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Morphism Properties of Digital Categories

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Abstract

In this paper we defined the Img_κ category and researched the properties of monomorphism, epimorphism and isomorphism for digital categories which are related with the categorical structure in [1]. Also initial and terminal objects in digital categories are defined by using $\kappa-$ adjacency relation. Hence we determined the initial and terminal objects of digital categories which have digital image with $\kappa-$ adjacency as objects. In addition to this we proved that the objects of the same type in a digital category are isomorphic.
Keywords — Digital Image, Digital Category, $\kappa-$ monomorphism, $\kappa-$ epimorphism, $\kappa-$ isomorphism.

1. Introduction

Digital Topology is a field of mathematical science which investigates the image processing and digital image processing. Many Researchers, for example Rosenfeld [2], Han [3], Kong [4], Boxer [5,6], Karaca [7] and others have contributed this area with their research. The notion of digital image, digital continuous map is studied in [2, 3, 4, 5, 6]. Their characterization and effective computation became a useful tool for our research. Then we carry this notion to category theory and we construct some fundamental category models in digital topology.

We introduce the Img_κ Category and give the monomorphism and epimorphism properties of Img_κ category. Also we define the initial and terminal objects in Img_κ and prove that initial and terminal objects are isomorphic in Img_κ.

2. Preliminaries

In this study we indicate the set of integers by $\mathbb{Z}$. Then $\mathbb{Z}^n$ denotes the set of lattice points in Euclidean $n-$dimensional spaces. A finite subset of $\mathbb{Z}^n$ with an adjacency relation is said to be digital image.

Definition 2.1. ([5] and [6])

(1) Two points p and q in $\mathbb{Z}$ are $2-adjacent$ if $|p-q|=1$.

(2) Two points p and q in $\mathbb{Z}^2$ are $8-adjacent$ if they are distinct and differ by at most 1 in each coordinate.

(3) Two points p and q in $\mathbb{Z}^2$ are $4-adjacent$ if they are $8-adjacent$ and differ by exactly one coordinate.

(4) Two points p and q in $\mathbb{Z}^3$ are $26-adjacent$ if they are distinct and differ by at most 1 in each coordinate.

(5) Two points p and q in $\mathbb{Z}^3$ are $18-adjacent$ if they are $26-adjacent$ and differ in at most two coordinates.

(6) Two points p and q in $\mathbb{Z}^3$ are $6-adjacent$ if they are $18-adjacent$ and differ by exactly one coordinate.

Suppose that κ be an $adjacency$ relation defined on $\mathbb{Z}^n$. A digital image $X \subset \mathbb{Z}^n$ is $\kappa-connected$ [5] if and only if for every pair of points $\{x, y\} \subset X$, $x \neq y$, there is a set $\{x_0, x_1, ..., x_c\} \subset X$ such that $x = x_0, x_c = y$ and x_i and x_{i+1} are $\kappa - neighbors, i \in \{0,1,...,c-1\}$.

Definition 2.2. Let X and Y are digital images such that $X \subset \square^{n_0}$, $Y \subset \square^{n_1}$. Then the digital function $f : X \to Y$ is a function which is defined between digital images.

Definition 2.3. ([5] and [6]) Let X and Y are digital images such that $X \subset \square^{n_0}$, $Y \subset \square^{n_1}$. Assume that $f : X \to Y$ be a function. Let κ_i be an $adjacency$ relation defined on $\square^{n_i}$, $i \in \{0,1\}$. f is called to be $(\kappa_0, \kappa_1) - continuous$ if the image under f of every $\kappa_0 - connected$ subset of X is $\kappa_1 - connected$.
A function satisfying Definition 2.3 is referred to be digitally continuous. A consequence of this definition is given below.

Proposition 2.4. ([5] and [6]) Let X and Y are digital images. Then the function $f : X \to Y$ is said to be $(\kappa_0, \kappa_1) - continuous$ if and only if for every $\{x_0, y_0\} \subset X$ such that x_0 and x_1 are $\kappa_0 - adjacent$, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are $\kappa_1 - adjacent$.

The basic notions of Category Theory is given in [8], [9]. Also digital and soft properties of categories are investigated in [1], [10].

Definition 2.5. [1] A category C consists of the following data:

- Objects: $A, B, C, ...$
- Arrows: $f, g, h, ...$
- For each arrow f, there are given objects

$$dom(f),\ cod(f)$$

called the domain and codomain of f. We write

$$f : A \to B$$

to indicate that $A = dom(f)$ and $B = cod(f)$

- Given arrows $f : A \to B$ and $g : B \to C$, that is, with

$$cod(f) = dom(g)$$

there is given an arrow

$$g \circ f : A \to C$$

called the composite of f and g.

- For each object A, there is given an arrow

$$1_A : A \to A$$

called the identity arrow of A.
This data required to satisfy the following laws:

Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : A \to B$, $g : B \to C$ and $h : C \to D$.
Unit: $f \circ 1_A = f = 1_B \circ f$ for all $f : A \to B$.

