Universal Journal of Mathematics and Applications, 6 (4) (2023) 155-161
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1376849

On Strongly Lacunary $\mathcal{I}_2^*$-Convergence and Strongly Lacunary $\mathcal{I}_2^*$-Cauchy Sequence

Erdinç Dündar[1*], Nimet Akın[2] and Esra Gülle[1]

[1] *Department of Mathematics, Afyon Kocatepe University, 03200 Afyonkarahisar, Türkiye*
[2] *Department of Mathematics Teaching, Afyon Kocatepe University, 03200 Afyonkarahisar, Türkiye*
[*] *Corresponding author*

Article Info

Keywords: *$\mathcal{I}_2$-Cauchy Sequence, $\mathcal{I}_2$-Convergence, Double sequence, Ideal, Lacunary sequence*
2010 AMS: *40A05, 40A35*
Received: *16 October 2023*
Accepted: *30 November 2023*
Available online: *4 December 2023*

Abstract

In the study conducted here, we have given some new concepts in summability theory. In this sense, firstly, using the lacunary sequence we have given the concept of strongly $\mathcal{I}_{\theta_2}^*$-convergence and we have examined the relations between $\mathcal{I}_{\theta_2}^*$-convergence and strongly $\mathcal{I}_{\theta_2}^*$-convergence and also between strongly $\mathcal{I}_{\theta_2}$-convergence and strongly $\mathcal{I}_{\theta_2}^*$-convergence. Also, using the lacunary sequence we have given the concept of strongly $\mathcal{I}_{\theta_2}^*$-Cauchy sequence and examined the relations between strongly $\mathcal{I}_{\theta_2}$-Cauchy sequence and strongly $\mathcal{I}_{\theta_2}^*$-Cauchy sequence.

1. Introduction and Definitions

In the mathematical literature, some types of convergence in summability theory and some applications and properties of these convergence types have been studied by many mathematicians. Especially recently, some types of convergence of double-indexed sequences have been frequently studied in summability theory. Also, the types of convergence defined in summability theory using the lacunary sequence have been studied by many authors. These studies were carried out by generalizing the theorems that give some similar properties in single-index sequences to double-index sequences. Classical convergence in real number sequences was generalized to the statistical convergence by Schoenberg [1] and Fast [2], independently. The ideal convergence, a generalization of statistical convergence that would later inspire many researchers, was first defined by Kostyrko et al. [3]. Nabiev [4] studied on the $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences with their characteristics. In the topology generated by n-normed spaces, the lacunary ideal convergence, lacunary ideal Cauchy sequence and their some important characteristics investigated by Yamancı and Gürdal [5]. The ideal lacunary convergence was introduced by Tripathy et al. [6]. In recent times, using the lacunary sequence, the $\mathcal{I}_\theta^*$-convergence, strongly $\mathcal{I}_\theta^*$-convergence, $\mathcal{I}_\theta^*$-Cauchy sequence and strongly $\mathcal{I}_\theta^*$-Cauchy sequence were introduced by Akın and Dündar [7, 8]. The concept of ideal convergence and some of its important characteristics defined for single-index sequences have also been defined for double-index sequences in the linear metric space by many mathematicians [9–11] and many useful works have been done in this sense. In addition, the ideal convergence and strong ideal convergence and some of its characteristic properties using the lacunary sequence for single-index sequences were also introduced to the literature by Hazarika [12], Dündar et al. [13] and Akın and Dündar [14] for double sequences and double set sequences in metric spaces and normed spaces.
In recently, some convergence types such as classical convergence, statistical convergence and ideal convergence in some metric spaces and normed spaces were studied in summability theory by a lot of mathematicians. In the study conducted here, using the lacunary sequence, we have given the concept of strongly $\mathcal{I}_{\theta_2}^*$-convergence and we have investigated the relations between $\mathcal{I}_{\theta_2}^*$-convergence and strongly $\mathcal{I}_{\theta_2}^*$-convergence and also between strongly $\mathcal{I}_{\theta_2}$-convergence and strongly $\mathcal{I}_{\theta_2}^*$-convergence. Furthermore, using the lacunary sequence, we have given the concept of strongly $\mathcal{I}_{\theta_2}^*$-Cauchy sequence and examined the relations between strongly $\mathcal{I}_{\theta_2}$-Cauchy sequence and strongly $\mathcal{I}_{\theta_2}^*$-Cauchy sequence.
Some basic definitions, concepts and characteristics that will be used throughout the study and are available in the literature will now be noted (see [3–8, 10, 11, 14–21]).
Firstly, we want to give the ideas of ideal, filter and some properties about these ideas are used in our study.
For $\mathcal{I} \subseteq 2^{\mathbb{N}}$, if the following propositions

