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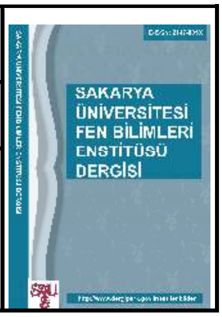

Bulanık Dizi Uzaylarının Hemen Hemen Entropisel Yakınsaklığı

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ÖZ

Bu çalışmada, bulanık sayı dizilerinin hemen hemen entropisel yakınsaklığı olarak adlandırılan yeni tip bir yakınsaklık tarifi verilmiştir. Buna ek olarak, hemen hemen entropisel yakınsaklığın birinci dereceden Cesàro matrisinin sütunlarının ötelenmesiyle, yani Cesàro matrisinin yakınsaklık alanlarının kesişimi kullanılarak tanımlanabileciği gösterilmiştir. Ve bu fikir herhangi bir $\boldsymbol{T}$ matrisine genişletilerek $\boldsymbol{T}$-entropisel yakınsaklık tanımını verilmiştir. Ayrıca, bulanık sayıların hemen hemen entropisel yakınsak ve sıfıra hemen hemen entropisel yakınsak dizilerinin kümeleri tanıtılmıştır. Son olarak, bu yeni kavram ile ilgili önemli görülen teorem ve ispatlarına yer verilmiştir. Örneğin, $\boldsymbol{E_f}$ ve $\boldsymbol{E_{f_0}}$ kümelerinin metrik uzay oldukları gösterilerek ve $\boldsymbol{E_c}, \boldsymbol{E_f}$ ve $\boldsymbol{E_b}$ uzayları arasındaki kapsama bağıntıları araştırılmıştır.

Anahtar Kelimeler: : bulanık sayı, dizi uzayı, entropi, hemen hemen entropisel yakınsaklık

Almost Entropy Convergence for Fuzzy Sequence Spaces

ABSTRACT

In this paper, we introduce a new type convergence called almost entropy convergence for sequences of fuzzy numbers. In addition this, we show that almost entropy convergence can be defined as the intersection of convergence field of Cesàro matrix. Besides, we generalize this idea to any matrix $\boldsymbol{T}$. By this way, we present the definition of $\boldsymbol{T}$-entropy convergence. After, the set of all almost entropy convergent and null almost entropy convergent sequences of fuzzy sets are defined. In addition this, we give some theorems, for example, we show that the sets $\boldsymbol{E_f}$ and $\boldsymbol{E_{f_0}}$ are complete metric spaces and give the inclusion relations between the spaces $\boldsymbol{E_c}, \boldsymbol{E_f}$ and $\boldsymbol{E_b}$, and proofs on this notion.

Keywords: fuzzy number, sequence space, entropy, almost entropy convergence

1. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [1]. After his innovation, mathematical structures were altered with fuzzy numbers. Recently, Şengönül [2], Şengönül and others [3] has made investigations on entropy concept for fuzzy sets. The notion of entropy was used by Şengönül and others [3] for computing some numerical values of P and T waves in ECG. Additionally, [4-14] used various disciplines of fuzzy set theory and entropy of fuzzy sets.

This paper is presented as in the following. In the first section, we recall some fundamental notions from fuzzy set theory, entropies of fuzzy sets and almost convergence. In the second section, we give the concept of almost entropy convergence. Also, we introduce almost entropy convergent and null almost entropy sequence spaces and prove some theorems about almost entropy convergence for sequences of fuzzy numbers.

Now, we recall some of the basic definitions and notions in the theory of fuzzy numbers and entropy concept for a better understanding of the subject.

Let X be a nonempty set. According to Zadeh a fuzzy subset of X is a nonempty subset $\{(x,u(x)): x \in X\}$ of $X \times [0,1]$ for some function $u\colon \mathrm{X} \to [0,1]$. Consider a function called as membership function, $u\colon \mathbb{R} \to [0,1]$ as a nonempty subset of $\mathbb{R}$ and denote the family of all such functions or fuzzy sets by E. Let us suppose that the function u satisfies the following properties:

(1) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,
(2) u is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and $\mu \in [0,1], u[\mu x + (1-\mu)y] \leq \min\{u(x), u(y)\}$,
(3) u is upper semi-continuous,
(4) The closure of $\{x \in \mathbb{R}: u(x) > 0\}$, denoted by u^0, is compact, [1].

