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Periods of Leonardo Sequences and Bivariate Gaussian Leonardo Polynomials

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Abstract

In this study, we investigate the periodic characteristics of Leonardo, Leonardo-Lucas, and Gaussian Leonardo sequences, presenting our findings through lemmas and theorems. Additionally, we introduce the concept of the power Leonardo like sequences and characterize the modules and integers within which these sequences exist. Furthermore, we conduct a comparative analysis between these power sequences and the power Fibonacci sequence under the same modulus. Lastly, we define a bivariate Gaussian Leonardo polynomial sequence and obtain specific properties associated with it.

Keywords: *Leonardo sequence, Period, Bivariate polynomials.*

Leonardo Dizisinin Periyotları ve İki değişkenli Gauss Leonardo Polinomları

Öz

Bu çalışmada, Leonardo, Leonardo-Lucas ve Gaussian Leonardo dizilerinin polinomları incelendi. Sonuçlar teoremler ve lemmalar yoluyla ifade edildi. Ayrıca, Leonardo kuvvet benzeri dizileri tanımlandı. Bu dizilerin var olduğu modüller ve sayıları karakterize edildi. Ek olarak, aynı modülde bu kuvvet dizilerinin periyotları ile Fibonacci kuvvet dizisinin periyotları karşılaştırıldı. Son olarak, iki değişkenli Gaussian Leonardo polinom dizisi tanımlandı ve belirli özellikleri elde edildi.

Anahtar Kelimeler: *Leonardo dizisi, periyot, iki değişkenli polinomlar.*

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I. INTRODUCTION

The Fibonacci numbers were first described by Leonardo Fibonacci in 1202 and since then, these numbers have many applications areas such as number theory, algebra, cryptography and geometry. Fibonacci sequences which composed of these numbers are denoted by $\{F_n\}_{n=0}^{\infty}$, where F_n denotes the nth Fibonacci number. This sequence is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions $F_0 = 0, F_1 = 1, n \geq 2$.

The first few terms of this sequence are

$$0,1,1,2,3,5,8,13,21, \ldots.$$

Lucas sequence $\{L_n\}_{n=0}^{\infty}$ has the same recurrence relation with the $\{F_n\}_{n=0}^{\infty}$. The first few Lucas numbers are

$$2,1,3,4,7,11,18,29, \ldots.$$

The characteristic equation of sequences $\{F_n\}$ and $\{L_n\}$ is

$$x^2 - x - 1 = 0.$$

The Binet's formula of $\{F_n\}$ and $\{L_n\}$ are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n$$

where α and β are roots of the characteristic equation of $\{F_n\}$ and $\{L_n\}$.

Ide and Renault defined power Fibonacci sequences modulo m, [1]. Authors obtained some results about the existence, numbers and periods of these sequences in the same study.

Definition 1.1. Let G be an integer sequence satisfying the recurrence relation $G_n = G_{n-1} + G_{n-2}$. If $G \equiv 1, \alpha, \alpha^2 \ldots (\text{mod } m)$ for some modulus m, then $\{G_n\}$ is called a power Fibonacci sequence modulo m, [1].

For $m = 19$, there are two power Fibonacci sequences

$$1,5,6,11,17,9, \ldots \text{ and } 1,15,16,12,9, \ldots.$$

Theorem 1.2. There is exactly one power Fibonacci sequence modulo 5. For $m \neq 5$, there exist power Fibonacci sequences modulo m precisely when m has prime factorization $m = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$ or $m = 5 \cdot p_1^{e_1} p_2{}^{e_2} \ldots p_k^{e_k}$, where each $p_i \equiv \mp 1(\text{mod}10)$, in either case there are exactly 2^k power Fibonacci sequences modulo m, [1].

Authors established a relationship between the periods of Fibonacci sequences and the periods of power Fibonacci sequences in the same modulo in the following lemma and theorem, recursively. Here, $\pi(m)$ denotes the period of the Fibonacci sequence and $|\alpha|, |\beta|$ the order of power Fibonacci sequences modulo m.

