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Research Article

On the Laplacian Spectrum of Desargues and Pappus Configurations

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Abstract

In this paper, we study the Laplacian spectrum of Desargues and Pappus configurations and present some basic spectral properties of these hypergraphs.

Keywords: Hypergraph, Desargues Configuration, Pappus Configuration, Laplacian spectrum

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Dezarg ve Pappus Konfigürasyonlarının Laplasyan Spektrumu Üzerine

Öz

Bu makalede, Desargues ve Pappus konfigurasyonlarının Laplasyan spektrumlarını inceliyor ve bu hipergrafların bazı temel spektrum özelliklerini sunuyoruz.

Anahtar Kelimeler: Hipergraf, Desargue Konfigürasyonu, Pappus Konfigürasyonu, Laplasyan spektrum.

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1. Introduction

Euler was the first to apply a graph technique to address the Seven Bridges of Königsberg puzzle in 1736. Sylvester first used the word "graph" in 1878. Graphs are widely utilized in many fields. Although graphs have undergone many developments in the first half of the 20th century, there is still much to learn about them and do with them in the domains of mathematics, physics, computer science, and many more. In Berge [1970, 1973] the author introduced hypergraphs as a tool to generalize the graph approach. While graphs contain only an edge between two points, hypergraphs can have multiple points on their edges. More details about hypergraphs can be found in most of the references cited throughout this article (Ouvrard, 2020).

A k-uniform hypergraph H is a pair (X, D) such that D is a subset of $P_k(\mathrm{X})$, the set of all k-subsets of a finite set X, where k is an integer and $k \geq 2$. Elements of X are called the vertices, while those of D are called the edges of H. A hypergraph H is said to be linear if every pair of distinct vertices of H is in at most one edge of H. The intersection graph of a hypergraph H =(X, E), denoted by G(H), is the graph where V(G(H)) = E, and E(G(H)) is the set of all unordered pairs $\{e, e'\}$ of distinct elements of E such that $|e \cap e'| = 1$ in H, where $|\mathrm{A}|$ for any set A denotes its cardinality. The intersection graphs of graphs are called line graphs (Naik et.al, 1982).

Steiner systems have close relation with the k-uniform hypergraphs. Consider integers t, k, and n that satisfy the conditions: $2 \leq t \leq k < n$. A Steiner system, denoted as $S(t;k;n)$, is defined as a k-uniform hypergraph $\mathrm{H} = (\mathrm{X}, \mathrm{D})$ comprising n vertices. In this hypergraph, for every subset $T \subseteq X$ consisting of t elements, there exists precisely one hyperedge $d \in D$ such that $T \subseteq d$. An illustrative example of a Steiner system is the complete graph K_n. Another noteworthy example is the Fano plane (Bretto, 2013), (Akça, et.al, 2006).

Researchers have explored the intersection of Steiner systems with algebraic geometry and commutative algebra (Braun et.al, 2016), (Ballico, 2020). They have established connections between two ideals in an appropriate polynomial ring, which define a Steiner configuration of points and its complement.

In 1953, Paige and Wexler introduced a form of the incidence matrix of a finite projective plane organized about a point line incident pair. This study is centered around examining the properties of the Laplacian matrix associated with a Steiner system. Laplacian matrices are semidefinite symmetric matrices and all of their eigenvalues are real numbers. The diagonal entries of each row are equal to the sum of the absolute value of the off-diagonal entries. Since each of the row sums of a Laplacian matrix is 0, the smallest eigenvalue of this matrix will always be 0 (Molitierno et al, 2011), (Paige et al, 1953).

The concept of the Laplacian integral hypergraphs get inspiration from Laplacian integral graphs. Research by Fallat et al. (2005) provided insights for describing graphs with distinctive integer eigenvalues in their exploration of the Laplacian spectrum of graphs. Also, it was shown that finite order projective planes are the Laplacian integrals (Zakiyyah, 2021).

