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Some new properties of the Meixner polynomials

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ABSTRACT

The present study deals with some new properties for the Meixner polynomials. In this manuscript we obtain a number of families of bilinear and bilateral generating functions, general properties and also some special cases for these polynomials. In addition, we derive a theorem giving certain families of bilateral generating functions for the generalized Lauricella functions and the Meixner polynomials. Finally, we get several interesting results of this theorem.

Keywords: Meixner polynomials, generating function, bilinear and bilateral generating function, recurrence relations, hypergeometric function.

Meixner polinomlarının bazı yeni özellikleri

ÖZ

Bu çalışma Meixner polinomlar için bazı yeni özellikler ele alınmıştır. Burada elde edilen sonuçlar Meixner polinomların bilineer ve bilateral doğurucu fonksiyonların çeşitli ailelerini, çeşitli özelliklerini ve bazı özel durumlarını içermektedir. Bunlara ek olarak genelleştirilmiş Lauricella fonksiyonları ve Meixner polinomları için bilateral doğurucu fonksiyon içeren teorem verildi. Son olarak, bu teoremin ilginç bazı sonuçları verildi.

Anahtar Kelimeler: Meixner polinomları, doğurucu fonksiyon, bilineer ve bilateral doğurucu fonksiyon, reküran bağıntıları, hipergeometrik fonksiyon.

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1. INTRODUCTION

The Meixner polynomials $M_n(z;\beta,x)$ are defined by the generating function relation (see, for example, [1], p. 449, Problem 20 (ii))

$$\sum_{n=0}^{\infty} M_n(z;\beta,x)\frac{u^n}{n!}=(1-u)^{-\beta-z}(1-\frac{u}{x})^z. \tag{1.1}$$

It is from (1.1) that (see, [2]):

$$M_n(z;\beta,x)=(-1)^n n!\sum_{k=0}^{n}\binom{z}{k}\binom{-z-\beta}{n-k}x^{-k} \tag{1.2}$$

In addition, we have the following relationship between the Meixner polynomials $M_n(z;\beta,x)$ and the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ ([1], p.443, Problem5(i)):

$$M_n(z;\beta,x)=n!P_n^{(\beta-1,-\beta-n-z)}(\frac{2}{x}-1).$$

The following for the Meixner polynomials generating function relationship holds true [1]:

$$\sum_{n=0}^{\infty} M_{n+m}(z;\beta,x)\frac{u^n}{n!}$$

$$=(1-u)^{-\beta-z-m}(1-\frac{u}{x})^z M_m(z;\beta,\frac{x-u}{1-u}). \tag{1.3}$$

First of all, some of the definitions and notations used in this paper are presented here as follows:

The four Appell functions of bivariate function, denoted by F_1, F_2, F_3 and F_4, (see [1], [4], [10], [14], [20]) were generalized by Lauricella functions of n variables which are denoted by $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ (see, [6], p. 60) and

$$F_A^{(2)}=F_2,\ F_B^{(2)}=F_3,\ F_C^{(2)}=F_4,\ F_D^{(2)}=F_1.$$

A further generalization of the well-known Kampé de Fériet hypergeometric function in bivariate function is owing to Srivastava and Daoust ([3], [4], [10], [14], [20]) defined the generalized Lauricella function as follows (see [3], [4] and [9, p. 37 et seq.]; see also [7, p. 106 et seq.] and [8, p. 143]):

$$F^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}}\left(\begin{matrix}\left[(a):\theta^{(1)},\ldots,\theta^{(n)}\right]:\left[(b^{(1)}):\phi^{(1)}\right];\ldots;\left[(b^{(n)}):\phi^{(n)}\right];\\ \left[(c):\psi^{(1)},\ldots,\psi^{(n)}\right]:\left[(d^{(1)}):\delta^{(1)}\right];\ldots;\left[(d^{(n)}):\delta^{(n)}\right];\end{matrix}\ z_1,\ldots,z_n\right)$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\Omega(m_1,\ldots,m_n)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!}$$

where, for convenience,

$$\Omega(m_1,\ldots,m_n):=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\ldots+m_n\theta_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\ldots+m_n\psi_j^{(n)}}}$$

$$\times\frac{\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}}{\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}}\cdots\frac{\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

the coefficients

$\theta_s^{(\alpha)}$ $(s=1,\ldots,A;\ \alpha=1,\ldots,n)$,

$\phi_s^{(\alpha)}$ $(s=1,\ldots,B^{(\alpha)};\ \alpha=1,\ldots,n)$,

$\psi_s^{(\alpha)}$ $(s=1,\ldots,C;\ \alpha=1,\ldots,n)$,

$\delta_s^{(\alpha)}$ $(s=1,\ldots,D^{(\alpha)};\ \alpha=1,\ldots,n)$

are real constants and $\left(b_{B^{(\alpha)}}^{(\alpha)}\right)$ abbreviates the array of $B^{(\alpha)}$ parameters $b_s^{(\alpha)}$ $(s=1,\ldots,B^{(\alpha)};\ \alpha=1,\ldots,n)$ with similar interpretations for other sets of parameters (see [4], [10], [14], [20]). Here, as usual, $(\mu)_\nu$ denotes the Pochhammer symbol and

