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ABOUT GROUP OF POINTWISE INNER AUTOMORPHISMS FOR NILPOTENCY CLASS FOUR

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Abstract. Let $L_{m,c}$ stand for the free metabelian nilpotent Lie algebra of class c of rank m over a field K of characteristic zero. Automorphisms of the form $\varphi(x_i) = e^{adu_i}(x_i)$ are called pointwise inner, where e^{adu_i}, is the inner automorphism induced by the element $u_i \in L_{m,c}$ for each $i = 1, \ldots, m$. The descriptions of the groups $\mathrm{PInn}(L_{m,2})$ and $\mathrm{PInn}(L_{m,3})$ of pointwise inner automorphisms are well known. In the present study, we investigate the group structure of $\mathrm{PInn}(L_{m,4})$ of pointwise inner automorphisms of $L_{m,4}$ that can be considered as the next step in this direction.

1. Introduction

Pointwise inner automorphisms of the free metabelian nilpotent Lie algebra $L_{m,c}$ forms a group shown by the author [4]. A generating set for the group $\mathrm{PInn}(L_{m,c})$ was provided, as well, in the same study: Each automorphism φ in $\mathrm{PInn}(L_{m,c})$ is of the form

$$\varphi(x_i) = e^{\mathrm{ad}(u_i)}(x_i) = (u_1, \ldots, u_m)$$

for some $u_i \in L_{m,c}$, $i = 1, \ldots, m$. Let us define the set

$$I_i = \{\varphi_u = (0, \ldots, 0, u, 0, \ldots, 0) \mid u \in L_{m,c}\}, \quad i = 1, \ldots, m,$$

consisting of m-tuples where each coordinate except for i-th position is necessarily filled by zero.

Theorem 1.1. [4] *The set I_i is a group for every $i = 1, \ldots, m$.*

Theorem 1.2. [4] *The set $\mathrm{PInn}(L_{m,c})$ of pointwise inner automorphisms of the free metabelian nilpotent Lie algebra $L_{m,c}$ forms a group generated by the set $I_1 \cup \cdots \cup I_m$.*

In the following theorems, the desription of $\mathrm{PInn}(L_{m,2})$ and $\mathrm{PInn}(L_{m,3})$ were given.

Theorem 1.3. [5] *Let the nilpotency class $c = 2$. Then the group $\mathrm{PInn}(L_{m,2})$ of pointwise inner automorphisms of the free metabelian Lie algebra $L_{m,2}$ is abelian, and the composition of two pointwise inner automorphisms is given by $\phi_u \phi_v = \phi_{u+v}$.*

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Theorem 1.4. [5] *Let $c = 3$. Then the group $\mathrm{PInn}(L_{m,3})$ of pointwise inner automorphisms of the free metabelian Lie algebra $L_{m,3}$ is abelian by nilpotent of class 2. That is, $[[\phi_u, \phi_v], \phi_w] = 0$, where $[\phi_u, \phi_v] = \phi_u\phi_v\phi_u^{-1}\phi_v^{-1}$. Furthermore, the compositon of two pointwise inner automorphisms is given by*

$$\varphi_u\varphi_v = \varphi_{u+v+\sum_i d_i[x_i,u_{i1}]+\frac{1}{2}[u_{j1},v_{j1}]}$$

where u_{j1}, v_{j1} are the linear parts of u_j, v_j in the expression of $\varphi_u = (u_1, \ldots, u_m)$, $\varphi_v = (v_1, \ldots, v_m)$, and

$$d_1x_1 + \cdots + d_mx_m$$

is the linear part of v.

In the current study, we investigate an analogue of the Theorems 1.3 and 1.4 for the nilpotency class $c = 4$.

Note: One may easily observe that a pointwise inner automorphism

$$\varphi(x_i) = (u_1, \ldots, u_m)$$

is inner if and only if $u_1 = \cdots = u_m$. In this respect, the group $\mathrm{Inn}(L_{m,c})$ of inner automorphisms is a normal subgroup of $\mathrm{PInn}(L_{m,c})$. We refer the reader for the structure of this group to the paper [3]. Additionally, each inner automorphism of $L_{m,c}$ preserves every ideal of the Lie algebra $L_{m,c}$, and by the paper [6] we have that such ideal preservative automorphisms are another generalization of inner automorphims.

