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Some Properties of Hypergeometric Meixner–Pollaczek Polynomials

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Abstract. Orthogonal polynomials appear in many areas of mathematics and have been the subject of interest of many mathematicians. The present study deals with some new properties for the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$. The results obtained here include various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials. Relevant connections of some of these families of generating functions with various known results are also indicated.

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1. Introduction

This study is mainly concerned about the Meixner-Pollaczek polynomials. These are the polynomials first discovered by Meixner [11] and are known in the literature as the Meixner polynomials of the second kind (see Chihara [4]). These polynomials were later studied by Pollaczek Chihara [15]. The polynomials are denoted by $P_n^{(\lambda)}(x;\phi)$, and have a hypergeometric representation:

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!} e^{in\phi}\, {}_2F_1\left(-n, \lambda + ix; 2\lambda; 1 - e^{-2i\phi}\right), \quad (\lambda > 0 \ \ and \ \ 0 < \phi < \pi, \ \ n = 0, 1, 2, 3, ...) \tag{1.1}$$

where

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

and

$$(a)_k := a(a + 1) \cdots (a + k - 1).$$

In addition, we have the following relationship between the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ and the Laguerre polynomials $L_n^{(\alpha)}(x)$ (see, [7] , pp.216):

$$\lim_{\phi \to 0} P_n^{\left(\frac{\alpha+1}{2}\right)}\left(-\frac{x}{2\phi};\phi\right) = L_n^{(\alpha)}(x).$$

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The polynomials are completely described by the recureence formula:

$$\begin{aligned} P_0^{(\lambda)}(x;\phi) &= 1, \quad P_1^{(\lambda)}(x;\phi) = 2(\lambda\cos\phi + x\sin\phi), \\ P_2^{(\lambda)}(x;\phi) &= x^2 + \lambda^2 + \left(\lambda^2 + \lambda - x^2\right)\cos 2\phi + (1+2\lambda)\,x\sin 2\phi, \end{aligned}$$

$$(n+1)\,P_{n+1}^{(\lambda)}(x;\phi) - 2\left[x\sin\phi + (n+\lambda)\cos\phi\right]P_n^{(\lambda)}(x;\phi) + (n+2\lambda-1)\,P_{n-1}^{(\lambda)}(x;\phi) = 0$$

for $n \geq 1$ [8] and have a generating function [1]

$$\sum_{n=0}^{\infty} P_n^{(\lambda)}(x;\phi)\,t^n = \left(1 - te^{i\phi}\right)^{-\lambda+ix}\left(1 - te^{-i\phi}\right)^{-\lambda-ix}. \tag{1.2}$$

The Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ also have the generating relations (see, for example, [7], p. 213),

$$\sum_{n=0}^{\infty} \frac{P_n^{(\lambda)}(x;\phi)}{(2\lambda)_n\,e^{in\phi}}t^n = e^t\,{}_1F_1\left(\begin{array}{c}\lambda+ix\\ 2\lambda\end{array};\left(e^{-2i\phi}-1\right)t\right) \tag{1.3}$$

and

$$\sum_{n=0}^{\infty} \frac{(z)_n\,P_n^{(\lambda)}(x;\phi)}{(2\lambda)_n\,e^{in\phi}}t^n = (1-t)^{-z}\,{}_2F_1\left(\begin{array}{c}z,\ \lambda+ix\\ 2\lambda\end{array};\frac{\left(1-e^{-2i\phi}\right)t}{t-1}\right). \tag{1.4}$$

Replacing z by 2λ in (1.4), we may write that

$$\sum_{n=0}^{\infty} \frac{1}{e^{in\phi}}P_n^{(\lambda)}(x;\phi)\,t^n = (1-t)^{-\lambda+ix}\left(1 - te^{-2i\phi}\right)^{-\lambda-ix}. \tag{1.5}$$

Theorem 1.1. *The following addition formula holds for the Meixner-Pollaczek polynomials* $P_n^{(\lambda)}(x;\phi)$*:*

$$P_n^{(\lambda_1+\lambda_2)}(x_1+x_2;\phi) = \sum_{m=0}^{n} P_{n-m}^{(\lambda_1)}(x_1;\phi)\,P_m^{(\lambda_2)}(x_2;\phi). \tag{1.6}$$