$(1)_n=n!$, $(0)_0:=1$, $(n\in\mathrm{N}_0)$ is defined by

$$(\mu)_\nu=\frac{\Gamma(\mu+\nu)}{\Gamma(\mu)}\ (\mu\in\mathbb{C}\setminus\mathbb{Z}_0^-)$$

$$=\begin{cases}1, & \text{if } \nu=0;\ \mu\in\mathbb{C}\setminus\{0\}\\ \mu(\mu+1)\ldots(\mu+n-1), & \text{if } \nu=n\in\mathbb{N};\ \mu\in\mathbb{C}\end{cases}.$$

For a suitably bounded non-vanishing multiple sequence $\{\Omega(m_1,m_2,...,m_s)\}_{m_1,...,m_s\in\mathbb{N}_0}$ of real or complex parameters, let $\phi_n(u_1;u_2,...,u_s)$ of s (real or complex) variables $u_1;u_2,...,u_s$ defined by

$$\phi_n(u_1;u_2,...,u_s):=\sum_{m_1=0}^{n}\sum_{m_2,...,m_s=0}^{\infty}\frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \times\Omega\big(f(m_1,...,m_s),m_2,...,m_s\big)\frac{u_1^{m_1}}{m_1!}\cdots\frac{u_s^{m_s}}{m_s!} \tag{1.4}$$

where, for convenience,

$$((b))_{m_1\phi}=\prod_{j=1}^{B}(b_j)_{m_1\phi_j} \text{ and } ((d))_{m_1\delta}=\prod_{j=1}^{D}(d_j)_{m_1\delta_j}.$$

The main target, is to study different properties of the Meixner polynomials. Miscellaneous properties and different families of bilinear and bilateral generating functions, and also some special cases for these polynomials are given. In addition, we derive a theorem giving certain families of bilateral generating functions for the generalized Lauricella functions and the Meixner polynomials. Nowadays, there are a lot of works related to Meixner polynomials and Lauricella functions theory and its applications (see [17], [18], [20]).

Lemma 1.1. *The following addition formula holds for the Meixner polynomials* $M_n(z;\beta,x)$:

$$M_n(z_1+z_2;\beta_1+\beta_2,x) =\sum_{p=0}^{n}\binom{n}{p}M_{n-p}(z_1;\beta_1,x)M_p(z_2;\beta_2,x). \tag{1.5}$$

Proof Replacing z by z_1+z_2 and β by $\beta_1+\beta_2$ in (1.1), we get

$$\sum_{n=0}^{\infty}M_n(z_1+z_2;\beta_1+\beta_2,x)\frac{u^n}{n!}$$

$$=(1-u)^{-\beta_1-\beta_2-z_1-z_2}(1-\frac{u}{x})^{z_1+z_2}$$

$$=\sum_{n=0}^{\infty}M_n(z_1;\beta_1,x)\frac{u^n}{n!}\sum_{p=0}^{\infty}M_p(z_2;\beta_2,x)\frac{u^p}{p!}$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}\frac{M_{n-p}(z_1;\beta_1,x)M_p(z_2;\beta_2,x)}{n!}u^n.$$

From the coefficients of u^n on the both sides of the last equality, one can get the desired result. ■

2. GENERATING FUNCTIONS FOR THE MEİXNER POLYNOMIALS $M_n(z;\beta,x)$

We study a number of families of bilinear and bilateral generating functions for the Meixner polynomials $M_n(z;\beta,x)$ which are generated by (1.1) and given by (1.2) using the similar method considered in (see [5], [10]-[16], [19], [21]-[25]).

We begin by stating the following theorem.