2. Preliminaries

The free metabelian nilpotent Lie algebra $L_{m,c}$ over a field K of characteristic zero is the free algebra of rank n in the variety of the Lie algebras satisfying the identities

$$[[x, y], [z, t]] = 0, \qquad \text{and} \qquad [y_1, y_2, \ldots, y_{c+1}] = 0$$

for all $x, y, z, t, y_1, y_2, \ldots, y_{c+1} \in L_{m,c}$. For more information on the Lie algebra $L_{m,c}$ we refer to the books [1, 2]. In this paper, we use the left normed commutators as below.

$$[u_1, \ldots, u_{n-1}, u_n] = [[u_1, \ldots, u_{n-1}], u_n], \quad n = 3, 4, \ldots$$

For each $v \in L_{m,c}$, the linear operator $\mathrm{ad}v : L_{m,c} \to L_{m,c}$ defined by

$$\mathrm{ad}v(u) = [u, v], \quad u \in L_{m,c},$$

is a derivation of $L_{m,c}$ which is nilpotent and $\mathrm{ad}^c v = (\mathrm{ad}v)^c = 0$ because $L_{m,c}^{c+1} = 0$, and thus the linear operator

$$e^{\mathrm{ad}(v)} = 1 + \frac{\mathrm{ad}v}{1!} + \frac{\mathrm{ad}^2 v}{2!} + \cdots + \frac{\mathrm{ad}^{c-1} v}{(c-1)!}$$

is well defined and is an automorphism of $L_{m,c}$. The set of all automorphisms are of the form $e^{\mathrm{ad}(v)}$, $v \in L_{m,c}$, is called the inner automorphism group of $L_{m,c}$ and is denoted by $\mathrm{Inn}(L_{m,c})$. The group $\mathrm{PInn}(L_{m,c})$ of pointwise inner automorphisms can be considered as a generalization of $\mathrm{Inn}(L_{m,c})$.

Our goal is to consider the group of pointwise inner automorphisms of $L_{m,4}$ and establish multiplication rule in this group for nilpotency class four.

3. Main Results

Theorem 3.1. *Let the nilpotency class $c = 4$. Then the group $\mathrm{PInn}(L_{m,4})$ of the free metabelian Lie algebra $L_{m,4}$ is metabelian. This means that*

$$[[\phi_u, \phi_v], [\phi_w, \phi_t]] = 0,$$

where $[\phi_u, \phi_v] = \phi_u\phi_v\phi_u^{-1}\phi_v^{-1}$.

Proof. In this case each element in $L_{m,4}$ is of the form

$$\sum_i c_i x_i + \sum_{i>j} c_{ij}[x_i, x_j] + \sum_{i>j\leq k} c_{ijk}[x_i, x_j, x_k].$$

Let's say

$$u_1 = \sum c_i x_i,\ u_2 = \sum_{i>j} c_{ij}[x_i, x_j],\ u_3 = \sum_{i>j\leq k} c_{ijk}[x_i, x_j, x_k] \text{ and}$$

$$v_1 = \sum_i d_i x_i,\ v_2 = \sum_{i>j} d_{ij}[x_i, x_j],\ v_3 = \sum_{i>j\leq k} d_{ijk}[x_i, x_j, x_k].$$

$$\phi_u(x) = x + [x, u] + \frac{1}{2}[x, u_1 + u_2, u_1] + \frac{1}{6}[x, u_1, u_1, u_1],$$

where $u = u_1 + u_2 + u_3$ and also let $v = v_1 + v_2 + v_3$. Hence we have

$$\phi_u\phi_v(x) = \phi_u(x + [x, v] + \frac{1}{2}[x, v_1 + v_2, v_1] + \frac{1}{6}[x, v_1, v_1, v_1]).$$

Consider the following elements:

$$w = w_1 + w_2 + w_3,\ \text{ where } w_1 = u_1 + v_1,$$

$$w_2 = u_2 + v_2 + d_1[x, u_1] + \frac{1}{2}[u_1, v_1] \text{ and}$$

$$w_3 = u_3 + v_3 + \sum_{1<i}[x, d_{i1}[x, u_1]] + \frac{1}{12}[v_1, u_1, u_1] + \frac{1}{12}[u_1, v_1, v_1].$$

Then we have

$$\phi_w(x) = x + [x, w_1] + [x, w_2] + \frac{1}{2}[x, w_1, w_1] + [x, w_3] + \frac{1}{2}[x, w_2, w_1] + \frac{1}{6}[x, w_1, w_1, w_1].$$

By some calculations we have the elements

$$w_3 = u_3 + v_3 + \sum_{i<j} d_{i1}[x_i, [x_1, u_1]] + \frac{1}{12}[v_1, u_1, u_1] + \frac{1}{12}[u_1, v_1, v_1] - \frac{1}{2}[u_1, v_1, v_1] + d_1[x_1, u_2] - \frac{1}{2}[v_2, u_1] + \frac{1}{2}[u_2, v_1].$$

And consequently we obtain $\phi_u\phi_v = \phi_{w_1+w_2+w_3}$. □

4. Conclusion

In this study, group structure of the group $\mathrm{PInn}(L_{m,4})$ was provided via multiplication rule in it. The next step might be extending the nilpotency class $c \geq 5$, and obtain new results.

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