb{R}^2_{\mathrm{M}} \cup \{O_\infty\}$ defined by $I_{\mathcal{C}}(O) = O_\infty$, $I_{\mathcal{C}}(O_\infty) = O$ and $I_{\mathcal{C}}(X) = X'$ for any point X different from O and O_∞, where the point X' lies on the ray OX and $d_M(O, X). d_M(O, X') = r^2$. $\mathcal{C}$ is termed the circle of the maximum circle inversion; O denotes the center of the maximum circle inversion; r signifies the radius of the maximum circle inversion; and the point X' is called the maximum circle inverse, briefly the inverse or the image of the point X under the map $I_{\mathcal{C}}$. Also, the maximum circle inversion has the property $I_{\mathcal{C}}{}^2(X) = X$.

Theorem 3.1. Any point (except the inversion center) inside of inversion circle is transformed to a point outside of it under the maximum circular inversion, and conversely, [9, 23].

Teorem 3.2. Let $\mathcal{C}$ be the maximum circle centered at the point $O = (0,0)$ with the radius r. If the points $P = (x, y)$ and $P' = (x', y')$ are inverses of each other with respect to the maximum circular inversion $I_{\mathcal{C}}$, then the following equality between the coordinates of P and P' holds

$$(x', y') = \frac{r^2}{(max\{|x|, |y|\})^2}(x, y),$$

[9, 23].

Corollary 3.3. Let $\mathcal{C}$ be the maximum circle centered at the point $O = (a, b)$ with the radius r. If the points $P = (x, y)$ and $P' = (x', y')$ are inverses of each other with respect to the maximum circular inversion $I_{\mathcal{C}}$ where $\mathrm{P} \neq \mathrm{O}$, then the following equalities between the coordinates of P and P' are valid

$$x' = a + \frac{r^2(x-a)}{(max\{|x-a|, |y-b|\})^2}$$
$$y' = b + \frac{r^2(y-b)}{(max\{|x-a|, |y-b|\})^2},$$

[9, 23].

4. IMAGES OF MAXIMUM CIRCLES UNDER THE MAXIMUM CIRCULAR INVERSION

It is well known that the inversion in Euclidean circle inverts circles not passing through the center of inversion to circles. Also, if these circles are completely inside of inversion circle, inverse images of them are outside of inversion circle and vice-versa. Besides, under the inversion in Euclidean circle, the images of circles passing through the inversion center are lines not passing through the inversion center. Additionally, if a circle is orthogonal to the inversion circle, then it is invariant under the inversion in Euclidean circle. In this section, inverses of the maximum circles under maximum circular inversion are studied and outcomes are given.

It is clear from the definition of maximum circular inversion that it leaves the inversion circle fixed. Images of other maximum circles are examined by considering the radii of the maximum circles and the positions of the lines joining their centers and the center of inversion. Firstly, the images of circles with the same center as the inversion circle are examined. And the following theorem shows that maximum circular inversion maps these to other maximum circles with the inversion center.

Since translations are isometries in maximum plane, no generality is lost to take the center of maximum circular inversion at the origin. Therefore, throughout this study, the center O is the origin unless otherwise stated.

Theorem 4.1. The maximum circular inversion transforms the concentric maximum circles centered on the inversion center to another concentric maximum circle.

Proof. Suppose that $I_{\mathcal{C}}$ is the inversion in the maximum circle $\mathcal{C}$ with the center O and the radius r. Let $\mathcal{K}$ denote a maximum circle with the center O and the radius r'. The image of $\mathcal{K}$ under $I_{\mathcal{C}}$ satisfies the equality

$$\max\{|\mathrm{x}'|, |\mathrm{y}'|\} = \frac{\mathrm{r}^2}{\mathrm{r}'}.$$

This means the maximum circle centered at O with radius $\frac{r^2}{r'}$, (Figure 1).

Moreover, if $\mathcal{K}$ is completely inside $\mathcal{C}$, then the maximum distance from the inversion center to the point X on $\mathcal{K}$ is less than r. From the definition of inversion, the maximum distance from the inversion center to the maximum circular inverse of the point X is greater than r. So, it is achieved that if the maximum circles sharing the same center with the inversion circle are completely inside the inversion circle, their maximum circular inverses are completely outside the inversion circle and vice-versa.

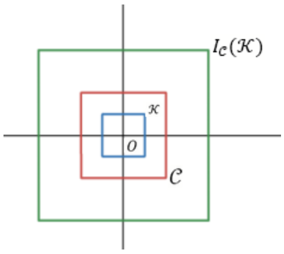

Figure 1. Maximum circular inverse of the concentric maximum circle centered on the inversion center

It is observed in the following theorems that the images of all other maximum circles are the curves other than the maximum circles and lines. Notice also that the images have different shapes and properties according to the positions of the maximum circles.

Theorem 4.2. The inversion in the maximum circle centered at the origin maps the maximum circle with the center on the coordinate axis to the curve having one of the following properties:

i) If the radius of the maximum circle is less than or equal to the maximum distance from its center to the separator lines passing through the inversion center, the image is a four-part closed curve such that two parts of the image are parallel line segments and the other two parts are parabolic arcs symmetrical about the coordinate axis, (Figure 2.a).

ii) If the radius of the maximum circle is greater than the maximum distance from its center to the separator lines passing through the inversion center, the image is an eight-part closed curve such that four parts of the image are four line segments parallel two by two and the others are four parabolic arcs symmetrical two by two about the coordinate axis, (Figure 2.b).

iii) If the radius of the maximum circle is equal to the maximum distance between its center and the inversion center, the image is the curve consisting of two rays, three line segments and two parabolic arcs, (Figure 2.c).

