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Existence and Uniqueness of Almost Periodic Solutions to Time Delay Differential Equations

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Abstract

In this study, a time-delayed relational neural network model was examined. The existence of almost periodic solutions of time delay differential equation has been investigated, and its uniqueness and stability have been proven with the help of Lyapunov point theorem and differential inequality. Since the signal transmission process of the neural network changes periodically in the real world, many scientists have focused on this issue and the almost periodic function has gained great importance because it is more suitable for practical events. The results found are different from and complementary to the previous ones.

Keywords: Almost Periodic; Lyapunov point theorem; Globally exponentially stable; Time-Varying Delays; Neural; Dynamic system.

Zaman Gecikmeli Diferansiyel Denklemlerin Yaklaşık Periyodik Çözümlerinin Varlığı ve Tekliği

Öz

Bu çalışmada zaman gecikmeli bir ilişkisel sinir ağı modeli incelenmiştir. Zaman gecikmeli diferansiyel denklemin neredeyse periyodik çözümlerinin varlığı araştırılmış, Lyapunov nokta teoremi ve diferansiyel eşitsizlik yardımıyla tekliği ve kararlılığı kanıtlanmıştır. Gerçek dünyada sinir ağının sinyal iletim süreci periyodik olarak değiştiğinden, birçok bilim adamı bu konuya ağırlık vermiş ve

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pratik olaylara daha uygun olması sebebi ile yaklaşık periyodik fonksiyon çok önem kazanmıştır. Bulunan sonuçlar öncekilerden farklı ve onları tamamlayıcı niteliktedir.

Anahtar Kelimeler: Yaklaşık periyodik; Lyapunov nokta teoremi; Küresel olarak kararlı; Zamanla Değişen Gecikmeler; Sinirsel; Dinamik sistem.

1. Introduction

Identifying and solving complex problems that cannot be solved with ordinary algorithms that we encounter in daily life is an interesting situation. Neural networks, which can both produce solutions to complex problems and are used in many multidisciplinary fields, have become a topic of current study and have attracted the attention of many writers, giving them the opportunity to be applied in different fields.

The first developed Hopfield artificial neural network model is considered to be the basis of neural networks. Hopfield proposed a Lyapunov function with positive values for this artificial neural network, which is a dynamic model [1]. This function is neural and is based on the presence of symmetric connection matrices in networks. Hopfield showed that the time derivative of this function is negative. This shows the stability of the system. The global stability of the system is proven by the symmetric connection matrix. Also, it is necessary to understand the properties of the activation functions and the values of the connection coefficients between neurons. Delay in the model is important for dynamic behavior. This delayed model was modeled using a circuit comprising neurons, an operational amplifier, and its connected resistance and capacitance elements. Dynamic neural network models have been frequently used in applications such as classification of examples, optimization and associative memory. Additionally, Cohen-Grossberg neural network models were examined and used to solve optimization problems, scientific fields and computational technology, and analog-digital converter design.

Cellular neural networks were discussed for the first time by L. O. Chua and L. Yang [2]. Cellular artificial neural networks consist of interconnected and mostly two-dimensional cells. The most important feature that distinguishes this structure from other models is that the connection weight coefficients create an invariant connection network on the studied plane. Cellular artificial neural networks have known features. In addition, because of their two-dimensional structure, they are frequently used in subjects such as image processing and pattern recognition. Relational memory networks are also important neural networks. Associative systems differ from those that can establish relationships between certain input and output vectors. Associative memory networks must remember the examples given to them during training, they memorize or store these examples. Associative memory networks are frequently used in applications such as pattern recognition, prediction, and completion of correct data from missing data.

For neural dynamic systems, the stability feature is a phenomenon that needs to be examined separately because it includes many dynamic behaviors. It affects some dynamic situations, such as the periodic oscillation behavior of the stability feature. The stability of delayed neural networks has many applications in many fields and has become an important argument in scientific studies. Application areas in the scientific field are image and signal processing, pattern recognition, optimization, etc. it is in the form. Time-varying delays in network activations are possible. The phenomenon of signal transmission in neural networks corresponds to periodic problems in applied sciences, and when there is a delay in the problems, stability analysis comes into play. To date, many researchers have addressed and examined the almost periodic problems we encounter in neural networks. In the neural network system, negative feedback can be called forgetting delay. Dependent population dynamics and neutral type time delays seen in vibrating bodies have recently become widely used. According to research, the number of papers on the stability and existence of an almost periodic solution in neutral neural networks with time-varying delays is small. Therefore, this study has an important place in the literature.