Then, the function u is called a fuzzy number.

Furthermore, we know that shape similarity of the membership functions does not reflect the conception of itself, but it will be used for examining the context of the membership functions. Whether a particular shape is suitable or not can be determined only in the context of a particular application. However, many applications are not overly sensitive to variations in the shape. In such cases, it is convenient to use a simple shape, such as the triangular shape of membership function. Let us define fuzzy set A on the set $\mathbb{R}$ with membership function as follows:

$$A(x) = \begin{cases} \dfrac{h_A(x-u_0)}{u_1-u_0}, & x \in [u_0, u_1) \\ \dfrac{-h_A(x-u_1)}{u_2-u_1}, & x \in [u_1, u_2] \\ 0, & \text{otherwise.} \end{cases}$$

Here h_A represents height of the fuzzy set A and $u_0, u_1, u_2 \in \mathbb{R}$. Additionally, we show fuzzy set A with the triple $(u_0, u_1 : h_A, u_2)$.

Let suppose that $\mathbb{N}, \mathbb{R}$ and F be the set of all positive integers, all real numbers and fuzzy sets in the form $(u_0, u_1 : h_A, u_2)$ on $\mathbb{R}$, respectively. We denote the set of all sequences with complex terms by w which is a linear space with addition and scalar multiplication of sequences. Each linear subspace of w is called a sequence space and write ℓ_∞, c and c_0 for the classical sequence spaces of all bounded, convergent and null sequences, respectively. For brevity in notation, through all the text, we shall write $\sum_n$, $\sup_n$, and $\lim_n$ instead of $\sum_{n=0}^{\infty}$, $\sup_{n\in\mathbb{N}}$ and $\lim_{n\to\infty}$.

Let λ and μ be two sequence spaces and $\mathbb{A} = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where n, $k \in \mathbb{N}$. Then, we can say that $\mathbb{A}$ defines a matrix mapping from λ to μ and we denote it by writing $\mathbb{A} \in (\lambda : \mu)$, if for every sequence $x = (x_k)$ is in λ and the sequence $\mathbb{A}x = \{(\mathbb{A}x)_n\}$, the $\mathbb{A}$- transform of x is in μ where k runs from 0 to ∞. The domain $\lambda_{\mathbb{A}}$ of an infinite matrix $\mathbb{A}$ in a sequence space λ is defined by

$$\lambda_{\mathbb{A}} = \{x = (x_k) \in w : \mathbb{A}x \in \lambda\}, \tag{1.1}$$

which is a sequence space. If assume λ as c, then $c_{\mathbb{A}}$ is called convergence field of $\mathbb{A}$. We write the limit of $\mathbb{A}x$ as $\mathbb{A} - \lim_n x_n = \lim_n \sum_{k=0}^{\infty} a_{nk} x_k$, and the $\mathbb{A}$ is called regular if $\lim_{\mathbb{A}} x = \lim x$ for every $x \in c$. A matrix $\mathbb{A} = (a_{nk})$ is called triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$.

The set

$$w(F) = \{\mathcal{A} = (A_k) : A : \mathbb{N} \to F, A(k) = (A_k) = (u_0^k, u_1^k : h_{A_k}, u_2^k)\}$$

is called sequence of fuzzy sets where $u_0^k, u_1^k, u_2^k \in \mathbb{R}$, $u_0^k \leq u_1^k \leq u_2^k$ and mean of notation $u_1^k: h_{A_k}$ is the kth term of the sequence (A_k) has maximum membership level at u_1^k and this membership level is equal to h_{A_k}. If for all $k \in \mathbb{N}$, $h_{A_k} = 1$ then the set $w(F)$ reduced to the sequence set of fuzzy numbers. Also, if we take $u_0^k = u_1^k = u_2^k$ and $h_{u_1^k} = 1$ then $w(F)$ reduced to the ordinary sequence space of real numbers.

