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Research Article

# On the Inversion in a Generalized Taxicab Circle

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**Abstract**
In this study, inversions in generalized taxicab circles are defined and their properties are presented. The results obtained by examining the images of points under inversions in generalized taxicab circles are provided. Additionally, the concept of the directed generalized taxicab length of a line segment is introduced. Based on this concept, the definitions of the generalized taxicab cross-ratio and the generalized taxicab harmonic conjugate are given, along with some of their properties. The impact of inversion in the generalized taxicab circle on these concepts is investigated.

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## 1. Introduction and preliminaries

Euclidean distance is a measure of the shortest distance between two points. Geometrically, Euclidean distance can also be expressed as the "as-the-crow-flies" distance between two points. Taxicab distance, on the other hand, measures the sum of the Euclidean lengths parallel to the coordinate axes between two points. Taxicab distance was introduced by Menger, developed by Krause and many researchers have contributed to the development of taxicab geometry through their studies on taxicab distance [1–6]. Minkowski distance, also known as $p$-norm, is controlled by the value of $p$. The taxicab metric, the Euclidean metric and the maximum metric, respectively, are the Minkowski metrics in the cases of $p = 1, 2$, and $\infty$. Wallen redefined the taxicab distance to eliminate potentially misleading symmetry. To calculate the generalized taxicab distance between two points, the Euclidean lengths parallel to the coordinate axes are multiplied by positive real numbers and then summed. This is why the generalized taxicab distance is also referred to as the weighted taxicab distance, creating an alternative means of measuring distances in non-Euclidean geometry. In recent years, metric geometry based on this distance has been extensively studied and developed. For further reading on the subject, refer to the studies [7–11].

The inversion in a circle is a geometric transformation that maps one point in the analytical plane to another point. It is an important tool used in geometry to study and solve various problems and theorems. Apollonius of Perga first introduced the concept of inversion in a circle. Since then, numerous researchers have studied and contributed to its development [12–14]. Also, many generalizations of the inversion map have been achieved by using geometric objects such as parallel lines and central cones instead of circles in the definition of inversion [13, 15–17]. Moreover, the inversion maps have been studied by considering different distance functions such as taxicab distance, maximum distance, Chinese Checkers distance, $\alpha$ -distance [4, 18–24]. Inversions in spheres and ellipsoids have been defined in Euclidean and non-Euclidean spaces and their properties have been presented [21, 25–27].

In this work, inversions in generalized taxicab circles are defined and their properties are presented. The results obtained by examining the images of points under inversions in the generalized taxicab circles are provided. Additionally, the concept of the directed generalized taxicab length of a line segment is introduced. Based on this concept, the definitions of the generalized taxicab cross-ratio and the generalized taxicab harmonic conjugate are given, along with some of their properties. The impacts of inversion in the generalized taxicab circle on these concepts are investigated. In the following, some concepts used throughout this work are mentioned.

**Definition 1.** *The generalized taxicab distance between points $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ in the analytical plane is*

$$d_G(A,B) = a|x_2 - x_1| + b|y_2 - y_1|,$$

*where the real numbers $a, b > 0$ [11].*

It is seen from definition that the generalized taxicab distance between the points $A_1$ and $A_2$ is equal to the sum of positive multiples $a$ and $b$ of the Euclidean lengths of the sides parallel to the coordinate axes in the right triangle with the hypotenuse $A_1A_2$. Indeed, $d_G$ is a family of distance functions depending on the positive numbers $a$ and $b$. In the special case of $a = b = 1$, $d_G$ is the taxicab distance. Throughout this paper we will assume that $a$ and $b$ are constant values given at the beginning, unless otherwise stated.

The generalized taxicab plane is the analytical plane equipped with the generalized taxicab distance and symbolized by $\mathbb{R}^2_G$. It is almost the same as the Euclidean plane except the distance function.