The existence of delayed almost periodic solutions for some energy-consuming systems in the generalized single-layer case in neural networks belongs to Kato [3]. Similarly, almost periodic solutions for nonlinear systems are given in [4,5]. There is also the theory of almost periodic functions for delayed differential equations and the first studies on the existence of ordinary or almost periodic solutions [6,7]. Li et al. [8] studied nearly periodic solutions and global exponential synchronization for delayed quaternion-valued neural networks. Researchers have recently produced interesting results on the stability of neural networks [9-12]. The almost periodic function, which is a continuation of the periodic function, was preferred by many scientists because it was more suitable for practical events and was introduced into mathematics by Bohr [13]. Stepanov processed periodicity without using continuity. Studies on the Bohr theory are given in general form in [14-19]. The results obtained have become one of the most special research topics in qualitative theory because of their importance and applications in physical science. The existence of almost periodic, asymptotically almost periodic and pseudo almost periodic solutions are among the most interesting topics of the qualitative theory of differential equations and difference equations, especially due to their applications in biology, economics and physics, and the researches have gained an important place in different fields [20-22].

In this study, the existence, stability and effects of an almost periodic solution for a time-delayed neural network model were investigated. Almost periodic solutions of a neutral bidirectional associative memory (BAM) neural network based on time-varying delays have been studied. In the light of the studies carried out, the delayed association neural network model was examined in this article. The existence and stability of nearly periodic solutions of the neutral BAM neural network based on varying delays were investigated. The existence of an almost periodic solution for this neural network was examined. Unlike other studies, behavioral solutions are examined directly in the field of memory. In the solutions of discontinuous neural networks, findings are created that will discover the chaos in the neuron and create inertia and nonlinear effects. These results extend neural networks and some previously known networks to some extent under given conditions. Thus, the study complements previous results. We initially describe the sufficient conditions for existence and stability, along with some definitions and preliminary results. We will use these to prove our main results. In section 3, we demonstrate the uniqueness and stability of the almost-periodic solution of the neutral neural network determined by time-varying delays.

2. Materials and Methods

2.1 Preliminaries

In the first part, we will give basic information about the subject, its features and the methods used in the analysis. We aim to prove our main results using these. First, let's give the properties of the almost periodic solution of a neutral neural network. What we consider in this study

$$u_i'(t) = -a_i u_i(t) + \sum_{j=1}^{n} w_{ij} g_j\left(u_j(t)\right) + \sum_{j=1}^{n} v_{ij} g_j\left(u_j(t-\tau)\right) + \sum_{j=1}^{n} z_{ij}\left(u_j(t-\tau)\right) + \varphi_i, \tag{2.1}$$

here $u_i(t)$ is the state vector of the time t. The constants a_i, w_{ij}, v_{ij}, z_{ij} are the connection weight parameters of the neural networks, φ_i is the output value, and a_i,$w_{ij}, v_{ij}, z_{ij}, \varphi_i$: $R \to R$ are almost periodic function for $i, j = 1,2, \dots, n$, $\tau > 0$ correspond to leakage and transmission delay respectively, g_j is the activation function of the ith neurons and the initial conditions with the equation (2.1) are of the form,

$$u_i(k) = \delta_i(k) \text{ for all } k \in [-\infty, 0], \ i = 1,2, \dots, n,$$

where $\delta_i(.)$ are continuous and real valued functions.

The aim of this study is to establish sufficient conditions for the existence and uniqueness of nearly periodic solutions for a given neural network. We can explain the theory of almost periodic functions as follows; for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\Upsilon = \Upsilon(\epsilon)$ in this interval such that,

$$| a_i(t + \Upsilon) - a_i(t)| < \epsilon, \left|w_{ij}(t + \Upsilon) - w_{ij}(t)\right| < \epsilon, \left|v_{ij}(t + \Upsilon) - v_{ij}(t)\right| < \epsilon,$$

$$|\varphi_i(t + \Upsilon) - \varphi_i(t)| < \epsilon, \left|z_{ij}(t + \Upsilon) - z_{ij}(t)\right| < \epsilon, \text{ for all } t \in R.$$

Equilibrium Point: If the condition $f(x_e,t) = 0$ is satisfied for every $t \geq t_0$, then x_e is the equilibrium point of the system $\dot{x} = f(x,t)$, $x(t_0) = x_0$.

