Symmetry Reductions, Exact Solutions and Conservation Laws for the Coupled Nonlinear Klein-Gordon System

Emrullah Yaşar[1*], İlker Burak Giresunlu[2]

[1] Uludağ Üniversitesi Fen-Edebiyat Fakültesi Matematik Bölümü, Nilüfer, Bursa, +90 224 294 1768
emrullah.yasar@gmail.com
[2] Bilecik Şeyh Edebali Üniversitesi Fen-Edebiyat Fakültesi Matematik Bölümü
Gülümbe, Bilecik, +90 228 214 2156
iburak.giresunlu@bilecik.edu.tr
*Corresponding author

Abstract

The Lie group method is applied to a coupled nonlinear Klein-Gordon system. The Klein-Gordon system is used to model many nonlinear phenomena including the propagation of dislocations in crystals and the behavior of elementary particles and the propagation of fluxons in Josephson junctions. The symmetry reductions and exact solutions which include the stationary and solitary waves are obtained. In addition, by using the multiplier method, we derive the local conservation laws of the coupled nonlinear Klein-Gordon system.

Keywords — Lie symmetry analysis, Coupled nonlinear Klein-Gordon system, Similarity reduction, Solitary wave solution

1 Introduction

The problem of finding the analytical or numerical solutions to nonlinear evolution equations (NLEEs) is quite active research area in nonlinear science**.** In the last two decades, several powerful approaches such as the inverse scattering, Hirota's bilinear method, the Darboux transformation method, (G'/G)-expansion method, homogeneous balance method, Adomian decomposition method, the functional variable method, the extended tanh function method, Lie group analysis, etc. have been proposed [1-8].

In the theory of NLEEs, the problem of revealing wave phenomena of some important process such as dispersion, dissipation and diffusion is quite important task [9]**.** By using the continuum models, one can study the nonlinear wave phenomena in heterogeneous media. As a result, we can investigate the long-wave dynamics of discrete models by the nonlinear Klein-Gordon system. This system is a physical model for long longitudinal waves in bilayers where nonlinearity comes only from the bonding material [10]. The system is described, in dimensionless variables, by the coupled Klein-Gordon equations [11]
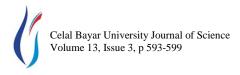
$$u_{tt} - u_{xx} = f_u(u,v), \qquad (1.1)$$

$$v_{tt} - c^2 v_{xx} = f_v(u,v)$$

where $v = v(x,t)$ is the scalar field, $u = u(x,t)$ represents the wave displacement at position x and time t, c is the ratio of the acoustic velocities of noninteracting components, and $f(u,v)$ describes the interaction between the chains of particles. We also observe this important system in the field of solid state physics, nonlinear optics and quantum field theory [12].

On the other hand, the n-coupled nonlinear Klein-Gordon equations with a scalar field v are described as ([13]-[14])

$$u_{lxx} - u_{ltt} - u_l + 2\left(\sum_{j=1}^{n} u_j^2 + v\right)u_l = 0,$$

$$v_x - v_t - 2\left(\sum_{j=1}^{n} u_j^2\right)_t = 0, \qquad l = 1,2,\dots n$$

For the case of $l = 1$ the one coupled nonlinear Klein-Gordon equations are Painleve integrable and are given as

$$u_{xx} - u_{tt} - u + 2u^3 + 2uv = 0, \qquad (1.2)$$
$$v_x - v_t - 4uu_t = 0.$$

Using the Hirota's theory, bilinear-form and one-soliton solutions of (1.2) are studied in [15]. We also observe that in the literature a plenty of studies about the some special types of exact solutions. Some of them are listed in [12], [16-21].

Among the mentioned above methods Lie group analysis has a special interest. As is mentioned in [22] by Anderson et. al, Lie's approach is known in the literature as one of the most powerful and effective method for obtaining exact solutions of partial differential equations (PDEs). In the last few decades Lie's group method (see, for example, Bluman and Kumei [23], Olver [8], Stephani [24], Vorobev [25], Winternitz [26]) successfully applied to differential equations arising in a variety of disciplines such as fluid dynamics, elasticity, boundary layer problems and so forth (see, for example, Ibragimov [27] or Rogers and Shadwick [28]).

It is well known that, conservation laws are the key instruments for describing the physical and mathematical properties of the considered model [29]. They are used for Lyapunov stability analysis and construction of numerical schemes. Moreover, conservation laws used in obtaining the new nonlocal symmetries, nonlocal conservation laws and linerization [30]. Because of the Noether's theorem limitations some efficient methods such as multiplier, adjoint symmetry, nonlocal conservation theorem method etc. were developed to investigate conservation laws of PDEs [8, 31-38].