The classification of lines in the generalized taxicab plane, similar to [6], is as follows:

**Definition 2.** *Let $m$ be the slope of the line $l$ in the generalized taxicab plane. The line $l$ is called the steep line, the gradual line and the separator (guide) line in the cases of $|m| > \frac{a}{b}$, $|m| < \frac{a}{b}$ and $|m| = \frac{a}{b}$, respectively. In the special cases that the line $l$ is parallel to x-axis or y-axis, $l$ is named as the horizontal line or the vertical line, respectively, [7].*

One of the basic objects in $\mathbb{R}^2_G$ is circle. The generalized taxicab circle centered at $M = (m_1, m_2)$ with the radius $r$ is a diamond with the equation $a|x - m_1| + b|y - m_2| = r$, whose vertices are $v_1 = (m_1 + \frac{r}{a}, m_2)$, $v_2 = (m_1, m_2 + \frac{r}{b})$, $v_3 = (m_1 - \frac{r}{a}, m_2)$ and $v_4 = (m_1, m_2 - \frac{r}{b})$.

Every Euclidean translation preserves the generalized taxicab distance. So, it is an isometry in $\mathbb{R}^2_G$. Reflections in the coordinate axes and the separator lines through the origin are isometries in the generalized taxicab plane. The set of axes of isometric reflections is

$$\{x = 0, y = 0, y = \frac{a}{b}x, y = -\frac{a}{b}x\}.$$

The set of isometric reflections in matrix form is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{b}{a} \\ \frac{a}{b} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{b}{a} \\ -\frac{a}{b} & 0 \end{bmatrix} \right\},$$

and the set of rotations about the origin that preserve the generalized taxicab distance in matrix form is as follows [8,9]:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{b}{a} \\ \frac{a}{b} & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{b}{a} \\ -\frac{a}{b} & 0 \end{bmatrix} \right\}.$$

## 2. The inversion in a generalized taxicab circle

In this section, the inversion in the generalized taxicab circle is firstly defined as an analog of the inversion in the Euclidean circle. It is well-known that the inversion in the Eucliden circle $\mathscr{C}$ centered at $O$ with the radius $r$ maps the point $X$ to the inverse point $X'$ located along the ray $\overrightarrow{OX}$ such that the product of the distances from the center $O$ to the point $X$ and the point $X'$ is equal to $r^2$. On the other hand, while the inversion leaves the points on circle $\mathscr{C}$ fixed, it maps points close to the center $O$ to points far from the center $O$ and conversely. Also, it is not defined at the inversion center $O$ and there is no point whose image is the inversion center $O$. Thus, by adding only one point called the "ideal point" or "point at infinity" to the Euclidean plane, the inversion can be defined at the inversion center $O$ and becomes a one-to-one map.

Now, the inversion in the generalized taxicab circle is defined as follows:

**Definition 3.** *Let $\mathscr{C}$ be the generalized taxicab circle centered at the point $O$ with the radius $r$ in $\mathbb{R}^2_G$ and $O_\infty$ be the ideal point added to the generalized taxicab plane. The inversion in the generalized taxicab circle $\mathscr{C}$ is the mapping $I_\mathscr{C}$ of $\mathbb{R}^2_G \cup \{O_\infty\}$ to $\mathbb{R}^2_G \cup \{O_\infty\}$ defined by $I_\mathscr{C}(O) = O_\infty$, $I_\mathscr{C}(O_\infty) = O$ and $I_\mathscr{C}(X) = X'$ for any point $X$ different from $O$, $O_\infty$ where the point $X'$ lies on the ray $\overrightarrow{OX}$ and $d_G(O,X).d_G(O,X') = r^2$. The point $X'$ is called the inverse or image of the point $X$ under the inversion $I_\mathscr{C}$. Here, $\mathscr{C}$ is termed the inversion circle; $O$ and $r$ denote the center with the radius of the inversion $I_\mathscr{C}$, respectively.*

It is a well-known fact that the inversions with respect to a circle in the Euclidean plane map the points inside of the inversion circle to the points outside of the circle and conversely. The following theorem provides that this property is also valid in the generalized taxicab plane.