Theorem 2.1 (Lyapunov Stability Theorem) ([20]). Let the equilibrium point be $x = 0$ for a system given as $\dot{x} = f(x)$. Let $V(x)$: $R^n \to R$ be a continuous and differentiable function. The time derivative of the function $V(x)$ is denoted by $\dot{V}(x)$ and is expressed as follows,

$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} f_i(x_i) = \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} f(x).$

i) If $V(0) = 0, V(x) > 0, \forall x \neq 0$ and $\dot{V}(x) \leq 0$, $\forall x \epsilon R^n$ if the condition is met, it is stable for $x = 0$.

ii) If the condition $\dot{V}(x) < 0, \forall x \neq 0$ is satisfied, $x = 0$ is asymptotically stable.

iii) If the condition $\dot{V}(x) > 0, \forall x \neq 0$ is satisfied, $x = 0$ is unstable.

It follows from this theorem that;

The function $\dot{V}(x)$ is negative semi-definite (i.e., $\dot{V}(x) \leq 0$, $\forall x \neq 0$), and $V(x)$ is a continuously differentiable positive definite function (i.e., $V(x) > 0$, $\forall x \neq 0$), the equilibrium point is stable.

If $\dot{V}(x)$ is strictly negative definite ($\dot{V}(x) < 0, \forall x \neq 0$), the equilibrium point is asymptotically stable.

If $\dot{V}(x)$ is positive definite ($\dot{V}(x) > 0, \forall x \neq 0$), the equilibrium point is unstable.

Lyapunov Stability Theorem is used to draw a conclusion about the stability properties of the equilibrium point and can characterize the stability of the equilibrium point by defining a positive energy function and examining its time derivative. One of the advantages of this theorem, called the Direct Method of Lyapunov Stability Theorem, allows us to determine the stability properties of the equilibrium point without solving the differential equation of the system.

Definition 2.2 ([6]). A continuous function $x : R \to R^n$ is said to be almost periodic on R if, for any $\epsilon > 0$, the set $T(u, \epsilon) = \{\Upsilon : \| u(t + \Upsilon) - u(t) \|\} < \epsilon$ for all $t \in R\}$ is relatively dense, for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$ with the property that, for any interval with length $l(\epsilon)$, there exists a number $\Upsilon = \Upsilon(\epsilon)$ in this interval such that

$\| u(t + \Upsilon) - u(t) \| < \epsilon$ for all $t \in R$.

Lemma 2.3 ([21]). Let $S^*(t) = (u_1^*(t), u_2^*(t), \dots, u_n^*(t))^T$ be an almost-periodic solution of equation (2.1) with initial value $(\phi_1^*(t), \phi_2^*(t), \dots, \phi_n^*(t))^T$. If there $\delta > 0$ and $K > 1$ such that for all solution $S(t) = (u_1(t), u_2(t), .., u_n(t))^T$ be the solution of equation (2.1) with initial value,

$|u_i(t) - u_i^*(t)| \leq K e^{-\delta t} \|\phi - \phi^*\|, t > 0, \ i = 1,2, \dots, n,$ where

$\|\phi - \phi^*\| = sup \max_{1 \leq j \leq n} |\phi_i(t) - \phi_i^*(t)|$. Then $S^*(t)$ is said to be globally exponentially stable.

In addition, we assume that the following conditions are met,

H1) $a_i > 0$, $w_{ij}, v_{ij}, z_{ij} : R \to R$ all continuous almost periodic functions, where a_i and $\tau > 0$ are constants and $i, j = 1,2, \dots, n$.