***Email addresses and ORCID numbers*: edundar@aku.edu.tr, 0000-0002-0545-7486 (E. Dündar), npancaroglu@aku.edu.tr, 0000-0003-4661-5388 (N. Akın), egulle@aku.edu.tr, 0000-0001-5575-2937 (E. Gülle)**
Cite as: E. Dündar, N. Akın, E. Gülle, *On Strongly Lacunary $\mathcal{I}_2^*$-Convergence and Strongly Lacunary $\mathcal{I}_2^*$-Cauchy Sequence*, Univers. J. Math. Appl., 6(4) (2023), 155-161.

(i) $\emptyset \in \mathscr{I}$,
(ii) If $T,U \in \mathscr{I}$, then $T \cup U \in \mathscr{I}$,
(iii) If $T \in \mathscr{I}$ and $U \subseteq T$, then $U \in \mathscr{I}$

are hold, then $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is named as an ideal.
If $\mathbb{N} \notin \mathscr{I}$, then $\mathscr{I}$ is named as a non-trivial ideal. Also, if $\{k\} \in \mathscr{I}$ for each $k \in \mathbb{N}$, then a non-trivial ideal is named as an admissible ideal.
For $\mathscr{F} \subseteq 2^{\mathbb{N}}$, if the following propositions

(i) $\emptyset \notin \mathscr{F}$,
(ii) If $T,U \in F$, then $T \cap U \in \mathscr{F}$,
(iii) If $T \in \mathscr{F}$ and $U \supseteq T$, then $U \in \mathscr{F}$

are hold, then $\mathscr{F} \subseteq 2^{\mathbb{N}}$ is named as a filter.
For a non-trivial ideal $\mathscr{I} \subseteq 2^{\mathbb{N}}$, $\mathscr{F}(\mathscr{I}) = \{T \subset \mathbb{N} : T = \mathbb{N} \setminus U \text{ for } \exists U \in \mathscr{I}\}$ is named as the filter corresponding with $\mathscr{I}$.
Here, we want to give the ideas of lacunary sequence and some properties about lacunary sequence are used in our studypaper.
The increasing integer sequence $\theta = \{k_r\}$ is named as a lacunary sequence when it satisfies the propositions $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$, $(r \to \infty)$. During the study, $I_r = (k_{r-1}, k_r]$ and q_r represent the intervals determined by $\{k_r\}$ and the ratio $\frac{k_r}{k_{r-1}}$, respectively.
Then after this, we regard $\theta = \{k_r\}$ as a lacunary sequence and $\mathscr{I} \subseteq 2^{\mathbb{N}}$ as a non-trivial admissible ideal.
For a sequence $(x_k) \subset \mathbb{R}$, if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - \ell| = 0$$

is hold, then (x_k) is strongly lacunary convergent to $\ell \in \mathbb{R}$.
For a sequence $(x_k) \subset \mathbb{R}$, if