Proof. Suppose that $I_{\mathcal{C}}$ is the inversion in the maximum circle $\mathcal{C}$ with the center $O = (0,0)$ and the radius r. Let K and K′ denote a maximum circle centered at M with the radius r' and its maximum inverse curve with respect to $I_{\mathcal{C}}$, respectively. Assume, without loss of generality, that the center M of K is on x-axis and its abcissa is x_0, where $x_0 > 0$. The maximum distance from the center M to the separator lines passing through O is $\frac{x_0}{2}$. And the maximum distance between the points M and O is equal to x_0. The maximum circle K is the set of points satisfying the equality

$$\max\{|x - x_0|, |y|\} = r'.$$

By applying the inversion map $I_{\mathcal{C}}$ to K, the points on K′ hold the equality

$$\max\{\left|x' - \frac{x_0}{r^2}(\max\{|x'|, |y'|\})^2\right|, |y'|\} = \frac{r'}{r^2}(\max\{|x'|, |y'|\})^2.$$

Firstly, in the case that $|x'| > |y'|$ and $\left|x' - \frac{x_0}{r^2}(x')^2\right| > |y'|$, one gets immediately the equality $\left|x' - \frac{x_0}{r^2}(x')^2\right| = \frac{r'}{r^2}(x')^2$. In the case $x' < 0$, $x' = \frac{r^2}{x_0 - r'}$. When r' is greater than the maximum distance between O and M, the line segment $x' = \frac{r^2}{x_0 - r'}, \frac{r^2}{x_0 - r'} \leq y' \leq -\frac{r^2}{x_0 - r'}$ is on K′. If $0 \leq x' < \frac{r^2}{x_0}$, the line segment $x' = \frac{r^2}{x_0 + r'}, -\left(\frac{r}{x_0 + r'}\right)^2 r' \leq y' \leq \left(\frac{r}{x_0 + r'}\right)^2 r'$ is on the image. In the case $x' \geq \frac{r^2}{x_0}$, the line segment $x' = \frac{r^2}{x_0 - r'}, -\left(\frac{r}{x_0 - r'}\right)^2 r' \leq y' \leq \left(\frac{r}{x_0 - r'}\right)^2 r'$ where $r' \leq \frac{x_0}{2}$ or the line segment $x' = \frac{r^2}{x_0 - r'}, -\frac{r^2}{x_0 - r'} \leq y' \leq \frac{r^2}{x_0 - r'}$ where $\frac{x_0}{2} < r' < x_0$ is on K′. In the case that $|x'| > |y'|$ and $\left|x' - \frac{x_0}{r^2}(x')^2\right| \leq |y'|$, the parabola arcs $|y'| = \frac{r'}{r^2}(x')^2, \frac{r^2}{x_0 + r'} \leq x' \leq \frac{r^2}{r'}$ for $r' > \frac{x_0}{2}$ and $|y'| = \frac{r'}{r^2}(x')^2, \frac{r^2}{x_0 + r'} \leq x' \leq \frac{r^2}{x_0 - r'}$ for $r' \leq \frac{x_0}{2}$ are in the solution region. If both $|x'| \leq |y'|$ and $\left|x' - \frac{x_0}{r^2}(y')^2\right| > |y'|$, then $\left|x' - \frac{x_0}{r^2}(y')^2\right| = \frac{r'}{r^2}(y')^2$. For the case $x' > \frac{x_0}{r^2}(y')^2$, the parabola arc $x' = \frac{x_0 + r'}{r^2}(y')^2, 0 \leq x' \leq \frac{r^2}{x_0 + r'}$ is not in the region determined by the inequality $\left|x' - \frac{x_0}{r^2}(y')^2\right| > |y'|$. For the other case, the equation $x' = \frac{x_0 - r'}{r^2}(y')^2$ is obtained. In special case of $r' = x_0$, the rays $x = 0, |y'| \geq \frac{r^2}{x_0}$ occur. If $r' < \frac{x_0}{2}$, then there is no solution. When $r' \geq \frac{x_0}{2}$, the parabola arc $x' = \frac{x_0 - r'}{r^2}(y')^2, (x_0 - r')\left(\frac{r}{r'}\right)^2 \leq x' \leq \frac{r^2}{x_0 - r'}$ is on the image. In the last case that $|x'| \leq |y'|$ and $\left|x' - \frac{x_0}{r^2}(y')^2\right| \leq |y'|$, then $|y'| = \frac{r'}{r^2}(y')^2$. When $r' \geq \frac{x_0}{2}$, the line segments $y' = \pm\frac{r^2}{r'}, (x_0 - r')\left(\frac{r}{r'}\right)^2 \leq x' \leq \frac{r^2}{r'}$ are on the image. By using the isometric reflections in the lines passing through the inversion center, It is immediately obtained similar results as the above analysis when $x_0 < 0$ or M on the y-axis.