H2) The constants $a_i, w_{ij}, v_{ij}, z_{ij}$ and φ_i are taken as follows:

$\overline{a_i} = \sup_{t \in R} | a_i(t)|, \ \overline{w_{ij}} = \sup_{t \in R} |w_{ij}(t)|, \overline{v_{ij}} = \sup_{t \in R} |v_{ij}(t)|,$

$\overline{z_{ij}} = \sup_{t \in R} |z_{ij}(t)|, \overline{\varphi_i} = \sup_{t \in R} |\varphi_i(t)| \ , \ i, j = 1,2, \dots, n.$

H3) For each $j \in \{1,2,\dots,n\}$, is with Lipschitz constant U_j,

$$|g_j(u_j) - g_j(v_j)| \leq U_j|u_j - v_j|,\ u_j, v_j \in R.$$

H4) There exist constants $\eta > 0$ and $i,j = 1,2,\dots,n$, such that for all $t \in R$, there holds

$$-a_i(t) + \xi_i^{-1}\sum_{j=1}^{n}[|w_{ij}| + |v_{ij}| + |z_{ij}|]\,U_j,\ \xi_j < -\eta < 0.$$

2.2. Existence of Almost Periodic Solutions

In this section, the existence and stability of the almost periodic solution of the equation (2.1) under the given conditions will be demonstrated.

Theorem 2.4. Assume that (H1)–(H4) are satisfied. In this case, the equation (2.1) has a unique, continuously differentiable, almost periodic solution.

Proof. Let's take $\bar{u}_1(t) = \xi_i^{-1}u_i(t)$.

Now we can convert (2.1) into the following equation,

$$\bar{u}_i'(t) = -a_i\,\bar{u}_i(t)$$

$$+\xi i^{-1}\sum_{j=1}^{n} w_{ij}g_j\xi_i\left(\bar{u}_j(t)\right) + \xi_i^{-1}\sum_{j=1}^{n} v_{ij}g_j\xi_i\left(\bar{u}_j(t-\tau)\right) + \xi_i^{-1}\sum_{j=1}^{n} z_{ij}\xi_i\left(\bar{u}_j(t-\tau)\right) + \xi_i^{-1}\varphi_i$$

Let $\forall\theta \in B$ be a Banach space and then, B is a Banach space, we consider the almost periodic solution $u^{\theta}(t)$ of nonlinear almost periodic differential equations,

$$\bar{u}_1'(t) = -a_i\,\bar{u}_i(t)$$

$$+\xi_i^{-1}\sum_{j=1}^{n} w_{ij}g_j\xi_i\left(\theta_j(t)\right) + \xi_i^{-1}\sum_{j=1}^{n} v_{ij}g_j\xi_i\left(\theta_j(t-\tau)\right) + \xi_i^{-1}\sum_{j=1}^{n} z_{ij}\xi_i\left(\theta_j(t-\tau)\right) + \xi_i^{-1}\varphi_i$$

Let us obtain an almost periodic solution of equation (2.1),

$$u^{\theta}(t) = (u_1{}^{\theta}(t), u_2{}^{\theta}(t), \dots, u_n{}^{\theta}(t))^T$$

$$=(e^{\int_{-k}^{t}u_1(k)dk}\left[\xi_n^{-1}\sum_{j=1}^{n} w_{1j}g_j\xi_n\left(\theta_j(k)\right) + \xi_n^{-1}\sum_{j=1}^{n} v_{1j}g_j\xi_n\left(\theta_j(k-\tau)\right) + \right.$$

$$\left.\xi_n^{-1}\sum_{j=1}^{n} z_{1j}\xi_n\left(\theta_j(k-\tau)\right) + \xi_n^{-1}\varphi_i(k)\right]dk, \dots, e^{\int_{-k}^{t}u_n(k)dk}\left[\xi_n^{-1}\sum_{j=1}^{n} w_{nj}g_j\xi_n\left(\theta_j(k)\right) + \right.$$

$$\left.\xi_n^{-1}\sum_{j=1}^{n} v_{nj}g_j\xi_n\left(\theta_j(k-\tau)\right) + \xi_n^{-1}\sum_{j=1}^{n} z_{nj}\xi_n\left(\theta_j(k-\tau)\right) + \xi_n^{-1}\varphi_i(k)\right]dk)^T. \quad (2.2)$$

Let's define the transformation $A : B \to B$ as follows:

$$A\,\theta(t) = u^{\theta}(t),\ \forall\,\theta \in B.$$

We can prove that the transformation of A corresponds to the contraction of B. We will prove that there is a mapping. Using Lemma (2.3), (H1-H4), for $\forall\,\theta,\psi \in B$,

$$|A(\theta(t)) - A(\psi(t))| = (|(A(\theta(t)) - A(\psi(t)))_1|, \dots, |(A(\theta(t)) - A(\psi(t)))_n|)^T$$