**Theorem 4.** *Except for the inversion center, any point inside of the inversion circle is transformed to a point outside of it under the inversion with respect to the generalized taxicab circle, and conversely.*

*Proof.* Suppose that $I_{\mathscr{C}}$ is the inversion in the generalized taxicab circle $\mathscr{C}$ with the center $O$ and the radius $r$.

Let the point $Y$ be inside of the circle $\mathscr{C}$, then $d_G(O,Y) < r$. If $Y'$ is the image of the point $Y$ under the inversion $I_{\mathscr{C}}$, then the equality

$$d_G(O,Y).d_G(O,Y') = r^2$$

is valid. One gets immediately

$$r^2 = d_G(O,Y).d_G(O,Y') < r d_G(O,Y').$$

Since $d_G(O,Y') > r$, the inverse point $Y'$ is outside of $\mathscr{C}$. Conversely, the proof is similar. ■

Since $I^2_{\mathscr{C}}(X) = X$ for any point $X$ in $\mathbb{R}^2_G$, $I_{\mathscr{C}}$ is an involutive map like reflections.

**Proposition 5.** *Let $I_{\mathscr{C}}$ be the inversion in the generalized taxicab circle $\mathscr{C}$ centered at the point $O$ with the radius $r$. If the point $X'$ is the inverse of the point $X$ under the inversion $I_{\mathscr{C}}$ where $X \neq O$ , then the following equality between the points $X$ and $X'$ holds*

$$X' - O = \frac{r^2}{(d_G(O,X))^2}(X - O).$$

*Proof.* Suppose that the points $X$ and $X'$ are a pair of inverse points with respect to the inversion $I_{\mathscr{C}}$ where $X \neq O$ , then $d_G(O,X).d_G(O,X') = r^2$ and $X'$ is on the ray $\overrightarrow{OX}$. Therefore, $X' - O = \lambda(X - O)$ where $\lambda \in \mathbb{R}^+$. It is immediately seen that

$$d_G(O,X') = \left|X' - O\right|_G = \lambda \left|X - O\right|_G = \lambda d_G(O,X)$$

and

$$\lambda = \frac{r^2}{(d_G(O,X))^2}.$$

When the obtained coefficient $\lambda$ is substituted, the desired result is achieved. ■

In the case that the inversion center $O$ is the origin, it is obvious from Proposition 5 that the equality $X = \lambda X'$ is valid, where $\lambda = \frac{r^2}{(d_G(O,X))^2}$ and $X \neq O$.

**Proposition 6.** *Let $I_{\mathscr{C}}$ be the inversion in the generalized taxicab circle $\mathscr{C}$ with the center $O = (a_1, a_2)$ and the radius $r$. If the points $X = (x,y)$ and $X' = (x',y')$ are a pair of inverse points with respect to the inversion $I_{\mathscr{C}}$ where $X \neq O$, then the following equalities are valid*

$$\begin{cases} x' &= a_1 + \dfrac{r^2(x-a_1)}{(a\left|x-a_1\right| + b\left|y-a_2\right|)^2}, \\ y' &= a_2 + \dfrac{r^2(y-a_2)}{(a\left|x-a_1\right| + b\left|y-a_2\right|)^2}. \end{cases}$$