In fuzzy set theory, the fuzziness of a fuzzy set is an important topic and there are many methods for measuring the fuzziness of a fuzzy set. Firstly, fuzziness was thought to be the distance between fuzzy set and its nearest nonfuzzy set. After, entropy was used instead of fuzziness [4, 11]. Now, we give the definition of entropy notion:

Let $u \in F$ and $u(x)$ be the membership function of the fuzzy set A and consider the function $H: F \to \mathbb{R}^+$. If the function H satisfies conditions below,

(5) $H(A) = 0$ iff A is crisp set,
(6) $H(A)$ has a unique maximum, if $A(x) = \frac{1}{2}$, for all $x \in \mathbb{R}$,
(7) Let $A, B \in F$. If $B(x) \leq A(x)$ for $A(x) \leq \frac{1}{2}$ and $A(x) \leq B(x)$ for $A(x) \geq \frac{1}{2}$ then $H(A) \geq H(B)$,
(8) $H(A^c) = H(A)$, where A^c is the complement of the fuzzy set A, then $H(A)$ is called entropy of the fuzzy set A, [13].

Let suppose that $A = A(x)$ be membership function of the fuzzy set A and the function $h: [0,1] \to [0,1]$ satisfies the following properties:

(9) Monotonically increasing at $\left[0, \frac{1}{2}\right]$ and decreasing at $\left[\frac{1}{2}, 1\right]$,
(10) $h(x) = 0$ if $x = 0$ and $h(x) = 1$ if $x = \frac{1}{2}$.

Then h is called entropy function and equality $H(A(x)) = h(A(x))$ holds for $x \in \mathbb{R}$.

Additionally, some well known entropy functions are given as in the following:

$h_1(x) = 4x(1-x)$, $h_2(x) = -x\ln x - (1-x)\ln(1-x)$, $h_3(x) = \{\min 2x, 2-2x\}$

and

$$h_4(x) = \begin{cases} 2x, & x \in \left[0, \frac{1}{2}\right] \\ 2(1-x), & x \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Here, h_1, h_2, h_3 are called logistic, Shannon and tent functions, respectively. Let X be a continuous universal set. The total entropy of the fuzzy set A on X is defined as follows:

$$e(A) = \int_{x \in X} h(A(x))p(x)dx \tag{1.2}$$

here $p(x)$ is the probability density function of the available data in X [15], [16]. If we take $p(x) = 1$ in (1.2) then $e(A)$ is called entropy of the fuzzy set A.

2. MATERIAL AND METHOD

Let us give the definitions of some triangle, regular matrices, which are necessary for the text. The Cesàro matrix of order one which is a lower triangular matrix defined by the matrix $C = (c_{nk})$ as follows:

$$c_{nk} = \begin{cases} \frac{1}{n+1}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$.

One of the best known regular matrix is $R = (r_{nk})$, the Riesz matrix which is a lower triangular matrix defined by

$$r_{nk} = \begin{cases} \frac{r_k}{R_n}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$, where $r = (r_k)$ is real sequence with $r_0 > 0$, $r_k \geq 0$ and $R_n = \sum_{k=0}^{n} r_k$, $(n \in \mathbb{N})$. The Riesz matrix R is regular if and only if $R_n \to \infty$ as $n \to \infty$, [17].

We begin with writing some required definitions and lemma by means of Lorentz [18].

The shift operator S on ℓ_∞ is defined by $(Sx)_n = x_{n+1}$ for all $n \in \mathbb{N}$. A Banach limit L is a nonnegative linear functional on ℓ_∞ satisfying $L(Sx)=L(x)$ and $L(e)=1$ where $e=(1,1,1,\dots)$. Any bounded sequence is called almost convergent to the generalized limit a if all Banach limits of the sequence x are equal to a [18]. This is denoted by $f-\lim x=a$. It is given by Lorentz [18] that $f-\lim x=a$ if and only if

$$\lim_p \frac{x_n+x_{n+1}+\cdots+x_{n+p-1}}{p}=a,$$

uniformly in n. By f, we denote the space of all almost convergent sequences, respectively, i.e.,

$$f=\left\{x=(x_k)\in\ell_\infty:\exists a\in\mathbb{C}\ni\lim_m\sum_{k=0}^{m}\frac{x_{n+k}}{m+1}=a,\text{uniformly in } n\right\}.$$

2.1. MAIN DEFINITIONS

Let h be an entropy function and $\mathcal{A}=(\mathrm{A}_k)$ be a sequence of fuzzy sets and $p_k(x)$ be the probability density function on $\mathbb{R}$ for every $k\in\mathbb{N}$.