$=(\left| e^{\int_{-k}^{t} u_1(k)dk} \left[\xi_n^{-1} \sum_{j=1}^{n} w_{1j}(g_j\xi_n\left(\theta_j(k)\right) -(g_j\xi_n\left(\psi_j(k)\right) + \xi_n^{-1} \sum_{j=1}^{n} v_{1j}(g_j\xi_n\left(\theta_j(k-\tau)\right) - g_j\xi_n\left(\psi_j(k-\tau)\right)) + \xi_n^{-1} \sum_{j=1}^{n} z_{1j}(\xi_n\left(\theta_j(k-\tau)\right) - \xi_n\left(\psi_j(k-\tau)\right)\right] dk\right|, \ldots,$

$\left|e^{\int_{-k}^{t} u_n(k)dk} \left[\xi_n^{-1} \sum_{j=1}^{n} w_{nj}(g_j\xi_n\left(\theta_j(k)\right) - g_j\xi_n\left(\psi_j(k)\right)) + \xi_n^{-1} \sum_{j=1}^{n} v_{nj}(g_j\xi_n\left(\theta_j(k-\tau)\right) - g_j\xi_n\left(\psi_j(k-\tau)\right)) + \xi_n^{-1} \sum_{j=1}^{n} z_{nj}(\xi_n\left(\theta_j(k-\tau)\right) - \xi_n\left(\psi_j(k-\tau)\right))\right] dk\right|)^T$

$\leq (e^{\int_{-k}^{t} u_1(k)dk} \left[\sum_{j=1}^{n} \overline{w_{1j}} U_j \left|\left(\theta_j(k)\right) - \left(\psi_j(k)\right)\right| + \sum_{j=1}^{n} \overline{v_{1j}} U_j \left|\left(\theta_j(k-\tau)\right) - \left(\psi_j(k-\tau)\right)\right| + \sum_{j=1}^{n} \overline{z_{1j}} U_j \left|\left(\theta_j(k-\tau)\right) - \left(\psi_j(k-\tau)\right)\right|\right] dk, \ldots, e^{\int_{-k}^{t} u_n(k)dk} \left[\sum_{j=1}^{n} \overline{w_{nj}} U_j \left|\left(\theta_j(k)\right) - \left(\psi_j(k)\right)\right| + \xi_n^{-1} \sum_{j=1}^{n} \overline{v_{nj}} U_j \left|\left(\theta_j(k-\tau)\right) - \left(\psi_j(k-\tau)\right)\right| + \sum_{j=1}^{n} \overline{z_{nj}} U_j \left|\left(\theta_j(k-\tau)\right) - \left(\psi_j(k-\tau)\right)\right|\right] dk)^T$

$$\leq \sum_{j=1}^{n} \overline{w_{1j}} + \overline{v_{1j}} + \overline{z_{1j}} U_j \, sup \left|\left(\theta_j(k)\right) - \left(\psi_j(k)\right)\right|, \ldots, \sum_{j=1}^{n} \overline{w_{nj}} + \overline{v_{nj}} + \overline{z_{nj}} U_j \, sup \left|\left(\theta_j(k)\right) - \left(\psi_j(k)\right)\right|)^T$$

$$=(sup \left|\left(\theta_j(k)\right) - \left(\psi_j(k)\right)\right|, \ldots, sup \left|\left(\theta_j(k)\right) - \left(\psi_j(k)\right)\right|)^T \tag{2.3}$$

Let m be a positive integer. Then, from (2.3), we get

$=(sup \left|A^m\left(\theta_j(k)\right) - A^m\left(\psi_j(k)\right)_1\right|, \ldots, sup \left|A^m\left(\theta_j(k)\right) - A^m\left(\psi_j(k)\right)_n\right|)^T$

$= (sup \left|A(A^{m-1}\left(\theta_j(k)\right) - A(A^{m-1}\left(\psi_j(k)\right)_1\right|, \ldots, sup \left|A(A^{m-1}\left(\theta_j(k)\right) - A(A^{m-1}\left(\psi_j(k)\right)_n\right|)^T$