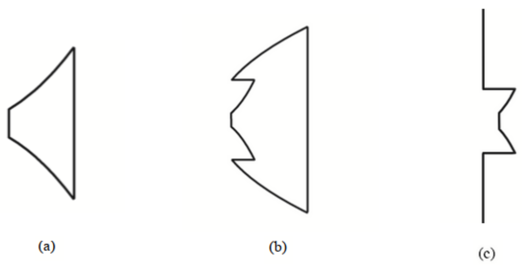

Figure 2. The image of the maximum circle whose the center is on the coordinate axis through the inversion center

Also, the images of line segments were examined according to the fact that their endpoints lie in regions determined by the separator lines passing through the inversion center as detailed in [11]. In the case that the radius of the maximum circle is less than or equal to the maximum distance from its center to the separator lines passing through the inversion center, all vertices and sides of maximum circle are in in the same region. Since the inverse images of two sides perpendicular to the coordinate axis in the region are two line segments parallel to them and the inverse images of the other two are parabola arc, the maximum circular inversion maps the maximum circle to a four-part closed curve, (Figure 2.a). In the case of ii, the vertices on one side of the maximum circle are in the alternate regions and the image of that side is a three-part curve consisting of a line segment parallel to it and two parabolic arcs. The vertices on two sides are in neighboring regions and the images of these sides are two-part curves comprising a line segment and a parabola arc. The vertices on last side are in same region and its image is a line segment parallel to it. Therefore, the maximum circular inversion maps the maximum circles in case (ii) to the eight-part closed curves as shown in Figure 2.b. If the radius of the maximum circle is equal to the maximum distance between its center and the inversion center, the side whose vertices are in alterne regions passes through the inversion center. Then, its image consists of two rays. The properties of the other sides are the same. So, in this case the image of maximum circle under the maximum circular inversion is a seven-part open curve, (Figure 2.c).

The following corollary is a result of theorem 4.2 for the case that the inversion center is not the origin:

Corollary 4.3. The maximum circular inversion maps the maximum circle with the center on the horizontal line or the vertical line passing through the inversion center to the curve having one of the following properties:

i) If the radius of the maximum circle is less than or equal to the maximum distance from its center to the separator lines passing through the inversion center, its image is a four-part closed curve such that two parts of the image are parallel line segments and the other two parts are the symmetric parabolic arcs about the line through centers,

ii) If the radius of the maximum circle is greater than the maximum distance from its center to the separator lines passing through the inversion center, the image is a eight-part closed curve such that four parts of the image are four line segments parallel two by two and the others are four parabolic arcs symmetrical two by two about the line through centers,

iii) If the radius of the maximum circle is equal to the maximum distance between its center and the inversion center, the image curve consists of two rays, three line segments and two parabolic arcs, (Figure 3).

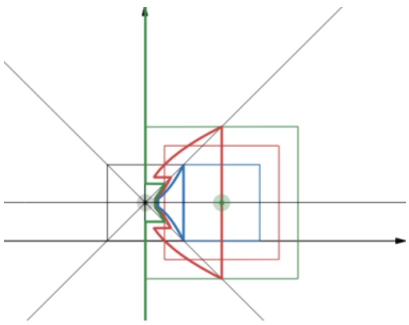

Figure 3. The images of the maximum circles with the centers on the horizontal line through the inversion center

Theorem 4.4. The maximum circular inversion maps the maximum circle with the center on the separator line passing through the inversion center to the curve having one of the following properties:

i) If the radius of the maximum circle is equal to the maximum distance between its center and the inversion center, then the image is a four-part open curve consisting of two rays and two line segments,

ii) If the radius of the maximum circle is less than the maximum distance between its center and the inversion center, then the image is a four-part close curve consisting of two parabola arcs and two line segments,

iii) If the radius of the maximum circle is greater than the maximum distance between its center and the inversion center, then the image is a six-part close curve consisting of two parabola arcs and four line segments.

Proof. Suppose that $I_{\mathcal{C}}$ is the inversion in the maximum circle $\mathcal{C}$ with the center $O = (0,0)$ and the radius r. Let K and K′ denote a maximum circle centered at M with the radius r' and its maximum inverse with respect to $I_{\mathcal{C}}$, respectively. Assume, without loss of generality, that the center M of K is on the separator line $y = \mathrm{x}$ and its abscissa is x_0, where $x_0 > 0$. The maximum distance from the center M to the other separator line passing through O and the maximum distance between the points M and O are equal to x_0. $I_{\mathcal{C}}$ maps the points on K to the points satisfying the equality

$$\max\{\left|\mathrm{x}' - \frac{x_0}{r^2}(\max\{|\mathrm{x}'|,|\mathrm{y}'|\})^2\right|, \left|\mathrm{y}' - \frac{x_0}{r^2}(\max\{|\mathrm{x}'|,|\mathrm{y}'|\})^2\right|\} = \frac{\mathrm{r}'}{r^2}(\max\{|\mathrm{x}'|,|\mathrm{y}'|\})^2.$$