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |x_k - \ell| \geq \varepsilon \right\} \in \mathscr{I}, \text{ (for every } \varepsilon > 0)$$

is hold, then (x_k) is strongly lacunary $\mathscr{I}$-convergent to $\ell \in \mathbb{R}$ and denoted with $x_k \to \ell[\mathscr{I}_\theta]$.
For a sequence $(x_k) \subset \mathbb{R}$, if for every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |x_k - x_N| \geq \varepsilon \right\} \in \mathscr{I},$$

then (x_k) is strongly lacunary $\mathscr{I}$-Cauchy sequence.
For a sequence $(x_k) \subset \mathbb{R}$, iff there exists a set $G = \{g_1 < g_2 < \cdots < g_k < \cdots\} \subset \mathbb{N}$ such that for the set $G' = \{r \in \mathbb{N} : g_k \in I_r\} \in \mathscr{F}(\mathscr{I})$

$$\lim_{\substack{r\to\infty \\ (r\in G')}} \frac{1}{h_r} \sum_{k\in I_r} |x_{g_k} - \ell| = 0$$

is hold, then (x_k) is strongly lacunary $\mathscr{I}^*$-convergent to $\ell \in \mathbb{R}$ and denoted with $x_k \to \ell[\mathscr{I}_\theta^*]$.
For a sequence $(x_k) \subset \mathbb{R}$, iff there exists a set $G = \{g_1 < g_2 < \cdots < g_k < \cdots\} \subset \mathbb{N}$ such that for the set $G' = \{r \in \mathbb{N} : g_k \in I_r\} \in \mathscr{F}(\mathscr{I})$

$$\lim_{\substack{r\to\infty \\ (r\in G')}} \sum_{k,p\in I_r} |x_{g_k} - x_{g_p}| = 0$$

is hold, then (x_k) is strongly lacunary $\mathscr{I}^*$-Cauchy sequence.
For each $k \in \mathbb{N}$ and a non-trivial ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$, if $\{k\} \times \mathbb{N} \in \mathscr{I}_2$ and $\mathbb{N} \times \{k\} \in \mathscr{I}_2$, then $\mathscr{I}_2$ is named as strongly admissible ideal.
$\mathscr{I}_2^0 = \{T \subset \mathbb{N}^2 : (\exists m(T) \in \mathbb{N})(i, j \geq m(T) \Rightarrow (i, j) \notin T)\}$ ideal is a strongly admissible ideal. Furthermore, it is clearly that $\mathscr{I}_2$ is a strongly admissible iff $\mathscr{I}_2^0 \subset \mathscr{I}_2$.
It can be clearly seen that a strongly admissible ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ is an admissible ideal.
The following set

$$\mathscr{F}(\mathscr{I}_2) = \{T \subset \mathbb{N}^2 : T = \mathbb{N}^2 \setminus U \text{ for } \exists U \in \mathscr{I}_2\}$$

is named a filter corresponding with $\mathscr{I}_2$.
For an admissible ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$, if for every countable mutually disjoint set family $\{T_1, T_2, ...\} \in \mathscr{I}_2$, there exists a countable set family $\{U_1, U_2, ...\}$ such that $T_k \Delta U_k \in \mathscr{I}_2^0$, that is, $T_k \Delta U_k$ is involved in finite union of rows and columns in $\mathbb{N}^2$ for each $k \in \mathbb{N}$ and $U = \bigcup_{k=1}^{\infty} U_k \in \mathscr{I}_2$ ($U_k \in \mathscr{I}_2$ for each $k \in \mathbb{N}$), then $\mathscr{I}_2$ is named satisfying the property $(AP2)$.
If for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - \ell| < \varepsilon$, for all $k, j > n_\varepsilon$, then the double sequence $x = (x_{kj}) \subset \mathbb{R}$ is convergent to $\ell \in \mathbb{R}$ and denoted with $\lim_{k,j\to\infty} x_{kj} = \ell$ or $\lim_{k,j\to\infty} x_{kj} = \ell$.
Now, we want to give the idea of double lacunary sequence and some properties about it are used in our manuscript.
The double sequence $\theta_2 = \{(k_r, j_u)\}$ is named as a double lacunary sequence when two increasing integer sequences satisfy the propositions