*Proof.* Suppose that the points $X = (x,y)$ and $X' = (x',y')$ are a pair of inverse points according to the inversion $I_{\mathscr{C}}$ where $X \neq O$. From Proposition 5,

$$X' - O = \frac{r^2}{(d_G(O,X))^2}(X - O).$$

Substituting the coordinates of the points,

$$\lambda = \frac{r^2}{(a\left|x-a_1\right| + b\left|y-a_2\right|)^2}$$

and

$$(x'-a_1,y'-a_2)=\frac{r^2}{(a\,|x-a_1|+b\,|y-a_2|)^2}(x-a_1,y-a_2)$$

are obtained. ■

**Corollary 7.** *Let $I_{\mathscr{C}}$ be the inversion in the generalized taxicab circle $\mathscr{C}$ centered at the point $O=(0,0)$ with the radius $r$. If the points $X=(x,y)$ and $X'=(x',y')$ are a pair of inverse points with respect to the inversion $I_{\mathscr{C}}$ where $X\neq O$, then the following equality between the coordinates of $X$ and $X'$ holds*

$$(x',y')=\frac{r^2}{(a\,|x|+b\,|y|)^2}(x,y).$$

*Proof.* By taking $a_1=a_2=0$ in Proposition 6, the equality is immediately gained. ■

**Example 8.** *Consider the generalized taxicab plane for $a=1$, $b=2$. Let $I_{\mathscr{C}}$ be the inversion in the unit generalized taxicab circle $\mathscr{C}$ . Assume that the point $X'=(x',y')$ is the inverse of the point $X=(3,5)$ under $I_{\mathscr{C}}$. From Corollary 7, the inverse point*

$$(x',y')=\frac{1}{(|3|+2\,|5|)^2}(3,5)=\left(\frac{3}{169},\frac{5}{169}\right)$$

*is obtained.*

**Theorem 9.** *Let $I_{\mathscr{C}}$ and $O$, $X$, $Y$ be the inversion in the generalized taxicab circle $\mathscr{C}$ centered at the point $O$ with the radius $r$ and any three collinear distinct points in the generalized taxicab plane, respectively . If the points $X$ , $X'$and $Y$, $Y'$ are two pairs of inverse points according to the inversion $I_{\mathscr{C}}$, then*

$$d_G(X',Y')=\frac{r^2 d_G(X,Y)}{d_G(O,X).d_G(O,Y)}.$$

*Proof.* Suppose that $O$, $X$, $Y$ are three collinear distinct points and the points $X'$, $Y'$ are the inverses of the points $X$, $Y$ under $I_{\mathscr{C}}$. Then, from the definition of $I_{\mathscr{C}}$, we have

$$d_G(O,X).d_G(O,X')=r^2=d_G(O,Y).d_G(O,Y').$$

It is well known from [7] that the ratios of the Euclidean and the generalized taxicab distances among three collinear points are same. Therefore, the required result follows as

$$\begin{aligned}
d_G(X',Y') &= \left|d_G(O,X')-d_G(O,Y')\right| \\
&= \left|\frac{r^2}{d_G(O,X)}-\frac{r^2}{d_G(O,Y)}\right| \\
&= \frac{r^2 d_T(P,Q)}{d_T(O,P).d_T(O,Q)}.
\end{aligned}$$

■

In the case that the points $O$, $X$, $Y$ are non collinear, the equality in Theorem 9 is not always valid.

**Example 10.** *Consider the generalized taxicab plane for $a=\frac{1}{2}$, $b=\frac{1}{3}$ and the points $X=(2,0)$, $Y=(1,2)$. Let $I_{\mathscr{C}}$ be the inversion in the unit circle $\mathscr{C}$. The inverse points of $X$ and $Y$ under the inversion $I_{\mathscr{C}}$ are $X'=(2,0)$ and $Y'=(\frac{36}{49},\frac{72}{49})$. The generalized taxicab distances from the inversion center to the point $X$ and to the point $Y$ are $1$ and $\frac{7}{6}$, respectively. Also, the generalized taxicab distances between $X$, $Y$ and between $X'$, $Y'$ are $\frac{7}{6}$ and $\frac{55}{49}$, respectively. Thus, it is clear that the equality in Theorem 9 does not hold for every point in the generalized taxicab plane. On the other hand, the following theorem clarifies the criteria to fulfil the equality postulated in Theorem 9.*