Theorem 2.1. *Corresponding to an identically non-vanishing function* $\Omega_\eta(s_1,...,s_k)$ *of* k *complex variables* $s_1,...,s_k$ $(k\in\mathbb{N})$ *and of complex order* η, *let*

$$\Lambda_{\eta,\psi}(s_1,...,s_k;\zeta):=\sum_{r=0}^{\infty}a_r\Omega_{\eta+\psi r}(s_1,...,s_k)\zeta^r$$

$$(a_r\neq 0,\ \eta,\psi\in\mathbb{C})$$

and

$$\Xi_{n,p}^{\eta,\psi}(z;\beta,x;s_1,...,s_k;\xi) :=\sum_{r=0}^{[n/p]}a_rM_{n-pr}(z;\beta,x)\Omega_{\eta+\psi r}(s_1,...,s_k)\frac{\xi^r}{(n-pr)!}.$$

Then, for $p\in\mathbb{N}$, *we have*

$$\sum_{n=0}^{\infty}\Xi_{n,p}^{\eta,\psi}\left(z;\beta,x;s_1,...,s_k;\frac{\mu}{u^p}\right)u^n =(1-u)^{-\beta-z}(1-\frac{u}{x})^z\Lambda_{\eta,\psi}(s_1,...,s_k;\mu) \tag{2.1}$$

provided that each member of (2.1) exists.

Proof Let S denote the first member of the assertion (2.1). Then,

$$S=\sum_{n=0}^{\infty}\sum_{r=0}^{[n/p]}a_rM_{n-pr}(z;\beta,x)\Omega_{\eta+\psi r}(s_1,...,s_k)\mu^r\frac{u^{n-pr}}{(n-pr)!}.$$

Replacing n by $n+pk$, we get

$$S=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}a_r\, M_n\left(z;\beta,x\right)\Omega_{\eta+\psi r}(s_1,...,s_k)\mu^r\frac{u^n}{n!}$$

$$=(1-u)^{-\beta-z}(1-\frac{u}{x})^z\Lambda_{\eta,\psi}(s_1,...,s_k;\mu)$$

which completes the proof. ■

By using a similar way, we can write the next result.

Theorem 2.2. *Corresponding to an identically non-vanishing function* $\Omega_\eta(s_1,...,s_r)$ *of* r *complex variables* $s_1,...,s_r$ $(r\in\mathbb{N})$ *and of complex order* η, *let*

$$\Lambda_{\eta,\psi}^{n,p}(z_1+z_2;\beta_1+\beta_2,x;s_1,...,s_r;\mu)$$

$$:=\sum_{i=0}^{[n/p]}a_iM_{n-pi}(z_1+z_2;\beta_1+\beta_2,x)\Omega_{\eta+\psi i}(s_1,...,s_r)\mu^i$$

$(a_i\neq 0,\ \ \eta,\ \psi\in\mathbb{C},\ n,\ p\in \mathrm{N})$

and the notation $[n/p]$ *means the greatest integer less than or equal* n/p *.*

Then, for $p\in\mathrm{N}$, *we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_l\binom{n-pl}{k-pl}M_{n-k}(z_1;\beta_1,x)$$

$$\times M_{k-pl}(z_2;\beta_2,x)\Omega_{\eta+\psi l}(s_1,...,s_r)\mu^l$$

$$=\Lambda_{\eta,\psi}^{n,p}(z_1+z_2;\beta_1+\beta_2,x;s_1,...,s_r;\mu)\qquad(2.2)$$

provided that each member of (2.2) exists.

Proof Let T denote the first member of the assertion (2.2). Then, upon substituting for the polynomials $M_n(z_1+z_2;\beta_1+\beta_2,x)$ from the (1.5) into the left-hand side of (2.2), we obtain

$$T=\sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl}a_l\binom{n-pl}{k}M_{n-k-pl}(z_1;\beta_1,x)$$

$$\times M_k(z_2;\beta_2,x)\Omega_{\eta+\psi l}(s_1,...,s_r)\mu^l$$

$$=\sum_{l=0}^{[n/p]}a_l\sum_{k=0}^{n-pl}\binom{n-pl}{k}M_{n-k-pl}(z_1;\beta_1,x)$$

$$\times M_k(z_2;\beta_2,x)\Omega_{\eta+\psi l}(s_1,...,s_r)\mu^l$$

$$=\Lambda_{\eta,\psi}^{n,p}(z_1+z_2;\beta_1+\beta_2,x;s_1,...,s_r;\mu).\quad ■$$

Theorem 2.3. *Corresponding to an identically non-vanishing function* $\Omega_\eta(s_1,...,s_r)$ *of* r *complex variables* $s_1,...,s_r$ $(r\in\mathbb{N})$ *and of complex order* η, *let*

$$\Lambda_{q,\eta,p}\left(z;\beta,x;s_1,...,s_r;t\right)$$

$$:=\sum_{n=0}^{\infty}a_nM_{m+qn}\left(z;\beta,x\right)\Omega_{\eta+pn}(s_1,...,s_r)\frac{t^n}{(nq)!}$$