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \text{ and } j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty,$$

for $r, u \to \infty$. We take the following screenings for double lacunary sequence:

$$k_{ru} = k_r j_u, \quad h_{ru} = h_r \bar{h}_u, \quad I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \quad \textit{and} \quad j_{u-1} < j \leq j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \ \ and \ \ q_u = \frac{j_u}{j_{u-1}}.$$

Then after this, we regard $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ as a strongly admissible ideal and $\theta_2 = \{(k_r, j_u)\}$ as a double lacunary sequence.
For a double sequence $(x_{kj}) \subset \mathbb{R}$, if

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| = 0$$

is hold, then (x_{kj}) is strongly lacunary convergent to $\ell \in \mathbb{R}$.
For a double sequence $(x_{kj}) \subset \mathbb{R}$, if

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| = 0$$

is hold, therefore (x_{kj}) is strongly lacunary Cauchy double sequence.
For a double sequence $(x_{kj}) \subset \mathbb{R}$, if for every $\varepsilon > 0$

$$\left\{(r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| \geq \varepsilon\right\} \in \mathscr{I}_2$$

is hold, then (x_{kj}) is strongly lacunary $\mathscr{I}_2$-convergent to $\ell \in \mathbb{R}$ and denoted with $x_{kj} \to \ell[\mathscr{I}_{\theta_2}]$.
For a double sequence $(x_{kj}) \subset \mathbb{R}$, if for every $\varepsilon > 0$, there exist $N = N(\varepsilon)$ and $S = S(\varepsilon)$ such that

$$\left\{(r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - x_{NS}\right| \geq \varepsilon\right\} \in \mathscr{I}_2,$$

then (x_{kj}) is strongly lacunary $\mathscr{I}_2$-Cauchy double sequence.
Now, let's give a useful lemma that we will use in our work.

Lemma 1.1 ([10]). *Let $\mathscr{F}(\mathscr{I}_2)$ be a filter corresponding with a strongly admissible ideal $\mathscr{I}_2$ with $(AP2)$. Thus, there exists a set $T \subset \mathbb{N}^2$ such that $T \in \mathscr{F}(\mathscr{I}_2)$ and the set $T\backslash T_k$ is finite for all k, where $\{T_k\}_{k=1}^{\infty}$ is a countable collection of subsets of $\mathbb{N}^2$ such that $T_k \in F(\mathscr{I}_2)$ for each k.*

2. Main Results

In the original part of our work, using the lacunary sequence, we will define for double-indexed sequences the definitions and concepts available in the literature for single-indexed sequences. For double sequences, we first defined lacunary $\mathscr{I}_2^*$-convergence and strongly lacunary $\mathscr{I}_2^*$-convergence with theorems examining the relationship between these new convergence types.

Definition 2.1. *A double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2^*$-convergent to $\ell \in \mathbb{R}$ iff there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have*

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} = \ell$$

and so we can write $x_{kj} \to \ell(\mathscr{I}_{\theta_2}^*)$.

Definition 2.2. *A double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly lacunary $\mathscr{I}_2^*$-convergent to $\ell \in \mathbb{R}$ iff there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have*

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| = 0$$

and so we can write $x_{kj} \to \ell[\mathscr{I}_{\theta_2}^*]$.

Theorem 2.3. *If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly $\mathscr{I}_{\theta_2}^*$-convergent to $\ell \in \mathbb{R}$, then it is $\mathscr{I}_{\theta_2}^*$-convergent to same point.*

Proof. As per our assumption, let $x_{kj} \to \ell[\mathscr{I}_{\theta_2}^*]$. Therefore, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., U = \mathbb{N}^2\backslash G' \in \mathscr{I}_2)$ and for every $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that for all $r,u > r_0$ we have

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| < \varepsilon, \ ((r,u) \in G').$$