In [19] the following sequence spaces named entropy bounded and entropy convergent sequences of fuzzy sets are introduced, respectively.

$$E_b=\left\{\mathcal{A}\in w(\mathrm{F}):\sup_k\int_{x\in\mathbb{R}}^{v}h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x<\infty\right\}\tag{2.1}$$

$$E_c=\left\{\mathcal{A}\in w(\mathrm{F}):\lim_k\int_{x\in\mathbb{R}}^{v}h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x=L,L\in\mathbb{R}\right\}\tag{2.2}$$

By taking inspiration [19], we define the set of almost and null almost entropy convergent sequence spaces as below. Now, we give the following definition that is useful for the text.

Definition 2.1.1. If the following limit

$$\frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x\to L,n\to\infty,\text{uni. in } p$$

exists then sequence of fuzzy sets $\mathcal{A}=(\mathrm{A}_k)$ is called almost entropy convergent sequence of fuzzy sets.

Let us show the set of all almost entropy convergent and null almost entropy convergent sequences of fuzzy sets as follows, respectively:

$$E_f=\left\{\mathcal{A}\in w(\mathrm{F}):\lim_n\frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x=L,L\in\mathbb{R},\text{uni. in } p\right\},$$

$$E_{f_0}=\left\{\mathcal{A}\in w(\mathrm{F}):\lim_n\frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}h(\mathrm{A}_k(x))p_k(\mathrm{x})\mathrm{d}x=0,\text{uni. in } p\right\}.$$

Example 2.1.1. Now, we give a sequence of fuzzy sets, $\mathcal{A}=(\mathrm{A}_k)$, by means of membership functions as written below:

$$A_k(x)=\begin{cases}\begin{cases}x-2, & x\in[2,3]\\ 4-x, & x\in(3,4]\\ 0, & \text{otherwise}\end{cases} & \text{, if } k \text{ is odd,}\\ \begin{cases}\dfrac{x-1}{2}, & x\in[1,3]\\ \dfrac{5-x}{2}, & x\in(3,5]\\ 0, & \text{otherwise}\end{cases} & \text{, if } k \text{ is even.}\end{cases}$$

It is clear that $(A_k)\in E_f$ and $\bar{0}$ fuzzy sequence is in E_{f_0}. From here, we conclude that E_f and E_{f_0} are not empty sets.

The definition of almost entropy convergence can be defined as the intersection of convergence field that is obtained by displacement of the lines of first-order Cesàro matrix. Let $\vartheta\in\mathbb{N}$ and $x=(x_k)\in\ell_\infty$. Let us define the matrix $S^\vartheta=(s_{nk}^v)$ as follows:

$$s_{nk}^{\vartheta}=\begin{cases}1, & n+\vartheta=k\\ 0, & others.\end{cases}$$

The sequence $(S^{\vartheta}x) = (S^0x, S^1x, S^2x, \dots, S^{\vartheta}x, \dots)$ named shifted transforms sequence of x, is obtained by S. Thus, almost entropy convergence has the same meaning with the convergence of first-order Cesàro average of the shifted transform sequence $(S^{\vartheta}x) = (S^0x, S^1x, S^2x, \dots, S^{\vartheta}x, \dots)$ to a fixed sequence for each ϑ. After these, we can generalize the set of almost entropy convergent and null almost entropy sequence spaces by the following sequence spaces called as the set of all T- entropy convergent and null T- entropy convergent sequences, respectively.

In other words, we can say that almost entropy and null almost entropy convergence can be defined as the intersection of convergence field of a matrix T. That is

$$E_T = \left\{\mathcal{A} \in \ell_\infty(\mathrm{F}): \lim_k \left[T(S^{\vartheta} \int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x)\right] = L, L \in \mathbb{R}, \vartheta = 0,1,2,\dots\right\},$$

$$E_{T_0} = \left\{\mathcal{A} \in \ell_\infty(\mathrm{F}): \lim_k \left[T(S^{\vartheta} \int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x)\right] = 0, \vartheta = 0,1,2,\dots\right\}.$$

By considering $T = C = (c_{nk})$ in the definitions of E_T and E_{T_0} we attain sequence spaces named almost entropy convergent and null almost entropy convergent sequences, respectively.