Lemma 1.3. Let p a prime of the form $p \equiv \mp 1(\text{mod}10), \alpha$ and β be two roots of $f(x) = x^2 - x - 1(\text{mod}p^e)$. Without loss of generality, assume $|\alpha| \geq |\beta|$.

- If $\pi(p^e) \equiv 2(\text{mod}4)$, then $|\alpha| = 2|\beta| \equiv 2(\text{mod}p^e)$.
- If $\pi(p^e) \equiv 0(\text{mod}4)$, then $|\alpha| = |\beta| \equiv 0(\text{mod}p^e)$, [1].

Theorem 1.4. Let $m = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$, a product of primes of the form $p_i \equiv \mp 1(\mathrm{mod}10)$.

- If $\pi(m) \equiv 0(\mathrm{mod}4)$, then modulo m, every power Fibonacci sequence has a period $\pi(m)$.
- If $\pi(m) \equiv 2(\mathrm{mod}4)$, then modulo m, one power Fibonacci sequence has (odd) period $\frac{1}{2}\pi(m)$ and all the others have period $\pi(m)$.
- If $\pi(m) \equiv 0(\mathrm{mod}4)$, then modulo $5m$, every power Fibonacci sequence have period $\pi(m)$.
- If $\pi(m) \equiv 2(\mathrm{mod}4)$, then modulo $5m$, every power Fibonacci sequence have period $2\pi(m)$, [1].

Catarino and Borges [1] defined the Leonardo sequences in 2013.

Definition 1.5. Leonardo sequence $\{Le_n\}$ is defined by the recurrence relation $Le_n = Le_{n-1} + Le_{n-2} + 1$ with initial conditions $Le_0 = Le_1 = 1$ for $n \geq 2$, where Le_n denotes the nth Leonardo number. Also, the Leonardo sequence has homogenous recurrence relation

$$Le_{n+1} = 2Le_n - Le_{n-2}, \qquad n \geq 2.$$

The first few terms of Leonardo sequence are

$$1,1,3,5,9,15,25,41, \ldots.$$

The third-order characteristic equation of the recurrence relation of the Leonardo sequence is

$$x^3 - 2x^2 + 1 = 0.$$

Soykan defined the Leonardo matrix A as follows:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

such that $\det A = -1$, [4]. He defined the Leonardo-Lucas sequence in the same study.

Definition 1.6. Leonardo-Lucas sequence is defined as $H_n = H_{n-1} + H_{n-2} - 1$ with $H_0 = 3, H_1 = 2$ for $n \geq 2$.

The first few terms of Leonardo-Lucas numbers are

$$3,2,4,5,8,12,19,30,48,77, \ldots.$$

This sequence $\{H_n\}$ satisfies the recurrence relation

$$H_n = 2H_{n-1} - H_{n-3}\,.$$

Mangueira etc. defined the Leonardo polynomials sequence in [5] as follows.

Definition 1.7. Leonardo polynomials $l_n(x)$ are defined by the recurrence relation

$$l_n(x) = 2xl_{n-1}(x) - l_{n-3}(x)$$

with $l_0(x) = l_1(x) = 1$ and $l_2(x) = 3$, for $n \geq 3$.

The first few terms of this sequence are

$$1,1,3,6x - 1,12x^2 - 2x - 1,24x^3 - 4x^2 - 2x - 3, \ldots.$$

The authors introduced bivariate Leonardo polynomials and complex Leonardo polynomials in the same study.

Definition 1.8. Bivariate Leonardo polynomials $l_n(x, y)$ satisfy the recurrence relation

$$l_n(x, y) = 2xl_{n-1}(x, y) - yl_{n-3}(x, y)$$

with initial conditions $l_0(x,y) = l_1(x,y) = 1$ and $l_2(x,y) = 3$, for $n \geq 3$.