In this study, the Laplacian spectrum of Desargues and Pappus configurations, which are examples of 3-uniform hypergraphs, is analyzed. It is shown that these hypergraphs are Laplacian integral, and the connectivity properties of these 3-uniform hypergraphs are provided.

2. Material and Method

This research is about the Laplacian spectrum of Desargues and Pappus configurations. First step is to analyze these hypergraphs, define their Laplacian matrices. The Laplacian matrix of a hypergraph provides information about the connectivity and relationships among vertices. Next step is to determine its spectrum. The spectrum of a matrix refers to its eigenvalues. The eigenvalues represent the important structural properties of the hypergraph. If all the eigenvalues of the Laplacian matrix are integers, it implies that the finite projective plane can be characterized as an integral Laplacian hypergraph.

3. The Laplacian Spectrum of Desargues Configuration

A Desargues configuration is a specific geometric configuration in projective geometry named after the French mathematician Girard Desargues. Although Desargues configurations and graph theory are two different mathematical concepts, they have connections and similarities, especially concerning the representation of geometric configurations as hypergraphs. A Desargues configuration consists of two triangles, typically labeled the reference triangle, the perspective triangle, and a central perspective point. These triangles are positioned so that certain lines connecting the corresponding vertices of the two triangles all intersect at the central point. In short, two triangles that are perspective from a center are also perspective from an axis. This is also known as Desargues' theorem (Grünbaum, 2009). The use of graph theory in this field has created a bridge between geometry and combinatorics, allowing mathematicians to construct, study and analyze Desargues configurations in a mathematical way.

The Desargues configuration has 10 vertices, 10 hyperedges, and each vertex is incident to 3 hyperedges, with every hyperedge containing 3 vertices, as illustrated in Figure 1. Because of these properties, the Desargues configuration is an example of a 3-uniform hypergraph.The point set of Desarques configuration is

$$P = \{v_1, v_2, \dots, v_{10}\}$$

and the lines are

L=$\{\{v_1, v_3, v_8\}, \{v_1, v_2, v_9\}, \{v_1, v_4, v_7\}, \{v_6, v_7, v_8\}, \{v_5, v_6, v_{10}\}, \{v_3, v_4, v_6\}, \{v_2, v_4, v_5\}, \{v_5, v_7, v_9\}, \{v_8, v_9, v_{10}\}, \{v_2, v_3, v_{10}\}\}$, (Figure 1).

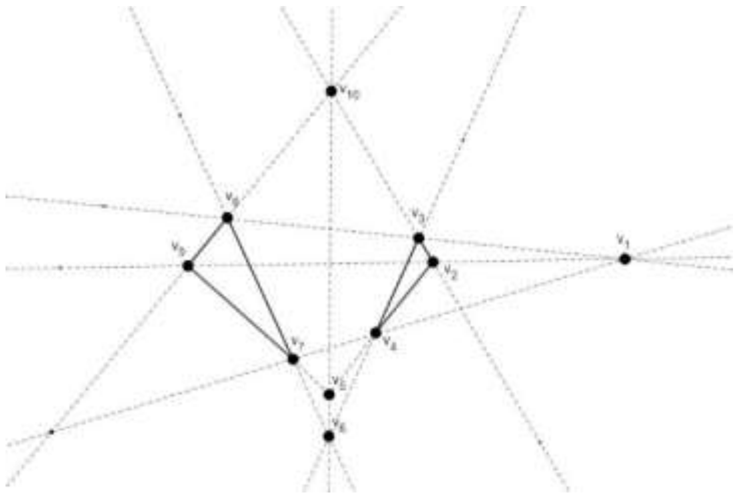

Figure 1. Desargues Configuration

Theorem 3.1. The spectrum of Desargues configuration consist of the eigenvalues -6 with multiplicity of 1 and 4 with multiplicity of 9.