Therefore, for every $\varepsilon > 0$ and all $r,u > r_0$ we have

$$\left|\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell\right| \leq \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| < \varepsilon, \ ((r,u) \in G')$$

and so $x_{kj} \to \ell(\mathscr{I}_{\theta_2}^*)$. □

Theorem 2.4. *If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly $\mathscr{I}^*_{\theta_2}$-convergent to $\ell \in \mathbb{R}$, then this sequence is strongly $\mathscr{I}_{\theta_2}$-convergent to same point.*

Proof. As per our assumption, let $x_{kj} \to \ell[\mathscr{I}^*_{\theta_2}]$. Therefore, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., U = \mathbb{N}^2 \backslash G' \in \mathscr{I}_2)$ there is a $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that for all $r,u > r_0$ we have

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| < \varepsilon, \ ((r,u) \in G').$$

Then,

$$A(\varepsilon) = \left\{(r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| \geq \varepsilon\right\} \subset U \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\})\right)\right].$$

Since $\mathscr{I}_2$ is a strongly admissible ideal, we have

$$U \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\})\right)\right] \in \mathscr{I}_2.$$

Thus $A(\varepsilon) \in \mathscr{I}_2$ and $x_{kj} \to \ell[\mathscr{I}_{\theta_2}]$. □

Theorem 2.5. *Let a strongly admissible ideal $\mathscr{I}_2$ with $(AP2)$. If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly $\mathscr{I}_{\theta_2}$-convergent to $\ell \in \mathbb{R}$, then this sequence is strongly $\mathscr{I}^*_{\theta_2}$-convergent to same point.*

Proof. As per our assumption, let $x_{kj} \to \ell[\mathscr{I}_{\theta_2}]$. Then, for every $\varepsilon > 0$,

$$\mathscr{M}(\varepsilon) = \left\{(r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| \geq \varepsilon\right\} \in \mathscr{I}_2.$$

Let us take

$$\mathscr{M}_1 = \left\{(r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| \geq 1\right\} \text{ and}$$

$$\mathscr{M}_\beta = \left\{(r,u) \in \mathbb{N}^2 : \frac{1}{\beta} \leq \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| < \frac{1}{\beta - 1}\right\},$$

for a natural number $\beta \geq 2$. Obviously, $\mathscr{M}_\alpha \cap \mathscr{M}_\gamma = \emptyset$ for $\alpha \neq \gamma$ and $\mathscr{M}_\alpha \in \mathscr{I}_2$ for each $\alpha \in \mathbb{N}$. Also, by $(AP2)$, there is a sequence $\{\mathscr{V}_\beta\}_{\beta\in\mathbb{N}}$ such that $\mathscr{M}_\alpha \Delta \mathscr{V}_\alpha$ is involved in finite union of rows and columns in $\mathbb{N}^2$ for each $\alpha \in \mathbb{N}$ and

$$\mathscr{V} = \bigcup_{\alpha=1}^{\infty} \mathscr{V}_\alpha \in \mathscr{I}_2.$$

We prove that

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| = 0,$$

for $G' = \mathbb{N}^2 \setminus \mathscr{V} \in \mathscr{F}(\mathscr{I}_2)$. For $\delta > 0$ select $q \in \mathbb{N}$ with the inequality $\frac{1}{q} < \delta$. Therefore,

$$\left\{(r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| \geq \delta\right\} \subset \bigcup_{j=1}^{q-1} \mathscr{M}_j.$$

Since $\mathscr{M}_\alpha \Delta \mathscr{V}_\alpha$ is a finite set for $\alpha \in \{1,2,\cdots,q-1\}$, there exists $r_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{q-1} \mathscr{M}_j\right) \cap \left\{(r,u) \in \mathbb{N}^2 : r \geq r_0 \wedge u \geq r_0\right\} = \left(\bigcup_{j=1}^{q-1} \mathscr{V}_j\right) \cap \left\{(r,u) \in \mathbb{N}^2 : r \geq r_0 \wedge u \geq r_0\right\}.$$

If $r \geq r_0$ and $(r,u) \notin V$, then

$$(r,u) \notin \bigcup_{j=1}^{q-1} \mathscr{V}_j \text{ and so } (r,u) \notin \bigcup_{j=1}^{q-1} \mathscr{M}_j.$$