Definition 1.9. Bivariate Leonardo complex polynomials $l_n(ix,y)$ are defined by the recurrence relation

$$l_n(ix,y) = 2xil_{n-1}(ix,y) - yl_{n-3}(ix,y)$$

with initial conditions $l_0(ix,y) = l_1(ix,y) = 1, l_2(ix,y) = 3$ and $i^2 = -1$, for $n \geq 3$.

II. THE PERIODS OF LEONARDO SEQUENCES AND LEONARDO LUCAS SEQUENCES MODULO k

Wall D. D. examined the Fibonacci sequence modulo m and obtained some results in 1960, [6].
In this section, we examine periods of Leonardo and Leonardo-Lucas sequences in the same modulo. The periods of the Leonardo and Leonardo-Lucas sequence are denoted with $l(k)$ and $l^*(k)$ modulo k, respectively.

Theorem 2.1. Leonardo sequence is periodic for modulo $k \geq 2$.

Proof. We have $Le_{n-2} = 2Le_n - Le_{n+1}$ from the recurrence relation of the Leonardo sequence. If $Le_{m+1} \equiv Le_{r+1}(\text{mod}k)$ and $Le_m \equiv Le_r(\text{mod}k)$, then $Le_{m-2} \equiv Le_{r-2}, \ldots Le_{m-r+1} \equiv Le_1$ and $Le_{m-r} \equiv Le_0$, so the sequence is periodic.

Similarly, we can see that the Leonardo-Lucas sequence is periodic because the recurrence relation of this sequence is the same as the Leonardo sequence.

Example 2.2. For modulo $k = 5$, Leonardo and Leonardo-Lucas sequences are as follows, recursively.

$$1,1,3,0,4,0,0,1,2,4,2,\ldots.$$
$$3,2,4,0,3,2,4,0,3,2,\ldots.$$

The following Theorems 2.3 and 2.4 can be proved as in [6].

Theorem 2.3. Let $gcd(k,t) = 1$. Then,

- $l(kt) = lcm(l(k), l(t))$,
- $l^*(kt) = lcm(l^*(k), l^*(t))$.

Theorem 2.4. Let k be a prime. Then, the following properties are satisfied.

- If $k \equiv \mp 3(\text{mod}10)$, then $l^*(k) = l(k) \mid 2 \cdot (k+1)$.
- If $k \equiv \mp 1(\text{mod}10)$, then $l^*(k) = l(k) \mid k-1$.

Lemma 2.5. For $k \geq 2$, we have

- $l(2^k) = 2l(2^{k-1})$,
- $l^*(2^k) = 2l^*(2^{k-1})$.

Theorem 2.6. For $k = 2^r, r \geq 2$, we have $l^*(k) = 2 \cdot l(k)$.

Proof. We prove this theorem with mathematical induction on r. For $r = 1$, we have $2 \cdot l(2^2) = l^*(2)$, and this statement is true. Let us assume that is true for $r = n$. So, we have $l^*(2^n) = 2 \cdot l(2^n)$. We will demonstrate that this is true for $r = n + 1$. We can write $l^*(2^{n+1}) = 2 \cdot l^*(2^n)$ from previous Lemma. Since $l^*(2^n) = 2 \cdot l(2^n)$, then we get $l^*(2^{n+1}) = 2 \cdot 2 \cdot l(2^n)$.

Finally, we obtain $l^*(2^{n+1}) = 2 \cdot l(2^{n+1})$ from previous Lemma.

We will give the relation between Leonardo matrix A and periods of Leonardo sequences with the following Lemma.

Lemma 2.7. $A^{l(k)} \equiv I(\mathrm{mod} k)$, where I is an identity matrix. Then, we can obtain $A^{l(k)} \equiv I + kC$, where $C = [c_{ij}]$ is a matrix such that $0 \leq c_{ij} \leq k - 1$ and c_{ij} are non-negative integers.