Definition 2.1.2. Let $\mathbb{A} = (a_{nk})$ be a lower triangular infinite matrix of real numbers, $p_k(x) = c_k \in (0,1]$ for all $k \in \mathbb{N}$ and

$$\frac{1}{n+1}\sum_j a_{nj} \sum_{k=0}^{j+p} \int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x, \to L \quad (2.3)$$

uniformly in p and for $n \to \infty$.

Then L is called $\mathbb{A}$- almost entropy limit of the sequence (A_k) of fuzzy sets if L is available. In this case, sequence (A_k) is called $\mathbb{A}$- almost entropy convergent to L. The set of all $\mathbb{A}$- almost entropy convergent sequences is represented by $E_{\mathbb{A}}$.

Let us write the R and C matrices instead of $\mathbb{A}$ in the statement (2.3) then we have

$$\lim_n \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=0}^{j+p}\frac{r_k}{R_j}\int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x = L, \quad (2.4)$$

and

$$\lim_n \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=0}^{j+p}\frac{1}{n+1}\int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x = L, \quad (2.5)$$

respectively, uniformly in p.

The statements (2.4) and (2.5) are named R- almost entropy and C- almost entropy of the sequence (A_k) of fuzzy sets and showed by E_R and E_C, respectively.

2.2. MAIN THEOREMS

Let us define the function M as follows:

$$M: \mathrm{E}_f \times \mathrm{E}_f \to \mathbb{R},$$

$$M(\mathcal{A},\mathcal{B}) = \sup_n \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} |[h_4(\mathrm{A}_k(x)) - h_4(\mathrm{B}_k(x))p_k(x)\mathrm{d}x] \quad (2.6)$$

where $\mathcal{A} = (A_k), \mathcal{B} = (B_k) \in \mathrm{E}_f$.

Theorem 2.2.1. The inclusions $\mathrm{E}_c \subset \mathrm{E}_f \subset \mathrm{E}_b$ hold.

Proof. Let suppose that $\mathcal{A} = (A_k) \in \mathrm{E}_f$. Then, we can write the following inequalities easily for every $\varepsilon > 0$ and $k > n_0 \in \mathbb{N}$:

$$\left|\frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x - \mathrm{L}\right| < \varepsilon$$

$$\Leftrightarrow L - \varepsilon < \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x < L + \varepsilon..$$

It is clear the definition of convergence that there are finite elements of the sequences (A_k) outside of the interval $(L - \varepsilon, L + \varepsilon)$.

Let us write

$$L_1 = \min\left\{\begin{array}{c}\frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_1(x)\mathrm{d}x, \dots, \\ \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_{n_0}(x)\mathrm{d}x, L-\varepsilon\end{array}\right\},$$

$$L_2 = \max\left\{\begin{array}{c}\frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_1(x)\mathrm{d}x, \dots, \\ \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_{n_0}(x)\mathrm{d}x, L+\varepsilon\end{array}\right\}.$$

Then, we can write

$$L_1 \le \left\{\frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x\right\} \le L_2$$

for every $k \in \mathbb{N}$. From here, $\mathcal{A} = (A_k) \in E_b$. It means that $\mathrm{E}_f \subset E_b$.

Now, let us take $\mathcal{A} = (A_k) \in E_c$. In this case $\lim_k \int_{x\in\mathbb{R}}^{v} h(\mathrm{A}_k(x))p_k(x)\mathrm{d}x = L$. Because of the fact that Cesàro matrix is regular, we can conclude that matrix Cesàro transforms convergent sequences to convergent ones, then $\mathcal{A} = (A_k) \in E_f$. Therefore, $\mathrm{E}_c \subset \mathrm{E}_f \subset \mathrm{E}_b$.

Theorem 2.2.2. The sets E_f and E_{f_0} are complete metric spaces with the metric (2.6).