**Theorem 11.** *Let $O$, $X$, $Y$ be any three distinct noncollinear points in the generalized taxicab plane and $I_{\mathscr{C}}$ be the inversion in the generalized taxicab circle $\mathscr{C}$ centered at the point $O$ with the radius $r$ such that it maps the points $X$ and $Y$ to the points $X'$ and $Y'$. If the directions of the rays $\overrightarrow{OX}$ and $\overrightarrow{OY}$ belong to one of the sets $D_i$, $i=1,2$, then*

$$d_G(X',Y')=\frac{r^2 d_G(X,Y)}{d_G(O,X).d_G(O,Y)},$$

*where $D_1=\{(1,0),(0,1)\}$, $D_2=\{(b,a),(b,-a)\}$.*

*Proof.* Since all translations preserve the generalized taxicab distance, the inversion center $O$ can be taken the origin without loss of generality. Consider the case that the directions of the rays $\overrightarrow{OX}$ and $\overrightarrow{OY}$ belong to the set $D_1$. Then the inverse points of $X = (x,0)$ and $Y = (0,y)$ under $I_{\mathscr{C}}$ are $X' = (\frac{r^2}{a^2x},0)$ and $Y' = (0,\frac{r^2}{a^2y})$. The generalized taxicab distances from the inversion center to the points $X$ and $Y$ are $a|x|$ and $b|y|$, respectively. Also, the generalized taxicab distance between the points $X'$ and $Y'$ is

$$d_G(X',Y') = a\left|\frac{r^2}{a^2x}\right| + b\left|\frac{r^2}{b^2y}\right| = r^2\frac{a|x|+b|y|}{(a|x|)(b|y|)}.$$

Since the generalized taxicab distance between the points $X$ and $Y$ is $a|x|+b|y|$, the equality

$$d_G(X',Y') = \frac{r^2 d_G(X,Y)}{d_G(O,X).d_G(O,Y)}$$

is acquired. Suppose that directions of the rays $\overrightarrow{OX}$ and $\overrightarrow{OY}$ belong to the set $D_2$. Then the inverse points of $X = (bx,ax)$ and $Y = (by,-ay)$ under $I_{\mathscr{C}}$ are $X' = \frac{r^2}{4a^2b^2x}(b,a)$ and $Y' = \frac{r^2}{4a^2b^2y}(b,-a)$. The generalized taxicab distances from the inversion center to the points $X$ and $Y$ are $2ab|x|$ and $2ab|y|$. Also, the generalized taxicab distance between the points $X'$ and $Y'$ is

$$d_G(X',Y') = \frac{abr^2(|x-y|+|x+y|)}{(2ab|x|)(2ab|y|)}.$$

Since the generalized taxicab distance between the points $X$ and $Y$ is

$$d_G(X,Y) = ab(|x-y|+|x+y|),$$

the equality

$$d_G(X',Y') = \frac{r^2 d_G(X,Y)}{d_G(O,X).d_G(O,Y)}$$

is attained. ■

## 3. The cross-ratio and the harmonic conjugate

In the context of the generalized taxicab plane, it is important to note that the inversion in a generalized taxicab circle does not qualify as an isometry, which means that distance preservation does not occur under this transformation. However, in relation to the notion of distance, it has been demonstrated that the cross-ratio remains invariant under the inversion in a generalized taxicab circle. Consequently, this section delves into an exploration of the cross-ratios and harmonic conjugates in the generalized taxicab plane.

Consider any two points $X$ and $Y$ lying on a directed line denoted by $l$. The directed generalized taxicab length of the line segment $XY$ is symbolized by $d_G[XY]$ . When the line segment $XY$ shares the same direction as $l$ , then the directed generalized taxicab length $d_G[XY]$ is equal to $d_G(X,Y)$. Conversely, if they are in opposite directions $d_G[XY]$ is equal to $-d_G(X,Y)$. If the points $X$, $Y$, $Z$ are on a directed line and the point $Z$ is between the points $X$ and $Y$, then these points are stated in the array $XZY$.