Example 2.8. For $k = 11$, the period of Leonardo sequence $l(k) = 10$. Then,

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{10} = \begin{bmatrix} 232 & -88 & -143 \\ 143 & -54 & -88 \\ 88 & -33 & -54 \end{bmatrix}$$

and we can see

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\mathrm{mod} 11).$$

Theorem 2.9. Let k be an arbitrary odd prime. Then $l(k^r) = k^{r-1} \cdot l(k)$, where $r \in \mathbb{Z}^+$.

For $k = 2$, we obtain the following Lemma.

Lemma 2.10. $l(2^{r+1}) = 2 \cdot l(2^r)$.

Theorem 2.11. Let $k = 2^r, r > 1$. Then, $l(2) = 1$ and $l(k) = 3 \cdot 2^{k-2}$.

Proof. It can be proven easily by induction and using the previous Lemma.

III. POWER LEONARDO LIKE SEQUENCES MODULO k

In this section, we describe power Leonardo like sequences Moreover, we examine periods of these sequences and compare them to periods of power Fibonacci sequences in some modulo.

Definition 3.1. Let L_p be an infinite sequence of integers satisfy the recurrence relation

$$L_{p_{n+2}} = 2L_{p_{n+1}} - L_{p_{n-1}}.$$

If $L_p \equiv 1, \theta, \theta^2, \theta^3, \ldots (\mathrm{mod} k)$, then L_p is called power Leonardo like sequence modulo k.

Example 3.2. For modulo $k = 55$, there are six power Leonardo like sequences.

$$\begin{array}{l} 1,1,1,1, \ldots, \\ 1,8,9,17,26,43, \ldots, \\ 1,23,34,12,1,23, \ldots, \\ 1,26,16,31,36,1, \ldots, \\ 1,41,31,6,26,21, \ldots, \\ 1,48,49,42,36, \ldots. \end{array}$$

Theorem 3.3. There are two power Leonardo like sequences for modulo 5. For $k \neq 5$, if k has prime factorization $k = t_1^{\phi_1} t_2^{\phi_2} \dots t_s^{\phi_s}$, then there are 3^s power Leonardo like sequences modulo k and if k has prime factorization $k = 5.t_1^{\phi_1} t_2^{\phi_2} \dots t_s^{\phi_s}$, then there are 2.3^s power Leonardo like sequences modulo k, where $t_j \equiv \mp 1(\text{mod}10),\ j = 1, \dots, s$.

Proof. Let $d(x)$ be a characteristic equation of the Leonardo sequence. Hence, the discriminant of $d(x)$ is 5 and $d(x)$ has three distinct roots. There is an apparent solution for each modulo due to one of the roots of $d(x)$ is 1 . For other solutions, $x^2 \equiv 5(\text{mod}5^e)$ has one solution only when $e = 1$ for modulo 5. Therefore, we can write two power Leonardo like sequences.

For $k \neq 5$, there are three solutions one of those solutions is 1 . Also, 5 is a quadratic residue for modulo $t_j \equiv \mp 1(\text{mod}10)$. Hence, by using Hensel's Lemma and Chinese Remainder Theorem, [8], it is seen that $d(x)$ has 3^s roots modulo k. Similarly, it can be seen that $d(x)$ has 2.3^s roots modulo $5k$.

Gaussian Leonardo numbers are defined as in [7]. The Gaussian Leonardo sequence $\{GLe_n\}_{n=0}^{\infty}$ is defined by the recurrence relation

$$GLe_n = GLe_{n-1} + GLe_{n-2} + (1+i), \qquad n \geq 2$$

with initial conditions $GLe_0 = 1 - i, GLe_1 = 1 + i$.

The homogenous form of this recurrence relation is

$$GLe_{n+1} = 2GLe_n - GLe_{n-2}, \qquad n \geq 2.$$

We noticed that Gaussian Leonardo sequence is periodic. The periods of the Gaussian Leonardo sequence are denoted with $\varphi_g(k)$ modulo k for the following numerical examples.