Proof. Replacing λ by $\lambda_1+\lambda_2$ and x by x_1+x_2 in (1.5), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{e^{in\phi}}P_n^{(\lambda_1+\lambda_2)}(x_1+x_2;\phi)\,t^n &= (1-t)^{-\lambda_1-\lambda_2+ix_1+ix_2}\left(1-te^{-2i\phi}\right)^{-\lambda_1-\lambda_2-ix_1-ix_2} \\ &= \left(\sum_{n=0}^{\infty}\frac{1}{e^{in\phi}}P_n^{(\lambda_1)}(x_1;\phi)\,t^n\right)\left(\sum_{m=0}^{\infty}\frac{1}{e^{im\phi}}P_m^{(\lambda_2)}(x_2;\phi)\,t^m\right) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{e^{i(n+m)\phi}}P_n^{(\lambda_1)}(x_1;\phi)\,P_m^{(\lambda_2)}(x_2;\phi)\,t^{n+m} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{e^{in\phi}}P_{n-m}^{(\lambda_1)}(x_1;\phi)\,P_m^{(\lambda_2)}(x_2;\phi)\,t^n. \end{aligned}$$

From the coefficients of t^n on the both sides of the last equality, one can get the desired result. □

The main object of this paper to study different properties of the Meixner-Pollaczek polynomials. Various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials are given. Nowadays, there are a lot of works related to Meixner-Pollaczek polynomials theory and its applications (see, [1–3, 5, 9, 10, 16]).

2. Bilinear and Bilateral Generating Functions

In this section, we derive several families of bilinear and bilateral generating functions for the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ which are generated by (1.1) and given explicitly by (1.2) using the similar method considered in (seee, [6, 12–14]).

We begin by stating the following theorem.

Theorem 2.1. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_s)$ of s complex variables $y_1,...,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_{1,...,}y_s;\tau) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_{1,...,}y_s)\,\tau^k, \quad (a_k \neq 0,\ \mu,\psi\in\mathbb{C})$$

and

$$\Theta_{n,p}^{\mu,\psi}(x;\phi;y_{1,...,}y_s;\xi) := \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(\lambda)}(x;\phi)\,\Omega_{\mu+\psi k}(y_{1,...,}y_s)\,\xi^k.$$

Then, for $p\in\mathbb{N}$; we have

$$\sum_{n=0}^{\infty}\theta_{n,p}^{\mu,\psi}\left(x;\phi;y_{1,...,}y_s;\frac{\eta}{t^p}\right)t^n = \left(1-te^{i\phi}\right)^{-\lambda+ix}\left(1-te^{-i\phi}\right)^{-\lambda-ix}\ \Lambda_{\mu,\psi}(y_{1,...,}y_s;\eta) \tag{2.1}$$

provided that each member of (2.1) exists.

Proof. For convenience, let S denote the first member of the assertion (2.1) of Theorem 2.1. Then,

$$S = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(\lambda)}(x;\phi)\,\Omega_{\mu+\psi k}(y_{1,...,}y_s)\,\frac{\eta^k}{t^{pk}}t^n.$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k P_n^{(\lambda)}(x;\phi)\,\Omega_{\mu+\psi k}(y_{1,...,}y_s)\,\frac{\eta^k}{t^{pk}}t^{n+pk} \\ &= \left(\sum_{n=0}^{\infty}P_n^{(\lambda)}(x;\phi)\,t^n\right)\left(\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_{1,...,}y_s)\,\eta^k\right) \\ &= \left(1-te^{i\phi}\right)^{-\lambda+ix}\left(1-te^{-i\phi}\right)^{-\lambda-ix}\ \Lambda_{\mu,\psi}(y_{1,...,}y_s;\eta), \end{aligned}$$

which completes the proof. □

By using a similar idea, we also get the next result immediately.

Theorem 2.2. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_s)$ of r complex variables $y_1,...,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{n,p,\mu,\psi}^{\lambda_1,\lambda_2}(x_1+x_2;\phi;y_{1,...,}y_s;z) := \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(\lambda_1+\lambda_2)}(x_1+x_2;\phi)\,\Omega_{\mu+\psi k}(y_{1,...,}y_s)\,z^k, \quad (a_k\neq 0,\ \mu,\psi\in\mathbb{C}).$$

Then, for $p\in\mathbb{N}$; we have

$$\sum_{k=0}^{n}\sum_{\ell=0}^{\left[\frac{k}{p}\right]} a_\ell P_{n-k}^{(\lambda_1)}(x_1;\phi)\,P_{k-p\ell}^{(\lambda_2)}(x_2;\phi)\,\Omega_{\mu+\psi\ell}(y_{1,...,}y_s)\,z^\ell = \Lambda_{n,p,\mu,\psi}^{\lambda_1,\lambda_2}(x_1+x_2;\phi;y_{1,...,}y_s;z) \tag{2.2}$$

provided that each member of (2.2) exists.