$(a_n\neq 0,\ \ \eta\in\mathbb{C})$

and

$$\theta_{\eta,p}(s_1,...,s_r;\mu)$$

$$:=\sum_{k=0}^{[n/q]}\binom{n}{n-qk}a_k\Omega_{\eta+pk}(s_1,...,s_r)\mu^k.$$

Then, for $p\in\mathbb{N}$, *we have*

$$\sum_{n=0}^{\infty}M_{n+m}\left(z;\beta,x\right)\theta_{\eta,p}(s_1,...,s_r;\mu)\frac{t^n}{n!}$$

$$=(1-t)^{-\beta-z-m}(1-\frac{t}{x})^z$$

$$\times\Lambda_{\eta,p,q}\left(z;\beta,\frac{x-t}{1-t};s_1,...,s_r;\mu\left(\frac{t}{1-t}\right)^q\right)\qquad(2.3)$$

provided that each member of (2.3) exists.

Proof Let Ψ denote the first member of the assertion (2.3) of Theorem 2.3. Then,

$$\Psi=\sum_{n=0}^{\infty}M_{n+m}\left(z;\beta,x\right)$$

$$\times\sum_{k=0}^{[n/q]}\binom{n}{n-qk}a_k\Omega_{\eta+pk}(s_1,...,s_r)\mu^k\frac{t^n}{n!}.$$

Replacing n by $n+qk$ and then using (1.3), we might write that

$$T=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+qk}{n}M_{n+m+qk}\left(z;\beta,x\right)$$

$$\times a_k\,\Omega_{\eta+pk}(s_1,...,s_r)\mu^k\frac{t^{n+qk}}{(n+qk)!}$$

$$=\sum_{k=0}^{\infty}a_k\left[(1-t)^{-\beta-z-m-qk}(1-\frac{t}{x})^zM_{m+qk}(z;\beta,\frac{x-t}{1-t})\right]$$

$$\times\Omega_{\eta+pk}(s_1,...,s_r)\frac{(\mu t^q)^k}{(kq)!}$$

$$=(1-t)^{-\beta-z-m}(1-\frac{t}{x})^{z}$$

$$\times\Lambda_{q,\eta,p}\left(z;\beta,\frac{x-t}{1-t};s_1,...,s_r;\mu\left(\frac{t}{1-t}\right)^{q}\right)$$

which completes the proof. ■

3. SPECIAL CASES

We can give many applications of our teorems obtained in the previous section with help of appropriate choices of the multivariable functions $\Omega_{\eta+\psi r}(s_1,...,s_k)$ $r\in\mathbb{N}_0$, $k\in\mathbb{N}$, in terms of simpler function of one and more variables.

For example, if we $k=1$, $s_1=x$ and

$$\Omega_{\eta+\psi r}(x)=g^{(s)}_{\eta+\psi r}(\lambda,x)$$

in Theorem 2.1, where the generalized Cesàro polynomials $g^{(s)}_{n}(\lambda,x)$ [5], generated by

$$\sum_{n=0}^{\infty}g^{(s)}_{n}(\lambda,x)t^{n}=(1-t)^{-s-1}(1-xt)^{-\lambda} \qquad (3.1)$$

The following result which provides a class of bilateral generating functions for the generalized Cesàro polynomials $g^{(s)}_{n}(\lambda,x)$ and the Meixner polynomials.

Corollary 3.1. *If*

$$\Lambda_{\eta,\psi}(\lambda,x;\zeta):=\sum_{r=0}^{\infty}a_r g^{(s)}_{\eta+\psi r}(\lambda,x)\zeta^{r}$$
$$(a_r\neq 0,\ \ \eta,\psi\in\mathbb{C})$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{r=0}^{[n/p]}a_r M_{n-pr}(z;\beta,x)g^{(s)}_{\eta+\psi r}(\lambda,x)\frac{\mu^{r}}{u^{pr}}\frac{u^{n}}{(n-pr)!}$$

$$=(1-u)^{-\beta-z}(1-\frac{u}{x})^{z}\Lambda_{\eta,\psi}(\lambda,x;\mu) \qquad (3.2)$$

provided that each member of (3.2) *exists.*

Remark *Using the generating relation* (3.1) *for the* generalized Cesàro polynomials $g^{(s)}_{n}(\lambda,x)$ *and getting* $a_r=1$, $\eta=0$, $\psi=1$ in Corollary 3.1, *we find that*

$$\sum_{n=0}^{\infty}\sum_{r=0}^{[n/p]}M_{n-pr}(z;\beta,x)g^{(s)}_{r}(\lambda,x)\mu^{r}\frac{u^{n-pr}}{(n-pr)!}$$
$$=(1-u)^{-\beta-z}(1-\frac{u}{x})^{z}(1-\mu)^{-s-1}(1-x\mu)^{-\lambda}.$$

If we set $r=1$, $s_1=z_3$ and

$$\Omega_{\eta+\psi i}(z_3)=M_{\eta+\psi i}(z_3;\beta_3,y)$$

in Theorem 2.2, we have the following bilinear generating functions for the Meixner polynomials.