In the case that $|\mathrm{x}'| > |\mathrm{y}'|$ and $\left|\mathrm{x}' - \frac{x_0}{r^2}(\mathrm{x}')^2\right| > \left|\mathrm{y}' - \frac{x_0}{r^2}(\mathrm{x}')^2\right|$, the equality $|r^2\mathrm{x}' - x_0(\mathrm{x}')^2| = \mathrm{r}'(\mathrm{x}')^2$ is obtained. For $\mathrm{x}' < 0$, the line segment $\mathrm{x}' = \frac{r^2}{x_0 - \mathrm{r}'}$, $\frac{r^2}{x_0 - \mathrm{r}'} \leq y \leq -\frac{r^2}{x_0 - \mathrm{r}'}$ where $x_0 < r'$ lies on the locus given by the above equation. For $0 \leq \mathrm{x}' < \frac{r^2}{x_0}$, the line segment $\mathrm{x}' = \frac{r^2}{x_0 + \mathrm{r}'}$, $(x_0 - \mathrm{r}')\left(\frac{r}{x_0 + \mathrm{r}'}\right)^2 \leq y' \leq \frac{r^2}{x_0 + \mathrm{r}'}$ is on the image. And the solution is not sought for the case $\mathrm{x}' \geq \frac{r^2}{x_0}$. Considering the case that $|\mathrm{x}'| > |\mathrm{y}'|$ and $\left|\mathrm{x}' - \frac{x_0}{r^2}(\mathrm{x}')^2\right| \leq \left|\mathrm{y}' - \frac{x_0}{r^2}(\mathrm{x}')^2\right|$, the parabola arc $\mathrm{y}' = \frac{x_0 - \mathrm{r}'}{r^2}(\mathrm{x}')^2, \frac{r^2}{x_0 + \mathrm{r}'} \leq x' \leq \frac{r^2}{|x_0 - \mathrm{r}'|}$, where $x_0 \neq r'$ or the ray $\mathrm{y}' = 0$, $\frac{r^2}{2x_0} \leq \mathrm{x}'$, where $x_0 = r'$ is obtained. If both $|\mathrm{x}'| \leq |\mathrm{y}'|$ and $\left|\mathrm{x}' - \frac{x_0}{r^2}(\mathrm{y}')^2\right| > \left|\mathrm{y}' - \frac{x_0}{r^2}(\mathrm{y}')^2\right|$, then $|r^2\mathrm{x}' - x_0(\mathrm{y}')^2| = \mathrm{r}'(\mathrm{y}')^2$. For $\mathrm{x}' < \frac{x_0}{r^2}(\mathrm{y}')^2$, one gets that the parabola arc $\mathrm{x}' = \frac{x_0 - \mathrm{r}'}{r^2}(\mathrm{y}')^2, \frac{r^2}{x_0 + \mathrm{r}'} \leq y' \leq \frac{r^2}{|x_0 - \mathrm{r}'|}$ where $x_0 \neq r'$ and the ray $\mathrm{x}' = 0$, $\frac{r^2}{2x_0} \leq \mathrm{y}'$, where $x_0 = r'$. Considering the case that $|\mathrm{x}'| \leq |\mathrm{y}'|$ and $\left|\mathrm{x}' - \frac{x_0}{r^2}(\mathrm{y}')^2\right| \leq \left|\mathrm{y}' - \frac{x_0}{r^2}(\mathrm{y}')^2\right|$, then $|r^2\mathrm{y}' - x_0(\mathrm{y}')^2| = \mathrm{r}'(\mathrm{y}')^2$. This equality signifies the line segments corresponding to $\mathrm{y}' = \frac{r^2}{x_0 - \mathrm{r}'}$, $\frac{r^2}{x_0 - \mathrm{r}'} \leq \mathrm{x}' \leq -\frac{r^2}{x_0 - \mathrm{r}'}$ where $x_0 < r$ in the case of $\mathrm{y}' < 0$ and $\mathrm{y}' = \frac{r^2}{x_0 + \mathrm{r}'}$, $(x_0 - \mathrm{r}')\left(\frac{r}{x_0 + \mathrm{r}'}\right)^2 \leq \mathrm{x}' \leq \frac{r^2}{x_0 + \mathrm{r}'}$ in the case of $0 \leq \mathrm{y}' < \frac{r^2}{x_0}$. The case of $x_0 < 0$ or M on the separator line $y = -x$ is similar to the above analysis.

In the case that the radius of the maximum circle is less than the maximum distance between its center and the inversion center, two sides of the maximum circle are in the same region while the other two sides are in neighboring region. The image of the side perpendicular to the coordinate axis in the region is a line segment, while the image of the side parallel to the coordinate axis is a parabolic arc. Thus, the image of the maximum circle consists of the line segments $\mathrm{x}' = \frac{r^2}{x_0 + \mathrm{r}'}$, $(x_0 - \mathrm{r}')\left(\frac{r}{x_0 + \mathrm{r}'}\right)^2 \leq y' \leq \frac{r^2}{x_0 + \mathrm{r}'}$; $\mathrm{y}' = \frac{r^2}{x_0 + \mathrm{r}'}$, $(x_0 - \mathrm{r}')\left(\frac{r}{x_0 + \mathrm{r}'}\right)^2 \leq \mathrm{x}' \leq \frac{r^2}{x_0 + \mathrm{r}'}$, and the parabolic arcs $\mathrm{y}' = \frac{x_0 - \mathrm{r}'}{r^2}(\mathrm{x}')^2, \frac{r^2}{x_0 + \mathrm{r}'} \leq x' \leq \frac{r^2}{|x_0 - \mathrm{r}'|}$, $\mathrm{x}' = \frac{x_0 - \mathrm{r}'}{r^2}(\mathrm{y}')^2, \frac{r^2}{x_0 + \mathrm{r}'} \leq y' \leq \frac{r^2}{|x_0 - \mathrm{r}'|}$, as shown in Figure 4.a.

In the case that the radius of the maximum circle is equal to the maximum distance between its center and the inversion center, two sides of the maximum circle pass through the inversion center. The images of these sides are two rays $\mathrm{x}' = 0$, $\frac{r^2}{2x_0} \leq \mathrm{y}'$ and $\mathrm{y}' = 0$, $\frac{r^2}{2x_0} \leq \mathrm{x}'$. The images of the other sides are the line segments $\mathrm{x}' = \frac{r^2}{x_0 + \mathrm{r}'}$, $(x_0 - \mathrm{r}')\left(\frac{r}{x_0 + \mathrm{r}'}\right)^2 \leq y' \leq \frac{r^2}{x_0 + \mathrm{r}'}$; $\mathrm{y}' = \frac{r^2}{x_0 + \mathrm{r}'}$, $(x_0 - \mathrm{r}')\left(\frac{r}{x_0 + \mathrm{r}'}\right)^2 \leq \mathrm{x}' \leq \frac{r^2}{x_0 + \mathrm{r}'}$, as shown in Figure 4.c.