We have

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| < \frac{1}{q} < \delta.$$

And so from this inequality, we have

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| = 0.$$

Therefore, we have $x_{kj} \to \ell[\mathscr{I}^*_{\theta_2}]$. □

Now, for double sequences, we have defined $\mathscr{I}^*_{\theta_2}$-Cauchy and strongly $\mathscr{I}^*_{\theta_2}$-Cauchy sequence with theorems examining the relationships between $\mathscr{I}^*_{\theta_2}$-Cauchy sequence and strongly $\mathscr{I}^*_{\theta_2}$-Cauchy sequence, and also between strongly $\mathscr{I}^*_{\theta_2}$-Cauchy sequence and strongly $\mathscr{I}_{\theta_2}$-Cauchy sequence.

Definition 2.6. *A double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}^*_2$-Cauchy sequence iff there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have*

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) = 0.$$

Definition 2.7. *A double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly lacunary $\mathscr{I}^*_2$-Cauchy sequence iff there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have*

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| = 0.$$

Theorem 2.8. *If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly lacunary $\mathscr{I}^*_2$-Cauchy sequence, then (x_{kj}) is lacunary $\mathscr{I}^*_2$-Cauchy sequence.*

Proof. As per our assumption, let (x_{kj}) is strongly lacunary $\mathscr{I}^*_2$-Cauchy sequence. Thus, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., H = \mathbb{N}^2 \setminus G' \in \mathscr{I}_2)$ and for every $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) \in \mathbb{N}$, we have

$$\frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| < \varepsilon, \ \ ((r,u) \in G')$$

for all $r,u > r_0$. Then, we have

$$\left| \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) \right| \le \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| < \varepsilon, \ \ ((r,u) \in G')$$

for every $\varepsilon > 0$ and all $r,u > r_0$ and so (x_{kj}) is lacunary $\mathscr{I}^*_2$-Cauchy sequence. □

Theorem 2.9. *If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly lacunary $\mathscr{I}^*_2$-Cauchy sequence, then (x_{kj}) is strongly lacunary $\mathscr{I}_2$-Cauchy sequence.*

Proof. As per our assumption, let (x_{kj}) is strongly lacunary $\mathscr{I}^*_2$-Cauchy sequence. Then, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., U = \mathbb{N}^2 \backslash G' \in \mathscr{I}_2)$ and for every $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) \in \mathbb{N}$, we have

$$\frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| < \varepsilon, \ \ ((r,u) \in G')$$

for all $r,u > r_0$. Let $N = N(\varepsilon) \in I_{r_0+1,u_0+1}$ and $S = S(\varepsilon) \in I_{r_0+1,u_0+1}$ Then, for every $\varepsilon > 0$ and all $r,u > r_0 = r_0(\varepsilon)$

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{NS}| < \varepsilon, \ \ ((r,u) \in G').$$

Now, let $U = \mathbb{N}^2 \setminus G'$. It is clear that $U \in \mathscr{I}_2$. Then,

$$A(\varepsilon) = \left\{ (r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{NS}| \ge \varepsilon \right\} \subset U \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\}) \right) \right].$$

Since $\mathscr{I}_2$ is a strongly admissible ideal, we have

$$U \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\}) \right) \right] \in \mathscr{I}_2$$

and so $A(\varepsilon) \in \mathscr{I}_2$. Hence, (x_{kj}) is strongly lacunary $\mathscr{I}_2$-Cauchy sequence. □

Theorem 2.10. *Let a strongly admissible ideal $\mathscr{I}_2$ with $(AP2)$. If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly lacunary $\mathscr{I}_2$-Cauchy sequence, then (x_{kj}) is strongly lacunary $\mathscr{I}^*_2$-Cauchy sequence.*