$\leq (sup \left|\left(\theta_j(k)\right) - \left(\psi_j(k)\right)_1\right|, \ldots, sup \left|\left(\theta_j(k)\right) - \left(\psi_j(k)\right)_n\right|)^T$

$$= (sup \left|\left(\theta_j(k)\right) - \left(\psi_j(k)\right)\right|, \ldots, sup \left|\left(\theta_j(k)\right) - \left(\psi_j(k)\right)\right|)^T \tag{2.4}$$

In view of (3.3) we have

$$\left|A^N\left(\left(\theta(t)\right)\right) - A^N\left(\psi(t)\right)_i\right| \leq \ sup \left|A^N\left(\theta(t)\right) - A^N\left(\psi(t)\right)_1\right| \leq$$

$$\sum_{j=1}^{n} l sup \ \left|\left(\left(\theta(t)\right)\right) - \left(\psi(t)\right)\right| \leq sup \max_{1\leq j\leq n} \left|\left(\left(\theta(t)\right)\right) - \left(\psi(t)\right)\right| \sum_{j=1}^{n} l \leq \left\|\left(\theta(t)\right) - \left(\psi(t)\right)\right\|$$

for all $t \in R, i, j = 1, 2, \ldots, n$. It follows that

$$\left\|\left(A^N\left(\theta(t)\right)\right) - A^N\left(\psi(t)\right)\right\| = sup \max_{1\leq j\leq n} \left|\left(A^N\left(\theta(t)\right)\right) - A^N\left(\psi(t)\right)\right| \leq \left\|\left(\theta(t)\right) - \left(\psi(t)\right)\right\| \tag{2.5}$$

Thus, the condition given in Lemma (2.3) is met. This guarantees the existence and uniqueness of the almost periodic solution of equation (2.1) expressed in Theorem (2.4) and the proof is completed.

2.3. Globally Exponentially Stability of the Almost Periodic Solution

Theorem 2.5. Assume that (H1)–(H4) are satisfied. Then equation (2.1) has exactly one almost periodic solution $x^*(t)$. Moreover, this solution is globally exponentially stable.

Proof. Let $u^{\theta}(t) = (u_1{}^{\theta}(t), u_2{}^{\theta}(t), \dots, u_n{}^{\theta}(t))^T$ be a solution of equation (2.1) with initial conditions

$$u_i(k) = \delta_i(k), \ \ |\delta_i(k)| < \rho, \ k \in [-\infty, 0]. \ i = 1,2,\dots,n,$$

Thus, according to Lemma (2.3), the solution $u_i(k)$ is bounded and $|u_i(t)| < \rho$,

for all $t \in R, i = 1, 2, \cdots, n.$

$S^*(t) = (x_1^*(t), x_2^*(t), \dots x_m^*(t))^T$ be an almost-periodic solution of equation (2.1) with initial value $(\phi_1{}^*(t), \phi_2{}^*(t), \dots, \phi_n{}^*(t))^T$. If there $\delta > 0$ and $K > 1$ such that for all solution $S(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ be the solution of equation (2.1) with initial value, $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))^T$.

$y_j(t) = x_j(t) - x_j^*(t) = S(t) - S^*(t)$. Then

$$y_i{}'(t) = -a_i y_i(t) + \sum_{j=1}^{n} w_{ij} g_j\left(y_j(t)\right) + \sum_{j=1}^{n} v_{ij} g_j\left(u_j(t-\tau)\right) - \left(x_j(t-\tau)\right)$$

$+g_j\left(x_j^*(t-\tau)\right) + \sum_{j=1}^{n} z_{ij}\left(y_j(t-\tau)\right) + \left(x_j(t-\tau)\right) - z_{ij}\left(x_j^*(t-\tau)\right),$

where $i = 1, 2, \cdots, n.$

$Y_i(t) = y_i(t)e^{it}, \ i = 1, 2, \cdots, n.$

Then

$$D^-|x_i(t) - x_i^*(t)| \le -a_i|x_i(t) - x_i^*(t)| + \sum_{j=1}^{n}|w_{ij}||g_j(x_j(t) - x_j^*(t))|$$

$$+\sum_{j=1}^{n}|v_{ij}|\left|g_j\left(u_j(t-\tau)\right) - \left(x_j(t-\tau)\right) + g_j\left(x_j^*(t-\tau)\right)\right|$$