3. The Category Img_κ

- The objects of Img_κ are $\kappa - adjacent$ digital images,
- The morphisms of Img_κ are $\kappa - continuous$ (*digitally continuous*) functions,
- For each morphism f in Img_κ

$$dom(f) \text{ and } cod(f)$$

are $\kappa - adjacent$ digital images.

- Let $(A,\kappa), (B,\kappa)$ and (C,κ) be $\kappa - adjacent$ digital images, i.e., objects in Img_κ. Given morphisms $f : (A,\kappa) \to (B,\kappa)$ and $g : (B,\kappa) \to (C,\kappa)$ with $cod(f) = dom(g)$. Then the function $g \circ f : (A,\kappa) \to (C,\kappa)$ is a $\kappa - continuous$ function, since the composition of two digital continuous ($\kappa - continuous$) is continuous. Thus $g \circ f : (A,\kappa) \to (C,\kappa)$ an $Img_\kappa -$ morphism.
- For each (A,κ), $Img_\kappa -$object, the identity map $1_{(A,\kappa)} : (A,\kappa) \to (A,\kappa)$ is a $\kappa - continuous$ function. Therefore $1_{(A,\kappa)}$ is an $Img_\kappa -$ morphism.
- Composition:
For all $f : (A,\kappa) \to (B,\kappa)$, $g : (B,\kappa) \to (C,\kappa)$ and $h : (C,\kappa) \to (D,\kappa)$ $Img_\kappa -$ morphisms, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- Identity:
For each $f : (A,\kappa) \to (B,\kappa)$ $Img_\kappa -$ morphism

$$f \circ 1_{(A,\kappa)} = f = 1_{(B,\kappa)}$$

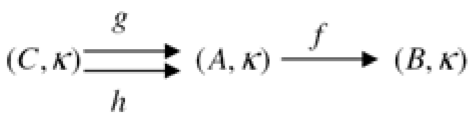
Thus we have construct the Img_κ category of digital images and digital continuous functions.
Let now investigate some properties of Img_κ category.

4. Epis and Monos in Img_κ

Definition 4.1: Let $(A,\kappa),(B,\kappa)$ and (C,κ) be $Img_\kappa-$objects which have the same cardinality. For given any $h,g:(C,\kappa)\to(A,\kappa)$ $Img_\kappa-$ morphism, if $f\circ g=f\circ h$ implies that $g=h$, then

$$f:(A,\kappa)\to(B,\kappa)$$

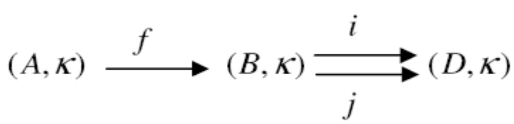
$Img_\kappa-$ morphism is called a $\kappa-$ monomorphism.



Definition 4.2: Let $(A,\kappa),(B,\kappa)$ and (D,κ) be $Img_\kappa-$objects which have the same cardinality. For given any $i,j:(B,\kappa)\to(D,\kappa)$ $Img_\kappa-$ morphism, if $i\circ f=j\circ f$ implies that $i=j$, then

$$f:(A,\kappa)\to(B,\kappa)$$

$Img_\kappa-$ morphism is called a $\kappa-$ epimorphism.



Definition 4.3: In category Img_κ, an $Img_\kappa-$ morphism

$$f:(A,\kappa)\to(B,\kappa)$$

is said to be an $\kappa-$isomorphism if there is a $g:(B,\kappa)\to(A,\kappa)$ $Img_\kappa-$ morphism such that

$$g\circ f=1_{(A,\kappa)} \text{ and } f\circ g=1_{(B,\kappa)}.$$

It is written as $(A,\kappa)\cong(B,\kappa)$.

Lemma 4.4: If $f:(A,\kappa)\to(B,\kappa)$ and $g:(B,\kappa)\to(C,\kappa)$ be two $\kappa-$monomorphisms in Img_κ category, then $g\circ f:(A,\kappa)\to(C,\kappa)$ is a $\kappa-$ monomorphism.

Proof: Let $f:(A,\kappa)\to(B,\kappa)$ and $g:(B,\kappa)\to(C,\kappa)$ be two $\kappa-$ monomorphisms. Suppose that $h,k:(C,\kappa)\to(A,\kappa)$ are $Img_\kappa-$ morphisms. If

$$(g\circ f)\circ h=(g\circ f)\circ k$$

then it follows that

$$g\circ(f\circ h)=g\circ(f\circ k)$$

from the associativity property of $Img_\kappa-$morphisms. Since g is an $\kappa-$monomorphism, we obtain that $f\circ h=f\circ k$. On the other hand, since f is an $\kappa-$ monomorphism, we have $h=k$. Therefore $g\circ f$ is a $\kappa-$ monomorphism. ■

Lemma 4.5: If $f:(A,\kappa)\to(B,\kappa)$ and $g:(B,\kappa)\to(C,\kappa)$ be two $\kappa-$epimorphisms in Img_κ category, then $g\circ f:(A,\kappa)\to(C,\kappa)$ is a $\kappa-$ epimorphism.