When the radius of the maximum circle is greater than the maximum distance between its center and the inversion center, the vertices on the two sides of the maximum circle are in the neighboring regions, so the image of each of these sides consists of a line segment and a parabolic arc. Each of the other two sides transforms to a line segment parallel to itself under inversion. Thus, the image of the maximum circle consists of the line segments $x' = \frac{r^2}{x_0-r'}, \frac{r^2}{x_0-r'} \leq y' \leq -\frac{r^2}{x_0-r'}$, $x' = \frac{r^2}{x_0+r'}, (x_0 - r')\left(\frac{r}{x_0+r'}\right)^2 \leq y' \leq \frac{r^2}{x_0+r'}$; $y' = \frac{r^2}{x_0-r'}, \frac{r^2}{x_0-r'} \leq x' \leq -\frac{r^2}{x_0-r'}$ and $y' = \frac{r^2}{x_0+r'}, (x_0 - r')\left(\frac{r}{x_0+r'}\right)^2 \leq x' \leq \frac{r^2}{x_0+r'}$, and the parabolic arcs $y' = \frac{x_0-r'}{r^2}(x')^2, \frac{r^2}{x_0+r'} \leq x' \leq \frac{r^2}{|x_0-r'|}$, $x' = \frac{x_0-r'}{r^2}(y')^2, \frac{r^2}{x_0+r'} \leq y' \leq \frac{r^2}{|x_0-r'|}$, as shown in Figure 4.b.

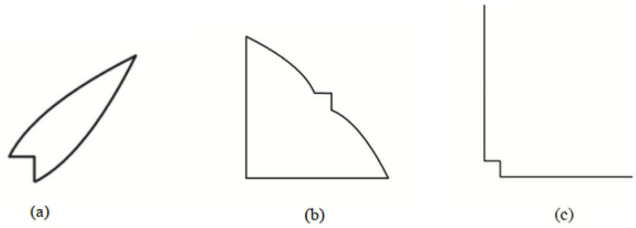

Figure 4. The image of the maximum circle with the center on the separator line through the inversion center

Theorem 4.5. The inversion in maximum circle maps the maximum circle with the center on a gradual line passing through the inversion center to the curve having one of the following properties:

i) If the radius of the maximum circle is less than the maximum distances from its center to the separator lines passing through the inversion center, then the image is a four-part closed curve such that two parts of the image are parallel line segments and the others are parabola arcs, (Figure 5.a),

ii) If the radius of the maximum circle is a value between the maximum distances from its center to the separator lines passing through the inversion center, then the image is a six-part closed curve consisting of three line segments and three parabola arcs, (Figure 5.b),

iii) If the radius of the maximum circle is greater than the maximum distances from its center to the separator lines passing through the inversion center, then the image is a eight-part closed curve consisting of four line segments and four parabola arcs, (Figure 5.c),

iv) If the radius of the maximum circle is equal to the maximum distance between its center and the inversion center, then the image is a seven-part curve consisting of two rays, three line segments and two parabola arcs, (Figure 5.d).

Proof. Suppose that $I_{\mathcal{C}}$ is the inversion in the maximum circle $\mathcal{C}$ with the center $O = (0,0)$ and the radius r. Let K and K′ denote a maximum circle centered at M with the radius r' and its maximum inverse with respect to $I_{\mathcal{C}}$, respectively. Assume, without loss of generality, that the coordinate of the center M is (x_0, y_0), where $0 < y_0 < x_0$. The maximum distances from the center M to the separator lines passing through the inversion center are $\frac{x_0-y_0}{2}$ and $\frac{x_0+y_0}{2}$. And the maximum distance between the center M and the inversion center is x_0. The points on K′ satisfy the equality

$$\max\{|r^2x' - x_0(\max\{|x'|, |y'|\})^2|, |r^2y - y_0(\max\{|x'|, |y'|\})^2|\} = r'(\max\{|x'|, |y'|\})^2.$$

Considering the case that $|x'| > |y'|$ and $\left|x' - \frac{x_0}{r^2}(x')^2\right| > \left|y' - \frac{y_0}{r^2}(x')^2\right|$, it is obtained the equality