$+ \sum_{j=1}^{n}|z_{ij}|\left|\left(y_j(t-\tau)\right) + \left(x_j(t-\tau)\right) - z_{ij}\left(x_j^*(t-\tau)\right)\right|$

$\le -a_i|x_i(t) - x_i^*(t)| + \sum_{j=1}^{n}(a_{ij}{}^+ + \alpha_{ij}{}^+ + \beta_{ij}{}^+) \times p_j\left|\left(x_j(t-\tau)\right) - \left(x_j^*(t-\tau)\right)\right|$ (2.6)

where D^- denotes the upper left derivative. If we let $y_i(t) = x_i(t) - x_i^*(t)$ then

$D^-|y_i(t)| \le -a_i y_i(t) + \sum_{j=1}^{n}(a_{ij}{}^+ + \alpha_{ij}{}^+ + \beta_{ij}{}^+)\, sup|y_j(s)|$

$= -a_i y_i(t) + \sum_{j=1}^{n}(a_{ij}{}^+ + \alpha_{ij}{}^+ + \beta_{ij}{}^+)p_j\, \overline{y}_j(t),$ (2.7)

where $\overline{y}_j(t) = sup|y_j(s)|$. From Lemma (2.3) we can see that a vector exists.

$\eta = (\eta_1, \eta_2, \dots, \eta_n)^T$ such that

$= -a_i y_i(t) + \sum_{j=1}^{n}(a_{ij}{}^+ + \alpha_{ij}{}^+ + \beta_{ij}{}^+)p_j\, \eta_j < 0,$

We can choose a small positive constant $\delta < 1$ such that, for $i = 1, 2, \dots, n.$

$$\delta\eta_i + \left[-a_i y_i(t) + \sum_{j=1}^{n}(a_{ij}{}^+ + \alpha_{ij}{}^+ + \beta_{ij}{}^+)p_j\, \eta_j e^{-\delta t}\right] < 0.$$

For constant value $\gamma > 1$,

$\gamma\eta_i e^{-\delta t} > 1, \forall t \in [-\tau, 0]$. For $\varepsilon > 0$,

$Yi(t) = \gamma\eta i\left[\sum_{j=1}^{n}\overline{y}_j(0)+\varepsilon\right]e^{-\delta t},\ i = 1,2,\cdot\cdot\cdot,n.$ (2.8)

From (2.7) and (2.8)

$$D_{-}\,Yi(t) = -\,\delta\,\gamma\eta_i\left[\sum_{j=1}^{n}\overline{y}_j(0)+\varepsilon\right]e^{-\delta t}$$

$$> \left[-a_i\eta_i + \sum_{j=1}^{n}(a_{ij}{}^{+} + \alpha_{ij}{}^{+} + \beta_{ij}{}^{+})p_j\,\eta_j e^{-\delta t}\right]\gamma \times \left[\sum_{j=1}^{n}\overline{y}_j(0)+\varepsilon\right]e^{-\delta t}$$

$$= -a_i\eta_i\,\gamma\left[\sum_{j=1}^{n}\overline{y}_j(0)+\varepsilon\right]e^{-\delta t} + \sum_{j=1}^{n}(a_{ij}{}^{+} + \alpha_{ij}{}^{+} + \beta_{ij}{}^{+})p_j\,\eta_j\,\gamma$$

$$\times\left(\sum_{j=1}^{n}\overline{y}_j(0)+\varepsilon\right)e^{-\delta(t-\tau)},$$

$Y_i(t) = \gamma\eta_i\left[\sum_{j=1}^{n}\overline{y}_j(0)+\varepsilon\right]e^{-\delta t} > \sum_{j=1}^{n}\overline{y}_j(0)+\varepsilon > |y_i(t)|.$ (2.9)

Here

$|y_i(t_i)| < Y_i(t_i)$, for $i = 1,2,\cdot\cdot\cdot,n$ and $t_i > 0$,

$|y_i(t_i)| - Y_i(t_i) = 0$,

We get

$0 \le D^{-}|y_i(t_i) - Y_i(t_i)| = D^{-}|y_i(t_i)| - D_{-}Y_i(t_i)$

and

$$D^{-}|y_i(t_i)| \le D_{-}Y_i(t_i).$$

Let $\varepsilon \to 0$ and $K = \max_{1\le j\le n}\{\eta_i\,\gamma + 1\}$. Then

$|x_i(t) - x_i^{*}(t)| = |y_i(t)| \le \eta_i\,\gamma\sum_{j=1}^{n}\overline{y}_j(0)e^{-\delta t} \le \eta_i\gamma e^{-\delta t}\|\phi-\phi^{*}\| \le Ke^{-\delta t}\|\phi-\phi^{*}\|.$

From the assumption Lemma (2.3), the solution $x^{*}(t)$ is globally exponentially stable.