Proof. As per our assumption, let (x_{kj}) is strongly lacunary $\mathscr{I}_2$-Cauchy sequence. Then, for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $S = S(\varepsilon)$ such that

$$A(\varepsilon) = \left\{ (r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{NS}| \ge \varepsilon \right\} \in \mathscr{I}_2.$$

Let us take

$$T_j = \left\{ (r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{m_j n_j}| \ge \frac{1}{j} \right\}, \ j = 1,2,\ldots,$$

where $m_j = P\left(\frac{1}{j}\right)$ and $n_j = S\left(\frac{1}{j}\right)$. It is clear that $T_j \in \mathscr{F}(\mathscr{I}_2)$ for $j = 1,2,\cdots$. Since $\mathscr{I}_2$ has the $(AP2)$ property, then by Lemma 1.1, there exists a set $T \subset \mathbb{N}^2$ such that $T \in \mathscr{F}(\mathscr{I}_2)$ and $T \setminus T_j$ is finite for all j. Now, we demonstrate that

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in T)}} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| = 0.$$

To show this, let $\varepsilon > 0$ and a natural number m be $m > \frac{2}{\varepsilon}$. If $(r,u) \in T$, then $T \setminus T_m$ is a finite set, so there exists $r_0 = r_0(m)$ such that $(r,u) \in T_m$ for all $r,u > r_0(m)$. Therefore, for all $r,u > r_0(m)$

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{s_m t_m}| < \frac{1}{m} \text{ and } \frac{1}{h_{ru}} \sum_{(s,t)\in I_{ru}} |x_{st} - x_{s_m t_m}| < \frac{1}{m}.$$

Hence, for all $r,u > r_0(m)$ it follows that

$$\frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| \leq \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{s_m t_m}| + \frac{1}{h_{ru}} \sum_{(s,t)\in I_{ru}} |x_{st} - x_{s_m t_m}| < \frac{1}{m} + \frac{1}{m} < \varepsilon.$$

Therefore, for any $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon)$ such that for $r,u > r_0(\varepsilon)$ and $(r,u) \in T \in \mathscr{F}(\mathscr{I}_2)$

$$\frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| < \varepsilon.$$

This demonstrates that (x_{kj}) is strongly lacunary $\mathscr{I}_2^*$-Cauchy sequence. □

Theorem 2.11. *If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly $\mathscr{I}_{\theta_2}^*$-convergent to ℓ, then (x_{kj}) is strongly $\mathscr{I}_{\theta_2}$-Cauchy double sequence.*

Proof. As per our assumption, let $x_{kj} \to \ell[\mathscr{I}_{\theta_2}^*]$. Therefore, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., U = \mathbb{N}^2 \backslash G' \in \mathscr{I}_2)$ there is a $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that for all $r,u > r_0$ we have

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - \ell| < \frac{\varepsilon}{2}, \ ((r,u) \in G').$$

Since

$$\frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| \leq \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - \ell| + \frac{1}{h_{ru}} \sum_{(s,t)\in I_{ru}} |x_{st} - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ ((r,u) \in G')$$

for all $r,u > r_0$ and so we have

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| = 0.$$

That is, (x_{kj}) is a strongly $\mathscr{I}_{\theta_2}^*$-Cauchy sequence. Thus,(x_{kj}) is a strongly $\mathscr{I}_{\theta_2}$-Cauchy sequence by Theorem 2.9. □

3. Conclusion

Using the lacunary sequence, for double sequences, we have first defined lacunary $\mathscr{I}_2^*$-convergence and strongly lacunary $\mathscr{I}_2^*$-convergence with theorems examining the relationship between these new convergence types. Furthermore, we have defined $\mathscr{I}_{\theta_2}^*$-Cauchy and strongly $\mathscr{I}_{\theta_2}^*$-Cauchy sequence with theorems examining the relationships between $\mathscr{I}_{\theta_2}^*$-Cauchy sequence and strongly $\mathscr{I}_{\theta_2}^*$-Cauchy sequence, and also between strongly $\mathscr{I}_{\theta_2}^*$-Cauchy sequence and strongly $\mathscr{I}_{\theta_2}$-Cauchy sequence. In the future, these studies are also debatable in terms of regularly convergence for double sequences.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Not applicable.