$$|r^2x' - x_0(x')^2| = r'(x')^2.$$

Firstly, suppose that $\mathrm{x}' < 0$, then the line segment with the equation $\mathrm{x}' = \frac{r^2}{x_0-\mathrm{r}'}$, $\frac{r^2}{x_0-\mathrm{r}'} \le y' \le -\frac{r^2}{x_0-\mathrm{r}'}$ where $x_0 < r'$ yields the above equality. If $0 \le \mathrm{x}' < \frac{r^2}{x_0}$, then it is obtained that the line segment $\mathrm{x}' = \frac{r^2}{x_0+\mathrm{r}'}$, $(y_0 - \mathrm{r}')\left(\frac{r}{x_0+\mathrm{r}'}\right)^2 \le y' \le (y_0 + \mathrm{r}')\left(\frac{r}{x_0+\mathrm{r}'}\right)^2$ is on the image. When $\mathrm{x}' \ge \frac{r^2}{x_0}$, the line segment $\mathrm{x}' = \frac{r^2}{x_0-\mathrm{r}'}$ where $x_0 > r'$ forms the part of K′. Depending on the values taken by the radius r' of the maximum circle K, the coordinates of the endpoints of the line segment change. If r' is a value between $\frac{x_0+y_0}{2}$ and x_0, then the endpoints of the line segment lie on the separator lines passing through the inversion center and the line segment is $\mathrm{x}' = \frac{r^2}{x_0-\mathrm{r}'}, -\frac{r^2}{x_0-\mathrm{r}'} \le y' \le \frac{r^2}{x_0-\mathrm{r}'}$. If r' is a value between $\frac{x_0-y_0}{2}$ and $\frac{x_0+y_0}{2}$, the endpoints of the line segment lie on the parabola and the separator line passing through the inversion center and the line segment is $\mathrm{x}' = \frac{r^2}{x_0-\mathrm{r}'}, \frac{r^2(y_0-\mathrm{r}')}{(x_0-\mathrm{r}')^2} \le y' \le \frac{r^2}{x_0-\mathrm{r}'}$. If r' is less than the value $\frac{x_0-y_0}{2}$, then the endpoints of the line segment lie on parabolas passing through the inversion center and the line segment is $\mathrm{x}' = \frac{r^2}{x_0-\mathrm{r}'}, \frac{r^2(y_0-\mathrm{r}')}{(x_0-\mathrm{r}')^2} \le y' \le \frac{r^2(y_0+\mathrm{r}')}{(x_0-\mathrm{r}')^2}$.

If both $|\mathrm{x}'| > |\mathrm{y}'|$ and $\left|\mathrm{x}' - \frac{x_0}{r^2}(\mathrm{x}')^2\right| \le \left|\mathrm{y}' - \frac{y_0}{r^2}(\mathrm{x}')^2\right|$, then $|r^2\mathrm{y}' - y_0(\mathrm{x}')^2| = \mathrm{r}'(\mathrm{x}')^2$.
Suppose that $\mathrm{y}' < \frac{y_0}{r^2}(\mathrm{x}')^2$, one gets that the parabola arc $\mathrm{y}' = \frac{(y_0-\mathrm{r}')}{r^2}(\mathrm{x}')^2, \frac{r^2}{x_0+\mathrm{r}'} \le x' \le \frac{r^2}{x_0-\mathrm{r}'}$ where $\mathrm{r}' \le \frac{x_0+y_0}{2}$ or $\frac{r^2}{x_0+\mathrm{r}'} \le x' \le -\frac{r^2}{y_0-\mathrm{r}'}$ where $\mathrm{r}' > \frac{x_0+y_0}{2}$ on the image. In the case of $\mathrm{y}' \ge \frac{y_0}{r^2}(\mathrm{x}')^2$, the equality $\mathrm{y}' = \frac{(y_0+\mathrm{r}')}{r^2}(\mathrm{x}')^2$ is obtained. If r' is less than the value $\frac{x_0-y_0}{2}$, then the endpoints of the parabola arc with the equation $\mathrm{y}' = \frac{(y_0+\mathrm{r}')}{r^2}(\mathrm{x}')^2$ lie on the parabolas passing through the inversion center such that this arc on K′ is $\mathrm{y}' = \frac{(y_0+\mathrm{r}')}{r^2}(\mathrm{x}')^2, \frac{r^2}{x_0+\mathrm{r}'} \le x' \le \frac{r^2}{x_0-\mathrm{r}'}$. If r' is greater than the value $\frac{x_0-y_0}{2}$, then the endpoints of the parabola arc lie on the parabola and the separator line passing through the inversion center such that it is $\mathrm{y}' = \frac{(y_0+\mathrm{r}')}{r^2}(\mathrm{x}')^2, \frac{r^2}{x_0+\mathrm{r}'} \le x' \le \frac{r^2}{y_0+\mathrm{r}'}$.