The plan of the paper is as follows. In Section 2, Lie symmetry method along with the simplest equation is employed to obtain exact solutions of (1.2). Then in Section 3, by using the multiplier method, conservation laws for (1.2) are constructed. Finally, in Section 4 conclusions are presented.

2 Symmetry Reductions and Exact Solutions of Eq.(1.2)
For the necessary notations and definitions about Lie group analysis see e.g., [8, 35-37, 39]. The symmetry group of the coupled nonlinear Klein-Gordon system will be generated by the vector field of the form

$$X = \xi(t,x,u,v)\frac{\partial}{\partial x} + \tau(t,x,u,v)\frac{\partial}{\partial t} + \eta(t,x,u,v)\frac{\partial}{\partial u}$$
$$+ \phi(t,x,u,v)\frac{\partial}{\partial x}$$

Performing the second order prolongation $pr^{(2)}X$ to (1.2) gives an overdetermined system of linear partial differential equations. The general solution of this system is

$$\tau(t,x,u,v) = f(x+t) - c_1 t + c_2,$$
$$\xi(t,x,u,v) = -c_1 x + f(x+t),$$
$$\eta(t,x,u,v) = c_1 u,$$
$$\phi(t,x,u,v) = (-2u^2 - 2v + 1)f'(x+t) + c_1(2v-1).$$

where f is arbitrary functions of $x + t$ For instance, taking the arbitrary function $x + t$ we obtain three-dimensional Lie algebra spanned by the following linearly independent operators:

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial x},$$
$$X_3 = -x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + (2v-1)\frac{\partial}{\partial v}. \qquad (2.1)$$

Having this generators, one has to solve the following Lagrange equations

$$\frac{dt}{\tau(t,x,u,v)} = \frac{dx}{\xi(t,x,u,v)} = \frac{du}{\eta(t,x,u,v)} = \frac{dv}{\phi(t,x,u,v)}$$

for getting symmetry reductions and exact solutions. For this aim, we have to construct optimal system of one-dimensional subalgebras for (1.2). We exploited the adjoint representation method for drawing the possible optimal system of one-dimensional subalgebras [8, 37]).

In adjoint representation, one needs the following well-known Lie series

$$Ad(\exp(\varepsilon X))Y = Y - \varepsilon[X,Y]$$
$$+ \frac{1}{2}\varepsilon^2[X,[X,Y]]$$
$$- \frac{1}{3}\varepsilon^3[X,[X,[X,Y]]] + \cdots$$

together with the commutation relations of the three operators which are

$$[X_1, X_2] = 0,$$
$$[X_1, X_3] = -X_1,$$
$$[X_2, X_3] = -X_2.$$

To compute the adjoint representation, we use commutator table of X_1, X_2 and X_3.

	X_1	X_2	X_3
X_1	0	0	$-X_1$
X_2	0	0	$-X_2$
X_3	X_1	X_2	0

In this manner, we construct the following table of adjoint representation

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 + \varepsilon X_1$
X_2	X_1	X_2	$X_3 + \varepsilon X_2$
X_3	$e^\varepsilon X_1$	$e^\varepsilon X_2$	X_3

For a nonzero vector

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3,$$

we have to simplify the coefficients a_i $(i = 1,2,3)$ as far as possible through adjoint maps to X. The calculations are easy and we obtain an optimal system of one-dimensional subalgebras is spanned by

$$X_1, X_2, X_1 \mp X_2, X_3 \tag{2.2}$$

We consider the following four cases:

Case 1. X_1
Considering the Lagrange system for the symmetry X_1, the group invariant solution corresponding to X_1 is

$$u = u(z), \quad v = v(z)$$

where $z = x$ is the group-invariant of X_1 By substituting of these solutions in (1.2), we find

$$u'' - u + 2u^3 + 2uv = 0, \tag{2.3}$$
$$v' = 0 \tag{2.4}$$

After integrating (2.4) and substituting $v = c_1$ into (2.3), we obtain

$$u'' - u + 2u^3 + 2uc_1 = 0. \tag{2.5}$$

Solving the Eq.(2.5) we obtain the solutions for (1.2) are given by

$$u(x,t) = c_2 \sqrt{\frac{1 - 2c_1}{-2c_1 + c_2^2}} \, sn\left(\times \sqrt{\frac{-1 + 2c_1}{-2c_1 + c_2^2}}, \frac{(\sqrt{2c_1}x + c_1)}{\sqrt{-2c_1}}\right),$$

$$v(x,t) = c_1$$

where c_1 and c_2 are arbitrary constants and sn is the sine-amplitude function.