References

[1] I.J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361–375.
[2] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244.
[3] P. Kostyrko, T. Šalát, W. Wilczyński, *$\mathscr{I}$-Convergence*, Real Anal. Exchange, **26**(2) (2000), 669–686.
[4] A. Nabiev, S. Pehlivan, M. Gürdal, *On $\mathscr{I}$-Cauchy sequence*, Taiwanese J. Math., **11**(2) (2007), 569–576.
[5] U. Yamancı, M. Gürdal, *On lacunary ideal convergence in random n-normed space*, J. Math., **2013** (2013), Article ID 868457, 8 pages.
[6] B. C. Tripathy, B. Hazarika, B. Choudhary, *Lacunary $\mathscr{I}$-convergent sequences*, Kyungpook Math. J., **52** (2012), 473–482.
[7] N. P. Akın, E. Dündar, Ş. Yalvaç, *Lacunary $\mathscr{I}^*$-convergence and lacunary $\mathscr{I}^*$-Cauchy sequence*, AKU J. Sci. Eng., (in press).
[8] N. P. Akın, E. Dündar, *Strongly lacunary $\mathscr{I}^*$-convergence and strongly lacunary $\mathscr{I}^*$-Cauchy sequence*, Math. Sci. Appl. E-Notes, (in press).
[9] P. Das, P. Kostyrko, W. Wilczyński, P. Malik,*$\mathscr{I}$ and $\mathscr{I}^*$-convergence of double sequences*, Math. Slovaca, **58**(5) (2008), 605-620.
[10] E. Dündar, B. Altay, *$\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy of double sequences*, Acta Math. Sci., **34**B(2) (2014), 343–353.
[11] E. Dündar, B. Altay, On some properties of $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy of double sequences, Gen. Math. Notes, **7**(1) (2011) 1–12.
[12] B. Hazarika, *Lacunary ideal convergence of multiple sequences*, J. Egyptian Math. Soc., **24** (2016), 54–59.
[13] E. Dündar, U. Ulusu, N. Pancaroğlu, *Strongly $\mathscr{I}_2$-lacunary convergence and $\mathscr{I}_2$-lacunary Cauchy double sequences of sets*, Aligarh Bull. Math., **35**(1-2) (2016), 1–15.
[14] N. P. Akın, E. Dündar, *On lacunary $\mathscr{I}_2^*$-convergence and lacunary $\mathscr{I}_2^*$-Cauchy sequence*, Commun. Adv. Math. Sci., **6**(4) (2023), 188–195.
[15] P. Das, E. Savaş, S. Kr. Ghosal, *On generalized of certain summability methods using ideals*, Appl. Math. Letter, **36** (2011), 1509–1514.
[16] P. Debnath, *Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces*, Comput. Math. Appl., **63** (2012), 708–715.
[17] E. Dündar, U. Ulusu, *On rough $\mathscr{I}$-convergence and $\mathscr{I}$-Cauchy sequence for functions defined on amenable semigroup*, Univer. J. Math. Appl., **6**(2) (2023), 86–90.
[18] A. R. Freedman, J. J. Sember, M. Raphael, *Some Cesàro type summability spaces*, Proc. Lond. Math. Soc., **37** (1978), 508–520.
[19] F. Nuray, E. Dündar, U. Ulusu, *Wijsman $\mathscr{I}_2$-convergence of double sequences of closed sets*, Pure Appl. Math. Lett., **2** (2014), 35–39.
[20] Y. Sever, U. Ulusu, E. Dündar, *On strongly $\mathscr{I}$ and $\mathscr{I}*$-lacunary convergence of sequences of sets*, AIP Conf. Proc., **1611** (2014), 357–362.
[21] U. Ulusu, F. Nuray, *On strongly lacunary summability of sequences of sets*, J. Appl. Math. Bioinform. **3**(3) (2013), 75–88.