Considering the inequalities $|\mathrm{x}'| \le |\mathrm{y}'|$ and $\left|\mathrm{x}' - \frac{x_0}{r^2}(\mathrm{y}')^2\right| > \left|\mathrm{y}' - \frac{y_0}{r^2}(\mathrm{y}')^2\right|$, then the point $(\mathrm{x}', \mathrm{y}')$ on K′ holds the equality $|r^2\mathrm{x}' - x_0(\mathrm{y}')^2| = \mathrm{r}'(\mathrm{y}')^2$. When $\mathrm{x}' < \frac{x_0}{r^2}\mathrm{y}'^2$, it is seen that the parabola arcs $\mathrm{x}' = \frac{x_0-\mathrm{r}'}{r^2}\mathrm{y}'^2, \frac{r^2}{x_0-\mathrm{r}'} \le y' \le \frac{r^2}{y_0-\mathrm{r}'}$ and $\mathrm{x}' = \frac{x_0-\mathrm{r}'}{r^2}\mathrm{y}'^2, \frac{r^2}{y_0+\mathrm{r}'} \le y' \le \frac{-r^2}{x_0-\mathrm{r}'}$ lie on K′ where $x_0 < \mathrm{r}'$. When the radius r' is less than x_0, all possible cases for r' have to be considered, as above. If r' is a value between $\frac{x_0+y_0}{2}$ and x_0, then the two parabola arcs determined by the equation $\mathrm{x}' = \frac{x_0-\mathrm{r}'}{r^2}\mathrm{y}'^2$ are on K′ such that their endpoints are on the separator line and the parabola passing through the inversion center: $\mathrm{x}' = \frac{x_0-\mathrm{r}'}{r^2}\mathrm{y}'^2, \frac{-r^2}{x_0-\mathrm{r}'} \le y' \le \frac{r^2}{y_0-\mathrm{r}'}$ and $\mathrm{x}' = \frac{x_0-\mathrm{r}'}{r^2}\mathrm{y}'^2, \frac{r^2}{y_0+\mathrm{r}'} \le y' \le \frac{r^2}{x_0-\mathrm{r}'}$. If r' is a value between $\frac{x_0-y_0}{2}$ and $\frac{x_0+y_0}{2}$, then only one parabolic arc, determined by the same equation, lies on K′ such that its endpoints are on the separator line and the parabola: $\mathrm{x}' = \frac{x_0-\mathrm{r}'}{r^2}\mathrm{y}'^2, \frac{r^2}{y_0+\mathrm{r}'} \le y' \le \frac{r^2}{x_0-\mathrm{r}'}$. If r' is less than the value $\frac{x_0-y_0}{2}$, then there is no solution. In the particular case of $x_0 = r'$, the rays $\mathrm{x}' = 0, \frac{r^2}{y_0+\mathrm{r}'} \le y'$ and $\mathrm{x}' = 0, y' \le \frac{r^2}{y_0-\mathrm{r}'}$ are on K′. When $\mathrm{x}' \ge \frac{x_0}{r^2}\mathrm{y}'^2$, no solution is sought.

The point (x', y') on K' satisfying the conditions $|x'| \leq |y'|$ and $\left|x' - \frac{x_0}{r^2}(y')^2\right| \leq \left|y' - \frac{y_0}{r^2}(y')^2\right|$ yields the equation $|r^2 y' - x_0 (y')^2| = r'(y')^2$. When the possible cases related to this equality are examined, the differences are observed according to the radius r'. If r' is greater than the value $\frac{x_0+y_0}{2}$, the intersection of the line $y = \frac{r^2}{y_0 - r'}$ and the region $\left\{(x', y') : x' > y' + \frac{x_0 - y_0}{r^2} y'^2, |x'| \leq |y'|\right\}$ is the line segment $y = \frac{r^2}{y_0 - r'}, \frac{(x_0 - r')r^2}{(y_0 - r')^2} \leq x' \leq -\frac{r^2}{y_0 - r'}$. If r' is less than $\frac{x_0+y_0}{2}$, there is no solution. If r' is greater than the value $\frac{x_0 - y_0}{2}$, the intersection of the line $y = \frac{r^2}{y_0 + r'}$ and the region $\left\{(x', y') : x' > -y' + \frac{x_0 + y_0}{r^2} y'^2, |x'| \leq |y'|\right\}$ is the line segment $y = \frac{r^2}{y_0 + r'}, \frac{(x_0 - r')r^2}{(y_0 + r')^2} \leq x' \leq \frac{r^2}{y_0 + r'}$. If r' is less than $\frac{x_0 - y_0}{2}$, there is no solution. Thus, when r' is greater than $\frac{x_0+y_0}{2}$, both of these line segments lie on K'. When r' is between the values $\frac{x_0 - y_0}{2}$ and $\frac{x_0+y_0}{2}$, only one of them lies on K'. When r' is smaller than $\frac{x_0 - y_0}{2}$, none of them lies on K'.

It is similar to the above when considering all possible placements of x_0 and y_0. In the case of (i), all vertices of the maximum circle are in the same region. Since the images of two sides perpendicular to the coordinate axis in the region are two line segments $x' = \frac{r^2}{x_0 + r'}, (y_0 - r')\left(\frac{r}{x_0 + r'}\right)^2 \leq y' \leq (y_0 + r')\left(\frac{r}{x_0 + r'}\right)^2$, $x' = \frac{r^2}{x_0 - r'}, \frac{r^2(y_0 - r')}{(x_0 - r')^2} \leq y' \leq \frac{r^2(y_0 + r')}{(x_0 - r')^2}$ and the images of others are two parabola arcs $y' = \frac{(y_0 - r')}{r^2}(x')^2, \frac{r^2}{x_0 + r'} \leq x' \leq \frac{r^2}{x_0 - r'}$, $y' = \frac{(y_0 + r')}{r^2}(x')^2, \frac{r^2}{x_0 + r'} \leq x' \leq \frac{r^2}{x_0 - r'}$, the maximum circular inversion maps the maximum circle to a four-part closed curve, as shown in Figure 5.a.

Considering situation (ii), the vertices on the two sides of the maximum circle are in the neighboring regions, so the images of these sides consist of line segments and parabolic arcs. Since the images of the other two sides are a line segment and a parabola arc, the maximum circular inversion maps the maximum circle to a six-part closed curve such that the line segments are $x' = \frac{r^2}{x_0 + r'}, (y_0 - r')\left(\frac{r}{x_0 + r'}\right)^2 \leq y' \leq (y_0 + r')\left(\frac{r}{x_0 + r'}\right)^2$, $x' = \frac{r^2}{x_0 - r'}, \frac{r^2(y_0 - r')}{(x_0 - r')^2} \leq y' \leq \frac{r^2}{x_0 - r'}$, $y = \frac{r^2}{y_0 + r'}, \frac{(x_0 - r')r^2}{(y_0 + r')^2} \leq x' \leq \frac{r^2}{y_0 + r'}$ and the parabola arcs are $y' = \frac{(y_0 - r')}{r^2}(x')^2, \frac{r^2}{x_0 + r'} \leq x' \leq \frac{r^2}{x_0 - r'}$, $y' = \frac{(y_0 + r')}{r^2}(x')^2, \frac{r^2}{x_0 + r'} \leq x' \leq \frac{r^2}{y_0 + r'}$, $x' = \frac{x_0 - r'}{r^2} y'^2, \frac{r^2}{y_0 + r'} \leq y' \leq \frac{r^2}{x_0 - r'}$, as shown in Figure 5.b.