3. Results and Discussion

The existence of almost periodic functions is especially important in applications such as time series analysis or waveform processing. At the same time, the global stability of artificial neural networks is also critical. The training process of the network occurs by optimizing the weights, which usually starts randomly at the beginning. Global stability means that this optimization can achieve similar results even if started from different starting points. This ensures that the learning process of the network is more reliable and repeatable.

Especially for the analysis of almost periodic functions, it is important that the network can consistently recognize and learn patterns over time. The existence and uniqueness of periodic solutions provide important information for understanding and analyzing the behavior of a system or equation. With stability analysis, the existence of almost periodic solutions can help evaluate the stability of equilibrium states or solutions in a system. Especially in mathematical modeling, these solutions are used to predict the behavior of the system. Almost periodic solutions can represent changes and cyclical behavior of a system over time. This is important for understanding and predicting system dynamics, especially for understanding the complexity of systems in fields such as engineering, physics, biology. Periodic solutions can form the basis for frequency analysis, especially in signal processing and control systems. It is used to understand, design and optimize the frequency response of systems. By measuring stability and reliability, almost uniqueness of periodic solutions means that a system behaves in a unique way under certain conditions. This is important to evaluate the stability of the system and predict how it will react to certain inputs or conditions. For these reasons, the existence and uniqueness of almost periodic solutions provide a powerful analysis tool for understanding and managing the complexity of systems in science, engineering and mathematics.

4. Conclusion

Artificial neural networks consist of interconnected and mostly two-dimensional cells. The most important feature that distinguishes this structure from other models is that the connection weight coefficients create an invariant connection network in the studied plane. Due to the limited speed of neurons, chaos, oscillation, instability, etc. may occur in the signal transmission between neurons. It occurs when the system is stable and manifests itself in some dynamics that will affect the stability of the system, but it also includes many dynamic behaviors such as periodic oscillatory behavior that is almost periodic. In artificial neural networks, not all cells are connected to each other, but are directly connected only to their neighbors, which reduces the complexity of the network structure and reduces energy consumption. Relational memory networks, one of the most important classes of artificial neural networks, are frequently used in applications such as pattern recognition, predicting and completing correct data from missing data. It also includes many dynamic behaviors such as periodic oscillatory behavior, almost periodic neural network models are widely used in solving optimization problems, in scientific fields and computing technology, in various engineering fields such as analog to digital converter design. It is very important to determine the stability of artificial neural networks designed in applications. The artificial neural network created in the type of neural network examined must have a single stable balance point. Repeating complex situations are represented by periodicity, and the dynamics and biological mechanisms of time-delayed periodic systems are discussed under the name of neural network construction. Many researchers who have addressed the near-periodic problems of neural networks have called the negative feedback state of the network system the forgetting delay. Dependent population dynamics and neutral type time delays occurring in vibrating masses have been widely used. However, there are very few papers focusing on the stability and existence of a nearly periodic solution for neutral neural networks with time-varying delays in terms of leakage. For this reason, the existence and stability of solutions to almost periodic problems is an issue that needs to be emphasized. Recently, some researchers have attached great importance to the one-way neutral type. Hopfield's dual associative memory model is of great importance for pattern recognition and automatic control applications. In this study, the existence and stability of the nearly periodic solution has been proven and its effects have been investigated. Near-periodic solutions of the neutral neural network based on time-varying delays have been studied. Although the results obtained cannot be directly applied to many arrangements, they do extend some known networks to some extent. The contribution of this article to science is in the solutions of discontinuous neural networks, findings that will reveal the chaos in the neuron and create inertia and nonlinear effects. Therefore, it is important that the results complement previous studies.

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