Proof: Let $f:(A,\kappa)\to(B,\kappa)$ and $g:(B,\kappa)\to(C,\kappa)$ be two $\kappa-$ epimorphisms. Assume that $h,k:(B,\kappa)\to(D,\kappa)$ are $Img_\kappa-$ morphisms. If

$$h\circ(g\circ f)=k\circ(g\circ f),$$

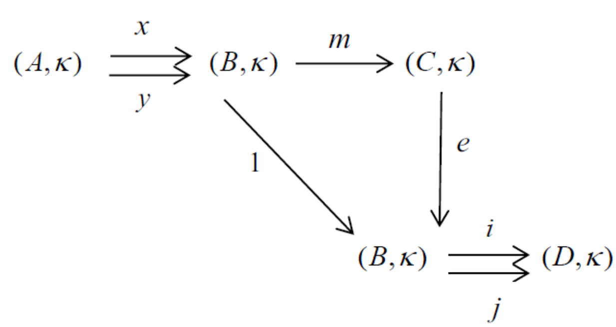
We have

$$(h\circ g)\circ f=(k\circ g)\circ f$$

From the associativity property. Since f is an $\kappa-$ epimorphism, we obtain that $h\circ g=k\circ g$, and since g is an $\kappa-$ epimorphism, we get $h=k$. Consequently $g\circ f$ is a $\kappa-$ epimorphism. ■

Theorem 4.6: Every isomorphism in Img_κ is both $\kappa-$ monomorphism and $\kappa-$ epimorphism.

Proof: Consider the following diagram:



$$i \circ e \circ m = j \circ e \circ m = i \circ 1_{(B,\kappa)} = j \circ 1_{(B,\kappa)}.$$

Thus we obtain that $i = j$. Consequently m is a $\kappa-$ epimorphism.

Definition 4.7: An object (U,κ) of a category Img_κ is said to be an initial object if for every object (X,κ) of Img_κ, the set $Mor_{Img_\kappa}((U,\kappa),(X,\kappa))$ is a singeleton. Dually (U,κ) is said to be a terminal object if for every object (X,κ) of Img_κ, the set $Mor_{Img_\kappa}((X,\kappa),(U,\kappa))$ is a singeleton.

Theorem 4.8: Let Img_κ be a category of digital images and digital continuous functions. Then any two initial objects in Img_κ are isomorphic.

Proof: Let (U_1,κ) and (U_2,κ) be initial objects in Img_κ. We must show $Mor_{Img_\kappa}((U_1,\kappa),(U_2,\kappa))$ contains an isomorphism. If (U_1,κ) is an initial object for Img_κ, then $Mor_{Img_\kappa}((U_1,\kappa),(U_2,\kappa)) = \{\alpha\}$ for any object (U_2,κ). If (U_2,κ) is an initial object for Img_κ, then $Mor_{Img_\kappa}((U_2,\kappa),(U_1,\kappa)) = \{\beta\}$ for any object (U_1,κ). We get $\alpha \circ \beta \in Mor_{Img_\kappa}((U_2,\kappa),(U_2,\kappa))$ by the composition property of morphisms. Therefore $\alpha \circ \beta = id_{U_2}$. Similarly we have $\beta \circ \alpha = id_{U_1}$. It follows that α is an isomorphism with $\beta = \alpha^{-1}$ and (U_1,κ) and (U_2,κ) are isomorphic.

Theorem 4.9: Let Img_κ be a category of digital images and digital continuous functions. Then any two terminal objects in Img_κ are isomorphic.

Proof: Assume that (O_1,κ) and (O_2,κ) are terminal objects in Img_κ. We want to show that $Mor_{Img_\kappa}((O_1,\kappa),(O_2,\kappa))$ contains an isomorphism. If (O_1,κ) is a terminal object for Img_κ, then $Mor_{Img_\kappa}((O_2,\kappa),(O_1,\kappa)) = \{f\}$ for any object (O_2,κ). If (O_2,κ) is a terminal object for Img_κ, then $Mor_{Img_\kappa}((O_1,\kappa),(O_2,\kappa)) = \{g\}$ for any object (O_1,κ). We have $g \circ f = id_{(O_2,\kappa)}$ and similarly $f \circ g = id_{(O_1,\kappa)}$ by the composition property of morphisms. Therefore f is an isomorphism with $f = g^{-1}$ and (O_1,κ) and (O_2,κ) are isomorphic.

Corollary 4.10: The objects of the same type in Img_κ are isomorphic.

Proof: The proof is obtained by Theorem 4.8 and Theorem 4.9.

5. Conclusion

We have construct the category Img_κ and worked on some properties of Img_κ in this paper. We have obtained some foldings deal with monomorphism and epimorphism properties of Img_κ. Also we concluded that the objects of the same type in the category Img_κ are isomorphic.

6. References

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