**Definition 12.** *The generalized taxicab cross-ratio of four distinct collinear points $N_i$, $i = 1,2,3,4$, lying on a directed line, denoted as $(N_1N_2,N_3N_4)_G$ is defined by*

$$(N_1N_2,N_3N_4)_G = \frac{d_G[N_1N_3]}{d_G[N_1N_4]}\cdot\frac{d_G[N_2N_4]}{d_G[N_2N_3]}.$$

**Corollary 13.** *Let $N_i$, $i = 1,2,3,4$ be four distinct points on the directed line. Then the generalized taxicab cross-ratio $(N_1N_2,N_3N_4)_G$ is positive, if both points $N_i$, $i = 3,4$ are between points $N_j$, $j = 1,2$ , or not.*

*Proof.* Suppose that both points $N_3$ and $N_4$ are between points $N_1$ and $N_2$. For the array $N_1N_3N_4N_2$, the generalized taxicab cross-ratio is

$$\begin{aligned}(N_1N_2,N_3N_4)_G &= \frac{d_G[N_1N_3]}{d_G[N_1N_4]}\cdot\frac{d_G[N_2N_4]}{d_G[N_2N_3]}\\ &= \frac{d_G(N_1,N_3)}{d_G(N_1,N_4)}\cdot\frac{(-d_G(N_2,N_4))}{(-d_G(N_2,N_3))}\\ &= \frac{d_G(N_1,N_3)}{d_G(N_1,N_4)}\cdot\frac{d_G(N_2,N_4)}{d_G(N_2,N_3)}\end{aligned}$$

and $(N_1N_2,N_3N_4)_G$ is positive. The proof is similar for other possible arrays. Now, suppose that neither point $N_3$ nor point $N_4$ lies between points $N_1$ and $N_2$. Considering the array $N_4N_1N_2N_3$, then the generalized taxicab cross-ratio follows as

$$\begin{aligned}(N_1N_2,N_3N_4)_G &= \frac{d_G[N_1N_3]}{d_G[N_1N_4]}\cdot\frac{d_G[N_2N_4]}{d_G[N_2N_3]}\\ &= \frac{d_G(N_1,N_3)}{(-d_G(N_1,N_4))}\cdot\frac{(-d_G(N_2,N_4))}{d_G(N_2,N_3)}\\ &= \frac{d_G(N_1,N_3)}{d_G(N_1,N_4)}\cdot\frac{d_G(N_2,N_4)}{d_G(N_2,N_3)}\end{aligned}$$

and $(N_1N_2,N_3N_4)_G$ is positive. It is similarly seen that the generalized taxicab cross-ratio is positive for other possible arrays. ■

**Corollary 14.** *Let $N_i$, $i=1,2,3,4$ be four distinct points on the directed line. If the pairs $\{N_1,N_2\}$ and $\{N_3,N_4\}$ separate each other, the generalized taxicab cross-ratio is negative.*

*Proof.* Suppose that the pairs $\{N_1,N_2\}$ and $\{N_3,N_4\}$ separate each other Considering the array $N_1N_3N_2N_4$, the generalized taxicab cross-ratio follows as

$$\begin{aligned}(N_1N_2,N_3N_4)_G &= \frac{d_G[N_1N_3]}{d_G[N_1N_4]}\cdot\frac{d_G[N_2N_4]}{d_G[N_2N_3]}\\ &= \frac{d_G(N_1,N_3)}{d_G(N_1,N_4)}\cdot\frac{d_G(N_2,N_4)}{(-d_G(N_2,N_3))}\\ &= -\frac{d_G(N_1,N_3)}{d_G(N_1,N_4)}\cdot\frac{d_G(N_2,N_4)}{d_G(N_2,N_3)}\end{aligned}$$

and $(N_1N_2,N_3N_4)_G$ is negative. It is similarly seen that the generalized taxicab cross-ratio is negative for other possible arrays. ■

It is well known that the cross ratio of four collinear points differently from the inversion center is invariant under the inversion in a Euclidean circle. The following theorem proves that this property is also valid for the inversion in a generalized taxicab circle.