Corollary 3.2. *If*

$$\Lambda^{n,p}_{\eta,\psi}(z_1+z_2;\beta_1+\beta_2,x;z_3;\beta_3,y;\mu)$$
$$:=\sum_{i=0}^{[n/p]}a_i M_{n-pi}(z_1+z_2;\beta_1+\beta_2,x)M_{\eta+\psi i}(z_3;\beta_3,y)\mu^{i}$$
$$(a_i\neq 0,\ \eta,\psi\in\mathbb{C})$$

then, we have

$$\sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl}a_l\binom{n-pl}{k}M_{n-k-pl}(z_1;\beta_1,x)$$
$$\times M_k(z_2;\beta_2,x)M_{\eta+\psi l}(z_3;\beta_3,y)\mu^{l}$$
$$=\Lambda^{n,p}_{\eta,\psi}(z_1+z_2;\beta_1+\beta_2,x;z_3;\beta_3,y;\mu) \qquad (3.4)$$

provided that each member of (3.4) exists.

Remark *Using (1.5) and taking* $a_l=\binom{n}{l}$, $\psi=1$, $x=y$, $p=1$, $\eta=0$, $\mu=1$ in Corollary 3.2, we *have*

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\binom{n-l}{k}\binom{n}{l}M_{n-k-l}(z_1;\beta_1,x)$$
$$\times M_{k-l}(z_2;\beta_2,x)M_l(z_3;\beta_3,x)$$
$$=M_n(z_1+z_2+z_3;\beta_1+\beta_2+\beta_3,x).$$

If we set $r=1$, $s_1=u$ and $\Omega_{\eta+pn}(u)=P_{\eta+pn}(u)$ in Theorem 2.3, where the Legendre polynomials $P_n(x)$ is generated by [1],

$$\sum_{n=0}^{\infty}P_n(x)t^{n}=\frac{1}{\sqrt{1-2xt+t^{2}}},$$

we get a family of the bilateral generating functions for the Legendre polynomials and the Meixner polynomials as follows:

Corollary 3.3. *If*

$$\Lambda_{q,\eta,p}(z;\beta,x;u;t) := \sum_{n=0}^{\infty} a_n M_{m+qn}(z;\beta,x) P_{\eta+pn}(u) \frac{t^n}{(nq)!}$$

$$\left(a_n \neq 0,\ m \in \mathrm{N}_0\right)$$

and

$$\theta_{\eta,p}(u;\mu) := \sum_{k=0}^{[n/q]} \binom{n}{n-qk} a_k P_{\eta+pk}(u) \mu^k$$

where $n, p \in \mathrm{N}$, *then we have*

$$\sum_{n=0}^{\infty} M_{m+n}(z;\beta,x) \theta_{\eta,p}(u;\mu) \frac{t^n}{n!}$$

$$= (1-t)^{-\beta-z-m} (1-\frac{t}{x})^z \times \Lambda_{q,\eta,p}\left(z;\beta,\frac{x-t}{1-t};u;\mu\left(\frac{t}{1-t}\right)^q\right) \tag{3.5}$$

provided that each member of (3.5) exists.

Furthermore, for each a_k $(k \in \mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(s_1,...,s_r)$, $r \in \mathbb{N}$, are expressed as an appropriate product of a lot of simpler functions, the assertions of Theorem 2.1, Theorem 2.2, Theorem 2.3 can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the Meixner polynomials given explicitly by (1.2).

4. FURTHER CONSEQUENCES

In this part we give some special properties for the Meixner polynomials $M_n(z;\beta,x)$ given by (1.2).

Theorem 4.1. *The Meixner polynomials* $M_n(z;\beta,x)$ *have the following integral representation:*

$$M_n(z;\beta,x) = \frac{1}{\Gamma(z+\beta)\Gamma(-z)} \times \int_0^\infty\int_0^\infty e^{-(u_1+u_2)} \left(u_1 + \frac{u_2}{x}\right)^n u_1^{z+\beta-1} u_2^{-z-1} du_1 du_2$$

where $\mathrm{Re}(z+\beta) > 0$, $\mathrm{Re}(-z) > 0$.