Case 2. X_2
Considering the Lagrange system for the symmetry X_2, the group invariant solution corresponding to X_2 is

$$u = u(z), \quad v = v(z)$$

where $z = t$ is the group-invariant of X_2 The substituting of these solutions in (1.2), we find

$$-u'' - u + 2u^3 + 2uv = 0, \tag{2.6}$$
$$v' + 4uu' = 0 \tag{2.7}$$

After integrating (2.7) and substituting $v = -2u^2$ into (2.6), we obtain

$$u'' + u + 2u^3 = 0 \tag{2.8}$$

Solving the Eq. (2.8) we obtain the solutions for (1.2) are given by

$$u(x,t) = c_2 \sqrt{\frac{-1}{-2 + c_2^2}}$$

$$\times sn\left((\sqrt{2}t + c_1)\sqrt{\frac{-1}{-2 + c_2^2}}, \frac{\sqrt{2}}{2}ic_2\right),$$

$$v(x,t) = -2\left[c_2 \sqrt{\frac{-1}{-2 + c_2^2}} \, sn\left(\times \sqrt{\frac{-1}{-2 + c_2^2}}, \frac{(\sqrt{2}t + c_1)}{2} \frac{\sqrt{2}}{2}ic_2\right)\right]^2$$

where c_1 and c_2 are arbitrary constans and sn is the sine-amplitude function.

Case 3. X_3
The symmetry X_3 yields the group invariant solution

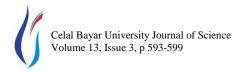$$u = \frac{E(z)}{x}, \quad v = \frac{1}{2} + \frac{F(z)}{2x^2} \tag{2.9}$$

where $z = {}^x/_t$ is an invariant of the symmetry X_3 Substituting (2.9) to (1.2), gives rise to system of ordinary differential equations (ODEs) where E and F satisfy

$$(z - z^4)E'' + 2z(z^2 - 1)E' + 2(E^3 + E) + EF = 0,$$
$$(z + z^2)F' - 2F + 8z^2 EE' = 0. \tag{2.10}$$

Case 4. $X_1 + cX_2$
Considering the related Lagrange systems corresponding to $X_1 + cX_2$, one obtains an invariant $z = x - ct$ and the group invariant solution of the form

$$u = u(z), \quad v = v(z) \tag{2.11}$$

and pursuing the same steps as before we get

$$(1 - c^2)u'' + 2u^3 - u + 2uv = 0, \quad (2.12)$$
$$(1 + c)v' + 4cuu' = 0 \quad (2.13)$$

where after integrating (2.13) and substituting $v = \frac{-2c}{1+c}u^2$ in (2.12), we deduce

$$(1 - c^2)u'' + 2\frac{1 - c}{1 + c}u^3 - u = 0. \quad (2.14)$$

Exact solitary wave solutions using simplest equation method

The simplest equation method, which was developed by Kudryashov [39-42] will be exploited for obtaining the exact solutions of evolution type systems. The first simplest equation that will be used is the Bernoulli equation

$$G'(x) = aG^2(z) + bG(z), \quad (2.15)$$

where a and b are arbitrary constants. The solutions of (2.15) can be expressed as

$$G(z) = b\frac{cosh[b(z + C)] + sinh[b(z + C)]}{1 - a\,cosh[b(z + C)] - a\,sinh[b(z + C)]}.$$

elementary functions where C is a constant of integration. Let us consider the solution of (2.14) of the form

$$u(z) = \sum_{i=0}^{M} A_i(G(z))^i, \quad (2.16)$$

where $G(z)$ satisfies the Bernoulli equation, M is a positive integer that can be determined by balancing procedure, and $A_0, A_1, \dots, A_M$ are parameters to be determined later.