**Theorem 15.** *The generalized taxicab cross-ratio of four distinct points on a directed line passing through the inversion center is invariant under inversion in a generalized taxicab circle.*

*Proof.* Assume that $I_{\mathscr{C}}$ is the inversion in the generalized circle $\mathscr{C}$ centered at the point $O$ with the radius $r$ and $N_i$, $i=1,2,3,4$ are four distinct points on a directed line passing through the inversion center. If the point $N_i'$ is the inverse point of $N_i$ under the inversion $I_{\mathscr{C}}$, then $I_{\mathscr{C}}$ maps the point $N_i$ close to the inversion center to the point $N_i'$ far from the center. Therefore, the directions of line segments $N_iN_j$ and $N_i'N_j'$ are opposite, where $i \neq j$.

Moreover, the inversion in the generalized circle preserves whether the pairs $\{N_1,N_2\}$ and $\{N_3,N_4\}$ are separated from each other or not.

Then, to complete the proof, it is sufficient to show that $\left|(N_1'N_2',N_3'N_4')_G\right| = |(N_1N_2,N_3N_4)_G|$. From Theorem 9, it follows as

$$\begin{aligned}\left|(N_1'N_2',N_3'N_4')_G\right| &= \left|\frac{d_G[N_1'N_3']}{d_G[N_1'N_4']}\cdot\frac{d_G[N_2'N_4']}{d_G[N_2'N_3']}\right|\\ &= \frac{d_G(N_1',N_3')}{d_G(N_1',N_4')}\cdot\frac{d_G(N_2',N_4')}{d_G(N_2',N_3')}\\ &= \frac{\left(\dfrac{r^2 d_G(N_1,N_3)}{d_G(O,N_1).d_G(O,N_2)}\right)}{\left(\dfrac{r^2 d_G(N_1,N_4)}{d_G(O,N_1).d_G(O,N_4)}\right)}\cdot\frac{\left(\dfrac{r^2 d_G(N_2,N_4)}{d_G(O,N_2).d_G(O,N_4)}\right)}{\left(\dfrac{r^2 d_G(N_2,N_3)}{d_G(O,N_2).d_G(O,N_3)}\right)}\\ &= \frac{d_G(N_1,N_3)}{d_G(N_1,N_4)}\cdot\frac{d_G(N_2,N_4)}{d_G(N_2,N_3)}\\ &= \left|\frac{d_G[N_1N_3]}{d_G[N_1N_4]}\cdot\frac{d_G[N_2N_4]}{d_G[N_2N_3]}\right|\\ &= |(N_1N_2,N_3N_4)_G|\,.\end{aligned}$$

■

Consider that the points $N_1, N_2, N_3$ are given on the directed line. If point $N_3$ is between points $N_1$ and $N_2$, the line segment $N_1N_2$ is internally divided by point $N_3$. Otherwise, it is externally divided. In the case of internal division, the ratio of the resulting directed generalized taxicab lengths is equal to

$$\frac{d_G(N_1,N_3)}{d_G(N_3,N_2)} = \gamma > 0,$$

In the other case, it is

$$\frac{d_G(N_1,N_3)}{d_G(N_3,N_2)} = \gamma < 0.$$

**Definition 16.** *In the generalized taxicab plane, the points $N_1$ and $N_2$ on a line $l$ are said to be harmonically divided by any pair of points $N_3$ and $N_4$ on the same line if*

$$\frac{d_G[N_1N_3]}{d_G[N_3N_2]} = \frac{d_G[N_1N_4]}{d_G[N_4N_2]}.$$

*The points $N_3$ and $N_4$ are referred to as the generalized taxicab harmonic conjugates with respect to $N_1$ and $N_2$, and the generalized taxicab harmonic set of points is denoted by $H(N_1N_2, N_3N_4)_G$.*