Proof. For convenince, let H denote the first member of the assetion (2.2). Then, upon subsituting for the the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ from the (1.6) into the left-hand side of (2.2), we obtain

$$\begin{aligned} H &= \sum_{\ell=0}^{[n/p]}\sum_{k=0}^{n-p\ell} a_\ell P_{n-k-p\ell}^{(\lambda_1)}(x_1;\phi)\,P_k^{(\lambda_2)}(x_2;\phi)\,\Omega_{\mu+\psi\ell}(y_{1,\dots,}y_s)\,z^\ell \\ &= \sum_{\ell=0}^{[n/p]}\left(a_\ell\left(\sum_{k=0}^{n-p\ell} P_{n-k-p\ell}^{(\lambda_1)}(x_1;\phi)\,P_k^{(\lambda_2)}(x_2;\phi)\right)\Omega_{\mu+\psi\ell}(y_{1,\dots,}y_s)\,z^\ell\right) \\ &= \sum_{\ell=0}^{[n/p]} a_\ell P_{n-p\ell}^{(\lambda_1+\lambda_2)}(x_1+x_2;\phi)\,\Omega_{\mu+\psi\ell}(y_{1,\dots,}y_s)\,z^\ell \\ &= \Lambda_{n,p,\mu,\psi}^{\lambda_1,\lambda_2}(x_1+x_2;\phi;y_{1,\dots,}y_s;z). \end{aligned}$$

□

Theorem 2.3. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_s)$ of s complex variables $y_1,...,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_{1,\dots,}y_s;\tau) := \sum_{k=0}^{\infty} a_k\Omega_{\mu+\psi k}(y_{1,\dots,}y_s)\,\tau^k$$

where $a_k\neq 0$, $\mu,\psi\in\mathbb{C}$ and

$$\theta_{n,p}^{\mu,\psi}(x;\phi;y_{1,\dots,}y_s;\xi) := \sum_{k=0}^{[n/p]} a_k\frac{1}{(2\lambda)_{n-pk}}\frac{1}{e^{i(n-pk)\phi}}P_{n-pk}^{(\lambda)}(x;\phi)\,\Omega_{\mu+\psi k}(y_{1,\dots,}y_s)\,\xi^k.$$

Then, for $p\in\mathbb{N}$; we have

$$\sum_{n=0}^{\infty}\theta_{n,p}^{\mu,\psi}\left(x;\phi;y_{1,\dots,}y_s;\frac{\eta}{t^p}\right)t^n = e^t\,{}_1F_1\left(\begin{matrix}\lambda+ix\\2\lambda\end{matrix};\left(e^{-2i\phi}-1\right)t\right)\Lambda_{\mu,\psi}(y_{1,\dots,}y_s;\eta), \tag{2.3}$$

provided that each member of (2.3) exists.

Proof. For convenience, let T denote the first member of the assertion (2.3) of Theorem 2.3. Then,

$$T = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k\frac{1}{(2\lambda)_{n-pk}}\frac{1}{e^{i(n-pk)\phi}}P_{n-pk}^{(\lambda)}(x;\phi)\,\Omega_{\mu+\psi k}(y_{1,\dots,}y_s)\,\frac{\eta^k}{t^{pk}}t^n.$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned} T &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k\frac{1}{(2\lambda)_n}\frac{1}{e^{in\phi}}P_n^{(\lambda)}(x;\phi)\,\Omega_{\mu+\psi k}(y_{1,\dots,}y_s)\,\frac{\eta^k}{t^{pk}}t^n t^{pk} \\ &= \left(\sum_{n=0}^{\infty}\frac{1}{(2\lambda)_n}\frac{1}{e^{in\phi}}P_n^{(\lambda)}(x;\phi)\,t^n\right)\left(\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_{1,\dots,}y_s)\,\eta^k\right) \\ &= e^t\,{}_1F_1\left(\begin{matrix}\lambda+ix\\2\lambda\end{matrix};\left(e^{-2i\phi}-1\right)t\right)\Lambda_{\mu,\psi}(y_{1,\dots,}y_s;\eta), \end{aligned}$$

which completes the proof. □

By using a similar idea, we also get the next result immediately.

Theorem 2.4. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_s)$ of s complex variables $y_1,...,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_{1,\dots,}y_s;\tau) := \sum_{k=0}^{\infty} a_k\Omega_{\mu+\psi k}(y_{1,\dots,}y_s)\,\tau^k$$

where $a_k\neq 0,\mu,\psi\in\mathbb{C}$ and

$$\theta_{n,p}^{\mu,\psi}(x;\phi;y_1,...,y_s;\xi) := \sum_{k=0}^{[n/p]} a_k \frac{(z)_{n-pk}\, P_{n-pk}^{(\lambda)}(x;\phi)}{(2\lambda)_{n-pk}\, e^{i(n-pk)\phi}} \Omega_{\mu+\psi k}(y_1,...,y_s)\,\xi^k.$$

Then, for $p \in \mathbb{N}$; *we have*

$$\sum_{n=0}^{\infty} \theta_{n,p}^{\mu,\psi}\left(x;\phi;y_1,...,y_s;\frac{\eta}{t^p}\right)t^n = (1-t)^{-z}\ {}_2F_1\left(\begin{array}{c} z,\ \lambda+ix \\ 2\lambda \end{array};\frac{\left(1-e^{-2i\phi}\right)t}{t-1}\right)\Lambda_{\mu,\psi}(y_1,...,y_s;\eta), \tag{2.4}$$

provided that each member of (2.4) exists.