When it comes to situation (iii), the image of the side with the vertices in alternate regions is a three-part curve consisting of a line segment parallel to it and two parabolic arcs. The images of two sides with the vertices in neighboring regions are two-part curves comprising a line segment and a parabola arc. The image of last side with the vertices in same region is a line segment. So, image of the maximum circle in case (iii) under the maximum circular inversion is eight-part closed curve such that the line segments are $x' = \frac{r^2}{x_0 + r'}, (y_0 - r')\left(\frac{r}{x_0 + r'}\right)^2 \leq y' \leq (y_0 + r')\left(\frac{r}{x_0 + r'}\right)^2$, $x' = \frac{r^2}{x_0 - r'}, -\frac{r^2}{x_0 - r'} \leq y' \leq \frac{r^2}{x_0 - r'}$, $y = \frac{r^2}{y_0 - r'}, \frac{(x_0 - r')r^2}{(y_0 - r')^2} \leq x' \leq -\frac{r^2}{y_0 - r'}$, $y = \frac{r^2}{y_0 + r'}, \frac{(x_0 - r')r^2}{(y_0 + r')^2} \leq x' \leq \frac{r^2}{y_0 + r'}$ and the parabola arcs are $y' = \frac{(y_0 - r')}{r^2}(x')^2, \frac{r^2}{x_0 + r'} \leq x' \leq -\frac{r^2}{y_0 - r'}$, $y' = \frac{(y_0 + r')}{r^2}(x')^2, \frac{r^2}{x_0 + r'} \leq x' \leq \frac{r^2}{y_0 + r'}$, $x' = \frac{x_0 - r'}{r^2} y'^2, \frac{-r^2}{x_0 - r'} \leq y' \leq \frac{r^2}{y_0 - r'}$, $x' = \frac{x_0 - r'}{r^2} y'^2, \frac{r^2}{y_0 + r'} \leq y' \leq \frac{r^2}{x_0 - r'}$, as shown in Figure 5.c.

When the case (iv) occurs, since the side with the vertices in alternate regions passes through the inversion center, its image consists of two rays $x' = 0, \frac{r^2}{y_0 + r'} \leq y'$, $x' = 0, y' \leq \frac{r^2}{y_0 - r'}$. The rest of the sides are the same as in case iii. And so, the image of the maximum circle in case (iv) under the maximum circular inversion is a seven-part curve such that the line segments are $x' = \frac{r^2}{x_0+r'}$, $(y_0 - r')\left(\frac{r}{x_0+r'}\right)^2 \leq y' \leq (y_0 + r')\left(\frac{r}{x_0+r'}\right)^2$, $y = \frac{r^2}{y_0-r'}, \frac{(x_0-r')r^2}{(y_0-r')^2} \leq x' \leq -\frac{r^2}{y_0-r'}$, $y = \frac{r^2}{y_0+r'}, \frac{(x_0-r')r^2}{(y_0+r')^2} \leq x' \leq \frac{r^2}{y_0+r'}$, the parabola arcs are $y' = \frac{(y_0-r')}{r^2}(x')^2, \frac{r^2}{x_0+r'} \leq x' \leq -\frac{r^2}{y_0-r'}$, $y' = \frac{(y_0+r')}{r^2}(x')^2, \frac{r^2}{x_0+r'} \leq x' \leq \frac{r^2}{y_0+r'}$ and the two rays mentioned above, as shown in Figure 5.c.

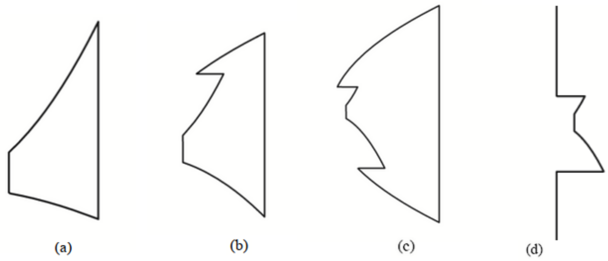

Figure 5. The image of the maximum circle with the center on the gradual line through the inversion center

It should be noted that Theorem 4.5 can be stated for the maximum circles with the centers on steep lines considering the isometric reflections in the separator lines passing through the inversion center.

5 CONCLUSION

In this study, the focus was on the examining of the images of maximum circles under the inversion in a maximum circle. Through a detailed analytical analysis, several observations were made. The study also introduced various properties related to the images of maximum circles under the maximum circle inversion. It was found that if the centers of maximum circles are different from the inversion center, their images do not form a maximum circle or a line but instead become curves comprising parabolic arcs, line segments, and rays. It was observed that if the maximum circle passes through the inversion center, the image is not closed curve. The resulting images were found to depend on the relative positions of the inversion circle and the maximum circle, leading to a classification of their images.. These findings provide to our understanding of how maximum circles are affected by maximum circular inversion. Consequently, it is thought that the results obtained in this study contribute to the literature including the subject of inversion in non-Euclidean geometry.

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