**Corollary 17.** *Two distinct points $N_3$ and $N_4$ are the generalized taxicab harmonic conjugates with respect to the points $N_1$ and $N_2$ if and only if*

$$(N_1N_2, N_3N_4)_G = -1.$$

**Theorem 18.** *Let $\mathscr{C}$ be the generalized taxicab circle with the center $O$ and the diameter that is the line segment $N_1N_2$ and the distinct points $N_3$ and $N_4$ on the ray $\overrightarrow{ON_2}$ divide the line segment $N_1N_2$ internally and externally, respectively. Then the points $N_3$ and $N_4$ are the generalized taxicab harmonic conjugates with respect to the points $N_1$ and $N_2$ if and only if $N_3$ and $N_4$ are the inverse points under the inversion $I_{\mathscr{C}}$.*

*Proof.* Suppose that the points $N_3$ and $N_4$ are the generalized taxicab harmonic conjugates with respect to the points $N_1$ and $N_2$ . Then

$$\begin{aligned} (N_1N_2, N_3N_4)_G &= -1 \\ \frac{d_G[N_1N_3]}{d_G[N_1N_4]} \cdot \frac{d_G[N_2N_4]}{d_G[N_2N_3]} &= -1. \end{aligned}$$

Since $N_3$ divides the line segment $N_1N_2$ internally and $N_3$ is on the ray $\overrightarrow{ON_2}$, then

$$d_G[N_2N_3] = -d_G(N_3,N_2) = d_G(O,N_3) - r$$

and

$$d_G[N_1N_3] = d_G(N_1,N_3) = r + d_G(O,N_3).$$

Similarly, since $N_4$ divides the line segment $N_1N_2$ externally and $N_4$ is on the ray $\overrightarrow{ON_2}$,

$$d_G[N_1N_4] = d_G(N_1,N_4) = d_G(O,N_4) + r$$

and

$$d_G[N_2N_4] = d_G(N_2,N_4) = d_G(O,N_4) - r.$$

From the equality

$$\frac{d_G(O,N_3)+r}{d_G(O,N_4)+r} \cdot \frac{d_G(O,N_4)-r}{d_T(O,N_3)-r} = -1,$$

$$d_G(O,N_3).d_G(O,N_4) = r^2$$

is obtained. Thus, the points $N_3$ and $N_4$ are the inverse points with respect to the inversion $I_{\mathscr{C}}$.

Conversely, suppose that the points $N_3$ and $N_4$ are the generalized taxicab inverse points. From the equality

$$d_G(O,N_3).d_G(O,N_4) = r^2$$

and the above directed generalized taxicab lenghts, it is similarly seen that the cross ratio $(N_1N_2, N_3N_4)_G$ is equal to $-1$. ■

## 4. Conclusions

This study investigates the notion of the inversion in a generalized taxicab circle, providing comprehensive definitions, theorems and proofs that illuminate its behavior in the non-Euclidean geometry. This study provides formulas for determining the coordinates of inverse points using the generalized taxicab circle inversion. These results demonstrates the relationships among the inversion center, the radius and the positions of points in the plane. Also, the study was extended to understand the preservation of other geometric properties, such as the cross-ratios and the harmonic conjugates. The generalized taxicab cross-ratio, a fundamental concept, was defined and analyzed. Notably, the study showed that, similar to the Euclidean case, the cross-ratio remains invariant under the inversion in a generalized taxicab circle proving a fundamental property that holds in this non-Euclidean context. Moreover, the research has introduced the concept of the harmonic division in the generalized taxicab plane. In the study, the harmonic conjugates have been defined and the relation between the harmonic division and the inversion has been given . Consequently, the study provides significant observations into the circle inversions in the generalized taxicab geometry, enriching understanding of non-Euclidean planes and facilitating more investigation in this field.

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