The balancing procedure gives $M = 1$ so the solution of (2.14) is of form

$$u(z) = A_0 + A_1 G(z). \quad (2.17)$$

Moreover, for the Riccati equation

$$G'(z) = aG^2(z) + bG(z) + d, \quad (2.18)$$

we shall use the following elementary solutions

$$G(z) = -\frac{b + \theta\,tanh\left(\frac{1}{2}\theta(z + C)\right)}{2a}$$

and

$$G(z) = -\frac{b + \theta\,tanh\left(\frac{1}{2}\theta(z + C)\right)}{2a}$$
$$+ \frac{sech\left(\frac{1}{2}\theta z\right)}{C\,cosh\left(\frac{1}{2}\theta z\right) - \frac{2a}{\theta}\,sinh\left(\frac{1}{2}\theta z\right)}$$

where $\theta^2 = b^2 - 4ad > 0$.

Solutions of Eq.(2.14) by using the Bernoulli equation as the simplest equation

Substituting (2.17) into (2.14) and making use of (2.15) and then equating the coefficients of the functions G^i to zero, we obtain an algebraic system of equations in term of A_i. Solving the obtained system of algebraic equations with the help of Maple, we get

$$A_0 = \frac{\sqrt{2 - 2c^2}}{2(-1 + c)},$$
$$A_1 = -ab\frac{(1 + c)(-1 + c^2)}{\sqrt{2 - 2c^2}},$$
$$A_2 = \frac{2}{\sqrt{-2 + 2c^2}}.$$

As a result, a solution of (1.2) is

$$u(x, t) = A_0 + A_1 b\left\{\frac{cosh[b(x-ct+C)]+sinh[b(x-ct+C)]}{1-a\,cosh[b(x-ct+C)]-a\,sinh[b(x-ct+C)]}\right\}$$
$$v(x, t) = \frac{-2c}{1+c}\left\{A_0+A_1 b\left[\frac{cosh[b(x-ct+C)]+sinh[b(x-ct+C)]}{1-a\,cosh[b(x-ct+C)]-a\,sinh[b(x-ct+C)]}\right]\right\}^2$$
$$(2.19)$$

Solutions of Eq.(2.14) by using the Riccati equation as the simplest equation

The solution of (2.14) is of the form (2.17). Substituting (2.17) into (2.14) and making use of (2.18), by equating all coefficients of the functions G^i to zero we yield again an algebraic system of equations in term of A_i. Solving this system, we obtain

$$A_0 = -\frac{ab(1 + c)}{2ai},$$
$$A_1 = a(1 + c)i,$$
$$b = 2i,$$
$$d = \frac{-1 + 2c^2}{a(2 - 2c^2)},$$

hence solutions of (1.2) are

$$u(x, t) = A_0 + A_1\left\{\frac{-b-\theta}{2a}\,tanh\left[\frac{1}{2}\theta(x-ct+C)\right]\right\},$$
$$v(x, t) = \frac{-2c}{1+c}\left\{A_0+A_1\left[\frac{-b-\theta}{2a}\,tanh\left(\frac{1}{2}\theta(x-ct+C)\right)\right]\right\}^2 \quad (2.20)$$

and

$$u(x,t) = A_0 + A_1 \left\{ \begin{array}{c} \frac{-b-\theta}{2a} tanh\left[\frac{1}{2}\theta(x-ct)\right] \\ + \frac{sech\left[\frac{1}{2}\theta(x-ct)\right]}{C \cosh\left[\frac{1}{2}\theta(x-ct)\right] - \frac{2a}{\theta} \sinh\left[\frac{1}{2}\theta(x-ct)\right]} \end{array} \right\},$$

$$v(x,t) = \frac{-2c}{1+c} \left[A_0 + A_1 \left\{ \begin{array}{c} \frac{-b-\theta}{2a} tanh\left[\frac{1}{2}\theta(x-ct)\right] \\ + \frac{sech\left[\frac{1}{2}\theta(x-ct)\right]}{C \cosh\left[\frac{1}{2}\theta(x-ct)\right] - \frac{2a}{\theta} \sinh\left[\frac{1}{2}\theta(x-ct)\right]} \end{array} \right\} \right]^2.$$

(2.21)

3 Conservation Laws

As well known, in the investigation of integrability properties of considered model the existence of a large number of (or infinity) conservation laws are the key instrument [8]. They are used for analysis, in particular, existence, uniqueness, stability analysis and construction of numerical schemes [30]. In addition, in the numerical integration of PDEs [43, 44], for example, to control numerical errors, conservation laws are also used.