Proof. The spectrum of the Desargues configuration is obtained from Laplacian matrix $\mathrm{L}(\mathcal{D}_{\mathcal{H}})$. $\mathrm{L}(\mathcal{D}_{\mathcal{H}})$ is calculated by subtracting the adjacency matrix $\mathrm{A}(\mathcal{D}_{\mathcal{H}})$ from the degree matrix $Deg(\mathcal{D}_{\mathcal{H}})$. The matrices $\mathrm{A}(\mathcal{D}_{\mathcal{H}})$ and $Deg(\mathcal{D}_{\mathcal{H}})$ are

$$\begin{pmatrix} 0&1&1&1&0&0&1&1&1&0\\ 1&0&1&1&1&0&0&0&1&1\\ 1&1&0&1&0&1&0&1&0&1\\ 1&1&1&0&1&1&1&0&0&0\\ 0&1&0&1&0&1&1&0&1&1\\ 0&0&1&1&1&0&1&1&0&1\\ 1&0&0&1&1&1&0&1&1&0\\ 1&0&1&0&0&1&1&0&1&1\\ 1&1&0&0&1&0&1&1&0&1\\ 0&1&1&0&1&1&0&1&1&0 \end{pmatrix} \text{ and } \begin{pmatrix} 3&0&0&0&0&0&0&0&0&0\\ 0&3&0&0&0&0&0&0&0&0\\ 0&0&3&0&0&0&0&0&0&0\\ 0&0&0&3&0&0&0&0&0&0\\ 0&0&0&0&3&0&0&0&0&0\\ 0&0&0&0&0&3&0&0&0&0\\ 0&0&0&0&0&0&3&0&0&0\\ 0&0&0&0&0&0&0&3&0&0\\ 0&0&0&0&0&0&0&0&3&0\\ 0&0&0&0&0&0&0&0&0&3 \end{pmatrix}$$

respectively. From the equality $\mathrm{L}(\mathcal{D}_{\mathcal{H}}) = \mathrm{Deg}(\mathcal{D}_{\mathcal{H}}) - \mathrm{A}(\mathcal{D}_{\mathcal{H}})$, the matrix $\mathrm{L}(\mathcal{D}_{\mathcal{H}})$ is determined as

$$\begin{pmatrix} 3&-1&-1&-1&-1&-1&-1&-1&-1&-1\\ -1&3&-1&-1&-1&-1&-1&-1&-1&-1\\ -1&-1&3&-1&-1&-1&-1&-1&-1&-1\\ -1&-1&-1&3&-1&-1&-1&-1&-1&-1\\ -1&-1&-1&-1&3&-1&-1&-1&-1&-1\\ -1&-1&-1&-1&-1&3&-1&-1&-1&-1\\ -1&-1&-1&-1&-1&-1&3&-1&-1&-1\\ -1&-1&-1&-1&-1&-1&-1&3&-1&-1\\ -1&-1&-1&-1&-1&-1&-1&-1&3&-1\\ -1&-1&-1&-1&-1&-1&-1&-1&-1&3 \end{pmatrix}.$$

Since the characteristic polynomial of $\mathrm{L}(\mathcal{D}_{\mathcal{H}})$ is $(x+6)(x-4)^9$, the eigenvalues of this hypergraph are $\lambda = -6$ with multiplicity of 1 and $\lambda = 4$ with multiplicity of 9. This completes the proof.

Corollary 3.1. The hypergraph of the Desargues configuration is a Laplacian integral.

Proof. It is seen that from the previous theorem that the eigenvalues are integers; therefore, the hypergraph of the Desargues configuration is Laplacian integral.

Theorem 3.2. The eigenvectors corresponding to the Laplacian spectrum of Desargues configuration spans a linear vector spaces of 1 and 9-dimensions.