Proof If we use the identity

$$a^{-v} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-at} t^{v-1} dt, \qquad \left(\mathrm{Re}(v) > 0\right)$$

on the left-hand side of the generating function (1.1), we have

$$\sum_{n=0}^{\infty} M_n(z;\beta,x) \frac{t^n}{n!}$$

$$= \frac{1}{\Gamma(z+\beta)} \int_0^\infty e^{-(1-t)u_1} u_1^{z+\beta-1} du_1 \frac{1}{\Gamma(-z)} \int_0^\infty e^{-(1-\frac{t}{x})u_2} u_2^{-z-1} du_2$$

$$= \frac{1}{\Gamma(z+\beta)\Gamma(-z)} \int_0^\infty\int_0^\infty e^{-(u_1+u_2)} \sum_{n=0}^{\infty} \frac{\left(u_1 + \frac{u_2}{x}\right)^n}{n!} t^n u_1^{z+\beta-1} u_2^{-z-1} du_1 du_2$$

From the coefficents of t^n on the both sides of the last equality, one can get the desired result. ■

We now focus on some miscellaneous recurrence relations of the Meixner polynomials. By differentiating each member of the generating function relation (1.1) with regard to x and using

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k),$$

we the following differential recurrence relation for the Meixner polynomials have been obtained:

$$M_n'(z;\beta,x) = \frac{z}{x^2} \sum_{m=0}^{n-1} \frac{(n-m)_{m+1}}{x^m} M_{n-m-1}(z;\beta,x), \quad n \geq 1.$$

Besides, by taking derivative each member of the generating function relation (1.1) with regard to t, we have the following another recurrence relation for these polynomials:

$$M_{n+1}(z;\beta,x) = (\beta+z) \sum_{m=0}^{n} M_{n-m}(z;\beta,x) - \frac{z}{x} \sum_{p=0}^{n} x^{-p} M_{n-p}(z;\beta,x).$$

5. THE GENERALIZED LAURICELLA FUNCTIONS

Now, we derive various families of bilateral generating functions for the generalized Lauricella (or the Srivastava-Daoust) functions and the Meixner polynomials.

Theorem 5.1. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} M_n(z;\beta,x) \phi_n(u_1;u_2,...,u_k) \frac{t^n}{n!}$$

$$=(1-t)^{-\beta-z}(1-\frac{t}{x})^{z}$$

$$\times \sum_{m_1,p,m_2,\ldots,m_k=0}^{\infty}\frac{((b))_{(m_1+p)\phi}(z+\beta)_{m_1}(-z)_p}{((d))_{(m_1+p)\delta}}$$

$$\times\Omega\big(f(m_1,\ldots,m_k),m_2,\ldots,m_k\big)\frac{(\frac{u_1t}{t-1})^{m_1}}{m_1!}\frac{\left(-\frac{u_1t}{x-t}\right)^p}{p!}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_k^{m_k}}{m_k!},$$

where $\phi_n(u_1;u_2,\ldots,u_k)$ *is given in* (1.4).

Proof With the help of the relationship (1.3), it can be easily observed that

$$\sum_{n=0}^{\infty}M_n(z;\beta,x)\phi_n(u_1;u_2,\ldots,u_k)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}M_n(z;\beta,x)\sum_{m_1=0}^{n}\sum_{m_2,\ldots,m_k=0}^{\infty}\frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}}$$

$$\times\Omega\big(f(m_1,\ldots,m_k),m_2,\ldots,m_k\big)\frac{u_1^{m_1}}{m_1!}\cdots\frac{u_k^{m_k}}{m_k!}\frac{t^n}{n!}$$

$$=\sum_{m_1,m_2,\ldots,m_k=0}^{\infty}\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\big(f(m_1,\ldots,m_k),m_2,\ldots,m_k\big)$$

$$\times\frac{(-u_1t)^{m_1}}{m_1!}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_k^{m_k}}{m_k!}(1-t)^{-\beta-z-m_1}(1-\frac{t}{x})^{z}M_{m_1}(z;\beta,\frac{x-t}{1-t})$$

$$=(1-t)^{-\beta-z}(1-\frac{t}{x})^{z}\sum_{m_1,m_2,\ldots,m_k=0}^{\infty}M_{m_1}(z;\beta,\frac{x-t}{1-t})\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}$$