Proof.

$$M(\mathcal{A},\mathcal{B}) = 0 \Leftrightarrow \sup_n \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} |[h_4(\mathrm{A}_k(x)) - h_4(\mathrm{B}_k(x))p_k(x)\mathrm{d}x]| = 0$$

$$\Leftrightarrow \sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} |[h_4(\mathrm{A}_k(x)) - h_4(\mathrm{B}_k(x))]p_k(x)\mathrm{d}x| = 0$$

$$\Leftrightarrow \int_{x\in\mathbb{R}}^{v} (|h_4(\mathrm{A}_1(x)) - h_4\mathrm{B}_1(x))| + \cdots + |h_4(\mathrm{A}_{n+p}(x)) - h_4(\mathrm{B}_{n+p}(x))|)p_k(x)\mathrm{d}x = 0$$

$$\Leftrightarrow h_4(\mathrm{A}_1(x)) = h_4(\mathrm{B}_1(x)) \wedge \dots \wedge h_4((\mathrm{A}_{n+p}(x)) = h_4(\mathrm{B}_{n+p}(x)) \Leftrightarrow \mathcal{A} = \mathcal{B}.$$

It is easy obvious that the equation $M(\mathcal{A},\mathcal{B}) = M(\mathcal{B},\mathcal{A})$ holds.

$$M(\mathcal{A},\mathcal{B}) = \sup_n \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} |[h_4(\mathrm{A}_k(x)) - h_4(\mathrm{B}_k(x))p_k(x)\mathrm{d}x]|$$

$$= \sup_n \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} |[h_4(\mathrm{A}_k(x)) - h_4(\mathrm{B}_k(x)) - h_4(\mathrm{C}_k(x)) + h_4(\mathrm{C}_k(x))]p_k(x)\mathrm{d}x|$$

$$\le \sup_n \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} |[h_4(\mathrm{A}_k(x)) - h_4(\mathrm{C}_k(x))]p_k(x)\mathrm{d}x| + \sup_n \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} |[h_4(\mathrm{C}_k(x)) - h_4(\mathrm{B}_k(x))]p_k(x)\mathrm{d}x|$$

$$= M(\mathcal{A},\mathcal{C}) + M(\mathcal{C},\mathcal{B})$$

with last stage we can say that M is metric on E_f. Let $(A_k(x))$ be a Cauchy sequence of convex fuzzy sets. Then for every $\varepsilon > 0$ existing a $n_0 \in \mathbb{N}$ such that

$$M(A_k^i, A_k^j) < \varepsilon$$

for $i,j \ge n_0$. By considering (2.7) we can write

$$\sup_n \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} \left|\left[h_4\left(A_k^i(x)\right) - h_4\left(A_k^j(x)\right)\right]p_k(x)\mathrm{d}x\right| < \frac{\varepsilon}{n+1}$$

$$\Rightarrow \sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} \left|\left[h_4\left(A_k^i(x)\right) - h_4\left(A_k^j(x)\right)\right]p_k(x)\mathrm{d}x\right| < \varepsilon \Rightarrow \left|h_4\left(A_k^i(x)\right) - h_4\left(A_k^j(x)\right)\right| < \varepsilon.$$

Therefore, we can conclude that $(h_4(A_k^i(x)))$ is a Cauchy sequence in $\mathbb{R}$. Because of the fact that every Cauchy sequence is convergent in $\mathbb{R}$, we can obtain

$$\lim_i h_4\left(A_k^i(x)\right) = A_k^0(x). \qquad (2.8)$$

Keeping in mind (2.8), we attain following equations:

$$\lim_j \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}\left|\left[h_4\left(A_k^i(x)\right) - h_4\left(A_k^j(x)\right)\right]p_k(x)\mathrm{d}x\right| = \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}\left|\left[h_4\left(A_k^i(x)\right) - h_4\left(A_k^0(x)\right)\right]p_k(x)\mathrm{d}x\right| < \varepsilon.$$

It means that $(A_k^i(x))$ is almost entropy convergent to $A_k^0(x)$ for all $i \to \infty$. Now, we will show that $(A_k^0(x)) \in E_f$:

$$\sup_n \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}\left|\left[h_4\left(A_k^0(x)\right)\right]p_k(x)\mathrm{d}x\right| = \sup_n \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}|[h_4(A_k^0(x)) - h_4(A_k^i(x)) + h_4(A_k^i(x))]p_k(x)\mathrm{d}x|$$

$$\leq \sup_n \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}\left|\left[h_4\left(A_k^0(x)\right) - h_4\left(A_k^i(x)\right)\right]p_k(x)\mathrm{d}x\right| + \sup_n \frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}\left|h_4\left(A_k^i(x)\right)p_k(x)\mathrm{d}x\right| < \infty.$$

From here, we obtain that E_f is complete metric space. The proof will repeated in the same way for E_{f_0}.