For $k = 3$ and $k = 3^2$, then the Gaussian Leonardo sequences are as follows, respectively.

$$1 - i, 1 + i, i, 2i, 0, 1, 2 + i, 1 + 2i, \dots,$$
$$1 - i, 1 + i, 3 + i, 5 + 3i, 5i, 6, 7 + 6i, 5 + 7i, \dots.$$

Then, we get $\varphi_g(9) = 3.\varphi_g(3)$.

For $k = 6$, the Gaussian Leonardo sequence is

$$1 - i, 1 + i, 3 + i, 5 + 3i, 3 + 5i, 3 + 3i, \dots$$

and it is obtained $\varphi_g(6) = 2 \cdot \varphi_g(3)$.

When we examined periods of power Leonardo like sequences, we noticed a similarity with power Fibonacci sequences. Although there is a resemblance between these sequences, we also observed some differences as follows.

i) There is no power Fibonacci sequence with the period of one.

ii) There is a power Leonardo like sequence with the period of one. This number sequence is 1,1,1, ...

Furthermore, the period lengths of these sequences are equal in the same modulo.

We get the results in the following table with numerical examples (see Table 1). Here, the period lengths of the power Leonardo like sequence and the period lengths of the power Fibonacci sequence are denoted by $\rho(k)$ and $|k|$ modulo k, respectively.

Table 1. *The period lengths of the power Leonardo like sequence and the period lengths of the power Fibonacci sequence*

k	$p(k)$	$\lvert k \rvert$
11	1,10,5	10,5
19	1,18,9	18,9
29	1,28,28	28,28
31	1,30,15	30,15
41	1,40,40	40,40
59	1,58,29	58,29
95	1,94,47	94,47
121	1,120,120	120,120

For $k = 15$, the Gaussian Leonardo sequence is

$$1 - i, 1 + i, 3 + i, 5 + 3i, 9 + 5i, 9i, 10, 11 + 10i, 7 + 11i, \ldots.$$

It can be seen that $\varphi_g(15) = [\varphi_g(3), \varphi_g(5)]$. Consequently, when the periods of Gaussian Leonardo sequences are examined, it is seen that these number sequences have some properties in terms of period lengths. Because; the coefficients of the Gaussian Leonardo numbers are the same as the Leonardo numbers.

IV. BIVARIATE GAUSSIAN LEONARDO POLYNOMIALS

In 2023, Gaussian Leonardo polynomial sequence is defined as in [9]. The Gaussian Leonardo polynomial sequence $\{GLe_n(x)\}_{n=1}^{\infty}$ is defined by

$$G\mathrm{Le}_{n+1}(x) = 2xG\mathrm{Le}_n(x) - G\mathrm{Le}_n(x)$$

for $n \geq 2$, with initial conditions $GLe_0(x) = 1 - i, GLe_1(x) = x + 1$.

The first few terms of this sequence are as follows

$$1 - i, x + i, 3x + i, 6x^2 + 2xi - 1 + i, \ldots.$$

In this section, we define bivariate Gaussian Leonardo polynomial sequences and explore some properties of this sequence.

Definition 4.1. The bivariate Gaussian Leonardo polynomial sequence $\{GLe_n(x,y)\}_{n=0}^{\infty}$ is defined by the recurrence relation

$$G\ \mathrm{Le}_{n+1}(x,y) = 2xGLe_n(x,y) - yGLe_{n-2}(x,y)$$

for $n \geq 2$, with initial conditions $GLe_0(x,y) = 1 - i, GLe_1(x,y) = x + i, GLe_2(x,y) = 3x + i$.