$$\times\Omega\big(f(m_1,\ldots,m_k),m_2,\ldots,m_k\big)\frac{(-\frac{u_1t}{1-t})^{m_1}}{m_1!}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_k^{m_k}}{m_k!}$$

$$=(1-t)^{-\beta-z}(1-\frac{t}{x})^{z}$$

$$\times \sum_{m_1,p,m_2,\ldots,m_k=0}^{\infty}\frac{((b))_{(m_1+p)\phi}(z+\beta)_{m_1}(-z)_p}{((d))_{(m_1+p)\delta}}$$

$$\times\Omega\big(f(m_1,\ldots,m_k),m_2,\ldots,m_k\big)\frac{(\frac{u_1t}{t-1})^{m_1}}{m_1!}\frac{\left(-\frac{u_1t}{x-t}\right)^p}{p!}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_k^{m_k}}{m_k!}$$

■

By appropriately choosing the multiple sequence $\Omega(m_1,m_2,\ldots,m_s)$ in Theorem 5.1, we get a number of interesting results as follows which give bilateral generating functions for the generalized Lauricella functions and the Meixner polynomials.

I. By letting

$$\Omega\big(f(m_1,\ldots,m_k),m_2,\ldots,m_k\big)$$

$$=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\ldots+m_k\theta_j^{(k)}}}{\prod_{j=1}^{E}(c_j)_{m_1\psi_j^{(1)}+\ldots+m_k\psi_j^{(k)}}}$$

$$\times\frac{\prod_{j=1}^{B^{(2)}}(b_j^{(2)})_{m_2\phi_j^{(2)}}}{\prod_{j=1}^{D^{(2)}}(d_j^{(2)})_{m_2\delta_j^{(2)}}}\cdots\frac{\prod_{j=1}^{B^{(k)}}(b_j^{(k)})_{m_k\phi_j^{(k)}}}{\prod_{j=1}^{D^{(k)}}(d_j^{(k)})_{m_k\delta_j^{(k)}}}$$

in Theorem 5.1, we obtain Corollary 5.1 below.

Corollary 5.1. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty}M_n(z;\beta,x)F^{A:B+1;B^{(2)};\ldots;B^{(k)}}_{E:D;D^{(2)};\ldots;D^{(k)}}\left(\begin{matrix}[(a):\theta^{(1)},\ldots,\theta^{(k)}]:[-n:1],[(b):\phi];[(b^{(2)}):\phi^{(2)}];\ldots;[(b^{(k)}):\phi^{(k)}];\\ [(c):\psi^{(1)},\ldots,\psi^{(k)}]:[(d):\delta];[(d^{(2)}):\delta^{(2)}];\ldots;[(d^{(k)}):\delta^{(k)}];\end{matrix}\ u_1,u_2,\ldots,u_k\right)t^n$$

$$=(1-t)^{-\beta-z}(1-\frac{t}{x})^{z}F^{A+B:1;1;B^{(2)};\ldots;B^{(k)}}_{E+D:0;0;D^{(2)};\ldots;D^{(k)}}\left(\begin{matrix}[(e):\varphi^{(1)},\ldots,\varphi^{(k+1)}]:[z+\beta:1],[-z:1];[(b^{(2)}):\phi^{(2)}];\ldots;[(b^{(k)}):\phi^{(k)}];\\ [(f):\xi^{(1)},\ldots,\xi^{(k+1)}]:[(d):\delta];[(d^{(2)}):\delta^{(2)}];\ldots;[(d^{(k)}):\delta^{(k)}];\end{matrix}\ (\frac{u_1t}{t-1}),\left(-\frac{u_1t}{x-t}\right),u_2,\ldots,u_k\right)$$

where the coefficients e_j, f_j, $\varphi_j^{(k)}$ *and* $\xi_j^{(k)}$ *are given by*

$$e_j=\begin{cases}a_j & (1\le j\le A)\\ b_{j-A} & (A<j\le A+B)\end{cases}$$

$$f_j=\begin{cases}c_j & (1\le j\le E)\\ d_{j-E} & (E<j\le E+D)\end{cases}$$

$$\varphi_j^{(r)} = \begin{cases} \theta_j^{(1)} & (1 \le j \le A; \quad 1 \le r \le 2) \\ \theta_j^{(r-1)} & (1 \le j \le A; \quad 2 < r \le k+1) \\ \phi_{j-A} & (A < j \le A+B; \quad 1 \le r \le 2) \\ 0 & (A < j \le A+B; \quad 2 < r \le k+1) \end{cases}$$

and

$$\xi_j^{(r)} = \begin{cases} \psi_j^{(1)} & (1 \le j \le E; \ 1 \le r \le 2) \\ \psi_j^{(r-1)} & (1 \le j \le E; \quad 2 < r \le k+1) \\ \delta_{j-E} & (E < j \le E+D; \quad 1 \le r \le 2) \\ 0 & (E < j \le E+D; \quad 2 < r \le k+1) \end{cases}$$

respectively.