Theorem 2.2.3. If

$$\frac{1}{n+1}\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v} h\big(B_k(x)\big)p_k(x)\mathrm{d}x$$

is increasing,

$$\lim_n \int_{x\in\mathbb{R}}^{v} h\left(B_{n+p}(x)\right)p_{n+p}(x)\mathrm{d}x = \infty$$

and

$$\lim_n \frac{\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}\big[h\big(A_k(x)\big)p_k(\mathrm{x}) - h\big(A_{k-1}(x)\big)p_{k-1}(\mathrm{x})\big]\mathrm{d}x}{\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}\big[h\big(B_k(x)\big)p_k(\mathrm{x}) - h\big(B_{k-1}(x)\big)p_{k-1}(\mathrm{x})\big]\mathrm{d}x} = L \tag{2.9}$$

then the following equation hold;

$$\lim_n \frac{\int_{x\in\mathbb{R}}^{v}\left[h\left(A_{n+p}(x)\right)p_{n+p}(\mathrm{x})\right]\mathrm{d}x}{\int_{x\in\mathbb{R}}^{v}\left[h\left(B_{n+p}(x)\right)p_{n+p}(\mathrm{x})\right]\mathrm{d}x} = L.$$

Proof. Let us suppose that (2.9) holds. Then, we can write

$$(L-\varepsilon)\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}[h(B_k(x))p_k(x) - h(B_{k-1}(x))p_{k-1}(x)]\mathrm{d}x < \sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}[h(A_k(x))p_k(x) - h(A_{k-1}(x))p_{k-1}(x)]\mathrm{d}x$$

$$< (L+\varepsilon)\sum_{k=0}^{n+p}\int_{x\in\mathbb{R}}^{v}[h(B_k(x))p_k(\mathrm{x}) - h(B_{k-1}(x))p_{k-1}(x)]\mathrm{d}x.$$

In this case we have the following inequalities:

$$(L-\varepsilon)\int_{x\in\mathbb{R}}^{v}\left[h\left(B_{n+p}(x)\right)p_{n+p}(\mathrm{x}) - h\big(B_0(x)\big)p_0(x)\right]\mathrm{d}x < \int_{x\in\mathbb{R}}^{v}\left[h\left(A_{n+p}(x)\right)p_{n+p}(x) - h\big(A_0(x)\big)p_0(x)\right]\mathrm{d}x$$

$$< (L+\varepsilon)\int_{x\in\mathbb{R}}^{v}\big[h(B_{n+p}(x))p_{n+p}(\mathrm{x}) - h(B_0(x))p_0(\mathrm{x})\big]\mathrm{d}x$$

If we divide every term of above inequalities by $\int_{x\in\mathbb{R}}^{v}\left[h\left(B_{n+p}(x)\right)p_{n+p}(\mathrm{x})\right]\mathrm{d}x$ and take limit of last inequality, for $p = 1,2,\ldots$ and $n \to \infty$ then we obtain the below inequalities:

$$(L-\varepsilon) < \lim_{n} \frac{\int_{x\in\mathbb{R}} \left[h\left(A_{n+p}(x)\right) p_{n+p}(\mathrm{x}) \right] dx}{\int_{x\in\mathbb{R}} \left[h\left(B_{n+p}(x)\right) p_{n+p}(x) \right] dx} < (L+\varepsilon)$$

This means that,

$$\lim_{n} \frac{\int_{x\in\mathbb{R}} \left[h\left(A_{n+p}(x)\right) p_{n+p}(x) \right] dx}{\int_{x\in\mathbb{R}} \left[h\left(B_{n+p}(x)\right) p_{n+p}(x) \right] dx} = L.$$

This step completes the proof.

3. DISCUSSION and CONCLUSION

Recently, Şengönül [2] and Şengönül and others [3] has made investigations on entropy concept for fuzzy sets. The concept entropy for sequences of fuzzy sets was used in [3] for computing some numerical values of P and T waves in ECG. They have also made similar investigations on veterinary discipline. In the present paper, almost and null almost entropy convergent sequence spaces are introduced. In this way, by combining the definitions of almost convergence and entropy, we define almost and null almost entropy convergent sequence spaces. By this way, we develop the definition of entropy convergence which is used in fuzzy set theory very often for measuring the fuzziness of fuzzy components.

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