The first few terms of this sequence as follows

$$1 - i, x + i, 3x + i, 6x^2 + 2xi + iy - y, 12x^3 + 4x^2i + 2xyi - 3xy - iy, \ldots.$$

Theorem 4.2. The generating function of bivariate Gaussian Leonardo polynomial sequence is

$$gl(t) = \sum_{n=0}^{\infty} GLe_n(x,y)t^n = \frac{-2x(xt+t-2)+1+i}{1-2xt+yt^3}.$$

Proof. We can write

$$\sum_{n=0}^{\infty} GLe_{n+3}(x,y)t^n = 2x\sum_{n=0}^{\infty} GLe_{n+2}(x,y)t^n - y\sum_{n=0}^{\infty} GLe_n(x,y)t^n$$

from recurrence relation.

Then, we get

$$\sum_{n=3}^{\infty} GLe_n(x,y)t^{n-3} = 2x\sum_{n=2}^{\infty} G\,\mathrm{Le}_n(x,y)\,t^{n-2} - y\sum_{n=0}^{\infty} GLe_n(x,y)$$

$$= \frac{1}{t^3}\sum_{n=3}^{\infty} G\,\mathrm{Le}_n(x,y)\,t^n$$

$$= 2x\frac{1}{t^2}\sum_{n=2}^{\infty} G\,\mathrm{Le}_n(x,y)\,t^n - y\sum_{n=0}^{\infty} G\,\mathrm{Le}_n(x,y)\,t^n .$$

We have $gl(t) = \sum_{n=0}^{\infty} G\mathrm{Le}_n(x,y)t^n$, so we obtain

$$gl(t)(1-2xt+yt^3) = -2x(xt+t-2)+1+i$$

and the proof is completed with the help of necessary operations.

Theorem 4.3. Binet's formula for bivariate Gaussian Leonardo polynomial sequence is

$$G\mathrm{Le}_n(x,y) = \alpha t_1^n + \beta t_2^n + \gamma t_3^n$$

where

$$\alpha = \frac{3+(-t_2-t_3)+t_2t_3}{t_1^2-t_1t_2-t_1t_3+t_2t_3},$$

$$\beta = \frac{3+(-t_1-t_3)+t_1t_3}{t_2^2-t_2t_3-t_1t_2+t_1t_3},$$

$$\gamma = \frac{3+(-t_1-t_2)+t_1t_2}{t_3^2-t_1t_2-t_1t_3-t_2t_3}$$

t_1, t_2, t_3 are the roots of characteristic equation

$$t^3 - 2xt^2 + yt = 0.$$

Lemma 4.4. For $n \geq 2$, the matrix form of the bivariate Gaussian Leonardo polynomial sequence is as follows.

$$\begin{bmatrix} 2x & 1 & 0 \\ 0 & 0 & 1 \\ -y & 0 & 0 \end{bmatrix}.$$

Then,

$$[3x+i\ \ x+1\ \ 1-i]\begin{bmatrix} 2x & 1 & 0 \\ 0 & 0 & 1 \\ -y & 0 & 0 \end{bmatrix}^n = [GLe_n(x,y)\ \ GLe_{n-1}(x,y)\ \ GLe_{n-2}(x,y)].$$

Theorem 4.5. For $k \geq 2$, we have

$$GLe_n(x,y) = \begin{bmatrix} GLe_n(x,y) & GLe_{n-1}(x,y) & GLe_{n-2}(x,y) \\ GLe_{n+1}(x,y) & GLe_n(x,y) & GLe_{n-1}(x,y) \\ GLe_{n+2}(x,y) & GLe_{n+1}(x,y) & GLe_n(x,y) \end{bmatrix}.$$

Proof. The proof of this theorem can be obtained by using the previous Lemma and induction easily.

V. CONCLUSION

In this paper, the periods of Leonardo and Leonardo-Lucas sequences are explored and established through the derivation of numerical values. Additionally, power Leonardo like sequences are defined, and certain properties of these sequences are derived. The periods of the Gaussian Leonardo sequence are also investigated. The results of these analyses reveal that the period properties of sequences based on the Leonardo sequence exhibit similarities. Finally, bivariate Gaussian Leonardo polynomial sequences are introduced. This study has the potential to extend into academic fields such as coding theory and cryptography, where the concept of period is frequently utilized.

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