Proof. From the previous theorem it was proved that Laplacian spectrum of Desargues configuration consists of the eigenvalues -6 and 4. The eigenspace spanned by the eigenvector (1,1,1,1,1,1,1,1,1,1) corresponding to the eigenvalue -6 is a 1-dimensional linear space. The eigenspace corresponding to eigenvalue 4 is a 9-dimensional linear space spanned by the following the set of eigen vectors

$\{(-1,0,0,0,0,0,0,0,0,1), (-1,0,0,0,0,0,0,0,1,0), (-1,0,0,0,0,0,0,1,0,0), (-1,0,0,0,0,0,1,0,0,0), (-1,0,0,0,0,1,0,0,0,0), (-1,0,0,0,1,0,0,0,0,0), (-1,0,0,1,0,0,0,0,0,0), (-1,0,1,0,0,0,0,0,0,0), (-1,1,0,0,0,0,0,0,0,0)\}$.

Here we can conclude the following: When a graph has a negative eigenvalue, especially for the Laplacian matrix, it means that the hypergraph is not fully connected. In other words, the hypergraph has at least two disjoint components or subgraphs.

4. The Laplacian Spectrum of Pappus Configuration

Like the Desargues configuration, the Pappus configuration is a well known geometric configuration in projective geometry, named after the Greek mathematician Pappus of Alexandria. It is a structure containing 9 points and 9 lines in the projective plane (Grünbaum, 2009). If we consider the Pappus configuration as a system of points and lines, we can see that every hyperedge contains 3 points and every point lies on 3 lines, which is an example of a 3-uniform hypergraph. The principles underlying the Pappus configuration, a classical problem in projective geometry, are relevant to a variety of fields including computer graphics, computer vision, engineering, robotics, architecture, CAD, network design and mathematical research. The Pappus configuration has an important place in these applications for understanding perspective, transformations and geometric relationships.

Pappus configuration has 9 vertices, 9 hyperedges, and each vertex is incident on 3 hyperlines and every hyperedge contains 3 vertices, as shown Figure 2. Because of these properties, the Pappus configuration is also an example of a 3-uniform hypergraph. The point set of Pappus configuration is $P = \{A, B, C, A', B', C', X, Y, Z\}$ and the set of lines L is $\{\{A, B, C\}, \{A', B', C'\}, \{X, Y, Z\}, \{A, X, B'\}, \{A, Y, C'\}, \{B, X, A'\}, \{B, Z, C'\}, \{C, Y, A'\}, \{C, Z, B'\}\}$.

Theorem 4.1. The spectrum of Pappus configuration consists of the eigenvalues -5 with multiplicity of 1 and 4 multiplicity of 8.

Proof. The spectrum of the Pappus configuration is obtained from Laplacian matrix $\mathrm{L}(\mathcal{P}_{\mathcal{H}})$. $\mathrm{L}(\mathcal{P}_{\mathcal{H}})$ is calculated by subtracting the adjacency matrix $\mathrm{A}(\mathcal{P}_{\mathcal{H}})$ from the degree matrix $Deg(\mathcal{P}_{\mathcal{H}})$. The matrices $\mathrm{A}(\mathcal{P}_{\mathcal{H}})$ and $Deg(\mathcal{P}_{\mathcal{H}})$ are

$$\begin{pmatrix} 0&1&1&1&1&1&1&1&1\\ 1&0&1&1&1&1&1&1&1\\ 1&1&0&1&1&1&1&1&1\\ 1&1&1&0&1&1&1&1&1\\ 1&1&1&1&0&1&1&1&1\\ 1&1&1&1&1&0&1&1&1\\ 1&1&1&1&1&1&0&1&1\\ 1&1&1&1&1&1&1&0&1\\ 1&1&1&1&1&1&1&1&0 \end{pmatrix} \text{ and } \begin{pmatrix} 3&0&0&0&0&0&0&0&0\\ 0&3&0&0&0&0&0&0&0\\ 0&0&3&0&0&0&0&0&0\\ 0&0&0&3&0&0&0&0&0\\ 0&0&0&0&3&0&0&0&0\\ 0&0&0&0&0&3&0&0&0\\ 0&0&0&0&0&0&3&0&0\\ 0&0&0&0&0&0&0&3&0\\ 0&0&0&0&0&0&0&0&3 \end{pmatrix}$$