In this section, we build conservation laws for (1.2). For the details see e.g., [8], [30], [31] and [34].
Consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, ..., x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$, namely

$$E_\alpha\left(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)}\right) = 0, \qquad \alpha = 1,2, ..., m \quad (3.1)$$

where $u_{(1)}, u_{(2)}, ..., u_{(k)}$ denote the collections of all first, second, ..., k th order partial derivatives, i. e., $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha)$, ..., respectively. The total derivative operator with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad j = 1,2, ..., n \quad (3.2)$$

where the summation conversion is used whenever appropriate. The $n-$tuple vector

$$T = (T^1, T^2, ..., T^n), \quad T^j \in A, \ j = 1,2, ..., n$$

is a conserved vector of (3.1) if T^i satisfied

$$D_i T^i|_{(3.1)} = 0 \quad (3.3)$$

The equation (3.3) is called a local conservation law of system.

The conservation laws of Eq.(3.1) will be generated by multipliers $Q^\alpha(x, u, u_{(1)}, ...)$ which they satisfy identically

$$Q^\alpha E_\alpha = D_i T^i \quad (3.4)$$

As demonstrated in [8], for constructing the associated multipliers, one takes the variational derivative of (3.4) that is,

$$\frac{\delta}{\delta u^\beta}(Q^\alpha E_\alpha) = 0, \quad (3.5)$$

holds for arbitrary functions of $u(x^1, x^2, ..., x^n)$. In our work, we confine ourselves to multipliers of the second order $\Lambda_\alpha = \Lambda_\alpha(x, t, u, v, u_x, v_x, u_{xx}, v_{xx})$. Once the multipliers are deduced the conserved vectors are calculated via a homotopy formula ([30, 34, 36]).

Conservation laws of Eq.(1.2)
For the coupled nonlinear Klein-Gordon system, we obtain that two second order multipliers (with the aid of GeM [30], see also [35, 36]), namely $\Lambda_1(x, t, u, v, u_x, v_x, u_{xx}, v_{xx})$, $\Lambda_2(x, t, u, v, u_x, v_x, u_{xx}, v_{xx})$ are given by

$$\Lambda_1 = c_1 u_x,$$
$$\Lambda_2 = \frac{1}{2}(2u^2 + v)c_1 + c_2, \quad (3.6)$$

where c_i, $i = 1,2$ are arbitrary constants.

We have the following two conserved vectors of (1.2) corresponding to the multipliers of Λ_1 and Λ_2, respectively:

$$C_1^t = -u^4 - u^2 v - u_t u_x - \frac{1}{4}v^2,$$
$$C_1^x = \frac{1}{2}u^4 + u^2 v + \frac{1}{2}u_x^2 + \frac{1}{4}v^2 - \frac{1}{2}u^2 + \frac{1}{2}u_t^2;$$

$$C_2^t = -2u^2 - v,$$
$$C_2^x = v. \quad (3.7)$$

4 Conclusions

In this work, we studied the nonlinear Klein-Gordon system which is one of the nonlinear second order evolution type system. We obtained some special type group invariant solutions via Lie group analysis. In Case 1 and Case 2, we constructed stationary type sine-amplitude group invariant solutions which corresponds space and time translations, respectively. In Case 3, second order nonlinear ordinary differential system is obtained which corresponds to the scaling transformation. Solving the analytically of the system (2.10) is quite intractable. It is possible to solve by some appropriate initial conditions. In Case 4, by using the wave invariant variable which corresponds to combination of space and time symmetry generators, we converted the original system (1.2) to second order NLODE. We solved this NLODE by the simplest equation method. We used Bernoulli and Riccati differential equations as simple equations which those have some special type kink-shaped tanh

and bell-shaped sech solutions. We note that by using the other auxiliary equations such as Jacobi elliptic equation, sub-equation method, etc., some new solutions also can be obtained. Some of the results are in agreement with the results obtained in the previous literature, and also new results are formally developed. We verified all the obtained solutions by putting them back into the original system (1.2) with the aid of Maple 2015.

The solutions obtained are solitary type traveling waves. These type waves on a water surface do not behave exactly as solitons. After the collisions of two solitary waves, in their amplitudes have been occurs some small changes and their oscillatory residuals are left behind.

In the last part of the work, we constructed conservation laws of the system. As the considered system is of evolution type, no recourse to a Lagrangian formulation is made. Therefore, we resorted to multiplier approach and two local conservation laws were deduced. The conservation laws C_2^t and C_2^x represent the conservation of energy of (1.2). In addition, one might interpret the vectors of C_1^t and C_1^x in terms of conservation of momentum. We note that the obtained conservation laws can be used in some well known numerical schemes for constructing the numerical solutions.

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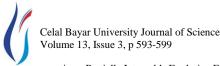
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