II. Upon setting

$$\Omega\big(f(m_1,...,m_k), m_2,...,m_k\big) = \frac{(a)_{m_1+...+m_k}(b_2)_{m_2}...(b_k)_{m_k}}{(c_1)_{m_1}...(c_k)_{m_k}}$$

and $\phi = \delta = 0$ (that is, $\phi_1 = ... = \phi_B = \delta_1 = ... = \delta_D = 0$) in Theorem 5.1, we get the following result.

Corollary 5.2. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} M_n(z;\beta,x) F_A^{(k)}[a,-n,b_2,...,b_k;c_1,...,c_k;u_1,u_2,...,u_k]t^n$$

$$= (1-t)^{-\beta-z}(1-\frac{t}{x})^z F^{1\,:\,1;1;1;...;1}_{1\,:\,0;0;1;...;1}$$

$$\left(\begin{matrix} [(a):1,...,1]: & [z+\beta\ :\ 1];[-z\ :\ 1];[b_2\ :\ 1]; \\ [(c_1):\psi^{(1)},...,\psi^{(k+1)}]: & -; \quad -; \quad [c_2\ :\ 1]; \end{matrix}\right.$$

$$\left.\begin{matrix} ...;[b_k\ :\ 1]; \\ ...;[c_k\ :\ 1]; \end{matrix}\ (\tfrac{u_1 t}{t-1}),\Big(-\tfrac{u_1 t}{x-t}\Big),u_2,...,u_k\right)$$

where the coefficients $\psi^{(s)}$ *are given by*

$$\psi^{(s)} = \begin{cases} 1, & (1 \le s \le 2) \\ 0, & (2 < s \le k+1) \end{cases}.$$

III. If we put

$$\Omega\big(f(m_1,...,m_k), m_2,...,m_k\big)$$

$$= \frac{(a_1^{(1)})_{m_2}...(a_1^{(k-1)})_{m_k}(a_2^{(1)})_{m_2}...(a_2^{(k-1)})_{m_k}}{(c)_{m_1+...+m_k}}$$

and $B=1,\ b_1 = b,\ \phi_1 = 1$ and $\delta = 0$ in Theorem 5.1, we get Corollary 5.3 below.

Corollary 5.3. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} M_n(z;\beta,x)$$

$$\times F_B^{(k)}\Big[-n,a_1^{(1)},...,a_1^{(k-1)},b,a_2^{(1)},...,a_2^{(k-1)};c;u_1,u_2,...,u_k\Big]t^n$$

$$= (1-t)^{-\beta-z}(1-\frac{t}{x})^z F^{1:1;1;2;...;2}_{1:0;0;0;...;0}$$

$$\left(\begin{matrix} [(b):\theta^{(1)},...,\theta^{(k+1)}]:[z+\beta\ :\ 1];[-z\ :\ 1][a^{(1)}\ :\ 1]; \\ [(c):1,...,1]: \quad -; \quad -; \quad -; \end{matrix}\right.$$

$$\left.\begin{matrix} ...;[a^{(k-1)}\ :\ 1]; \\ ...; \quad -; \end{matrix}\ (\tfrac{u_1 t}{t-1}),\Big(-\tfrac{u_1 t}{x-t}\Big),u_2,...,u_k\right),$$

where the coefficients $\theta^{(s)}$ *are given by*

$$\theta^{(s)} = \begin{cases} 1, & (1 \le s \le 2) \\ 0, & (2 < s \le k+1) \end{cases}.$$

IV. By letting

$$\Omega\big(f(m_1,...,m_k), m_2,...,m_k\big) = \frac{(a)_{m_1+...+m_k}(b_2)_{m_2}...(b_s)_{m_k}}{(c)_{m_1+...+m_k}}$$

an $\phi = \delta = 0$, in Theorem 5.1, we get the following result.

Corollary 5.4. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} M_n(z;\beta,x) F_D^{(k)}[a,-n,b_2,...,b_k;c;u_1,u_2,...,u_k]t^n$$

$$= (1-t)^{-\beta-z}(1-\frac{t}{x})^z$$

$$\times F_D^{(k+1)}\left[a,z+\beta,-z,b_2,...,b_k;c;(\frac{u_1 t}{t-1}),\left(-\frac{u_1 t}{x-t}\right),u_2,...,u_k\right].$$

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