respectively. From the equality $\mathrm{L}(\mathcal{P}_{\mathcal{H}}) = \mathrm{Deg}(\mathcal{P}_{\mathcal{H}}) - \mathrm{A}(\mathcal{P}_{\mathcal{H}})$, the matrix $\mathrm{L}(\mathcal{P}_{\mathcal{H}})$ is found as

$$\begin{pmatrix} 3&-1&-1&-1&-1&-1&-1&-1&-1\\ -1&3&-1&-1&-1&-1&-1&-1&-1\\ -1&-1&3&-1&-1&-1&-1&-1&-1\\ -1&-1&-1&3&-1&-1&-1&-1&-1\\ -1&-1&-1&-1&3&-1&-1&-1&-1\\ -1&-1&-1&-1&-1&3&-1&-1&-1\\ -1&-1&-1&-1&-1&-1&3&-1&-1\\ -1&-1&-1&-1&-1&-1&-1&3&-1\\ -1&-1&-1&-1&-1&-1&-1&-1&3 \end{pmatrix}.$$

Since the characteristic polynomial of $\mathrm{L}(\mathcal{P}_{\mathcal{H}})$ is $(x+5)(x-4)^8$, the eigenvalues of this hypergraph are $\lambda = -5$ with multiplicity of 1 and $\lambda = 4$ with multiplicity of 8. This completes the proof.

Theorem 4.2. The eigenvectors corresponding to the Laplacian spectrum of Pappus configuration spans the linear vector spaces of 1 and 8-dimensions.

Proof. From the previous theorem, it was proved that the Laplacian spectrum of Desargues configuration consists of eigenvalues -5 and 4. The eigenspace spanned by the eigenvector $\{1,1,1,1,1,1,1,1,1\}$ corresponding to eigenvalue -5 is a 1-dimensional linear space. The eigenspace corresponding to eigenvalue 4 is a 8-dimensional linear space spanned by the vectors $\{-1,0,0,0,0,0,0,0,1\}$, $\{-1,0,0,0,0,0,0,1,0\}$, $\{-1,0,0,0,0,0,1,0,0\}$, $\{-1,0,0,0,0,1,0,0,0\}$, $\{-1,0,0,0,1,0,0,0,0\}$, $\{-1,0,0,1,0,0,0,0,0\}$, $\{-1,0,1,0,0,0,0,0,0\}$, $\{-1,1,0,0,0,0,0,0,0\}$.

While the Pappus and Desargues configurations share some similarities, they differ in their geometric structure, incidence properties and the spaces they span. The Pappus configuration shows the perspective of points on axes, while the Desargues configuration builds a graph model about two perspective triangles. While these configurations are widely used in the field of projective geometry, in graph theory applications they form an example of a 3-uniform hypergraph.

5. Conclusions and Recommendations

In this study, the Laplacian spectrum and the properties of Desargues and Pappus configurations are investigated.

As a result, since the Laplacian eigenvalues of these structures are integers, they are Laplacian integral structures. Moreover, for the Desargues configuration, the eigenspace corresponding to the eigenvalue -6 is a 1-dimensional linear vector space, while the eigenspace associated with the eigenvalue 4 is a 9-dimensional linear vector space. Similarly, in the case of a Pappus configuration, our analysis shows that the eigenspace for eigenvalue -5 is 1-dimensional, while the eigenspace corresponding to eigenvalue 4 is a 8-dimensional linear vector space.

Since the eigenvalues of the Laplacian matrices provide information about the connectivity of graph models, it was observed that as the negative eigenvalues grow numerically, the connectivity of the graph model increases. Therefore, Pappus configuration was found to be more connected than Desargues configuration.

These results will contribute to the investigation of spectral properties in areas such as combinatorics, algebraic geometry, and coding theory.

6. Acknowledge

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