Proof. For convenience, let T denote the first member of the assertion (2.4) of Theorem 2.4. Then,

$$T = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k \frac{(z)_{n-pk}\, P_{n-pk}^{(\lambda)}(x;\phi)}{(2\lambda)_{n-pk}\, e^{i(n-pk)\phi}} \Omega_{\mu+\psi k}(y_1,...,y_s)\,\frac{\eta^k}{t^{pk}} t^n.$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned} T &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k \frac{(z)_n\, P_n^{(\lambda)}(x;\phi)}{(2\lambda)_n\, e^{in\phi}} \Omega_{\mu+\psi k}(y_1,...,y_s)\,\frac{\eta^k}{t^{pk}} t^n t^{pk} \\ &= \left(\sum_{n=0}^{\infty} \frac{(z)_n\, P_n^{(\lambda)}(x;\phi)}{(2\lambda)_n\, e^{in\phi}} t^n\right)\left(\sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1,...,y_s)\,\eta^k\right) \\ &= (1-t)^{-z}\ {}_2F_1\left(\begin{array}{c} z,\ \lambda+ix \\ 2\lambda \end{array};\frac{\left(1-e^{-2i\phi}\right)t}{t-1}\right)\Lambda_{\mu,\psi}(y_1,...,y_s;\eta), \end{aligned}$$

which completes the proof. □

3. Special Cases

As an application of the above theorems, when the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_s)$, $k \in \mathbb{N}_0, s \in \mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems.

We first set

$$\Omega_{\mu+\psi k}(y_1,...,y_s) = u_{\mu+\psi k}^{(\alpha_1,...,\alpha_s)}(y_1,...,y_s)$$

in Theorem 2.1, where the Erkus–Srivastava polynomials $u_{\mu+\psi k}^{(\alpha_1,...,\alpha_s)}(x_1,...,x_s)$ [6], generated by

$$\prod_{j=1}^{s}\left\{\left(1-x_j t^{m_j}\right)^{-\alpha_j}\right\} = \sum_{n=0}^{\infty} u_n^{(\alpha_1,...,\alpha_s)}(x_1,...,x_s)t^n,\ \ (\alpha_j \in \mathbb{C}\ \ (j=1,...,s);\ \ |t| < \min\left\{|x_1|^{-1/m_1},...,|x_s|^{-1/m_s}\right\}. \tag{3.1}$$

We are thus led to the following result which provides a class of bilateral generating functions for the Erkus–Srivastava polynomials and the family of multivariable polynomials given explicitly by (3.1).

Corollary 3.1. *If*

$$\Lambda_{\mu,\psi}(y_1,...,y_s;w) := \sum_{k=0}^{\infty} a_k u_{\mu+\psi k}^{(\alpha_1,...,\alpha_s)}(y_1,...,y_s)w^k,\ \ (a_k \neq 0,\ \mu,\psi \in \mathbb{C}),$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(\lambda)}(x;\phi)\, u_{\mu+\psi k}^{(\alpha_1,...,\alpha_s)}(y_1,...,y_s)w^k t^{n-pk} = \left(1-te^{i\phi}\right)^{-\lambda+ix}\left(1-te^{-i\phi}\right)^{-\lambda-ix}\Lambda_{\mu,\psi}(y_1,...,y_s;w) \tag{3.2}$$

provided that each member of (3.2) exists.

Remark 3.2. Using the generating relation (3.1) for the Erkuş-Srivastava polynomials and getting $a_k = 1$, $\mu = 0$, $\psi = 1$, we find that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} P_{n-pk}^{(\lambda)}(x;\phi)\, u_k^{(\alpha_1,\dots,\alpha_r)}(z_1,\dots,z_r) w^k t^{n-pk} = \left(1-te^{i\phi}\right)^{-\lambda+ix}\left(1-te^{-i\phi}\right)^{-\lambda-ix}\prod_{j=1}^{r}\left\{\left(1-z_j w^{m_j}\right)^{-\alpha_j}\right\}$$

where $|w| < \min\left\{|z_1|^{-1/m_1},\dots,|z_r|^{-1/m_r}\right\}$.

Now we set

$$s = 1 \text{ and } \Omega_{\mu+\psi k}(y_1) = P_{\mu+\psi k}^{(\lambda_1)}(x_1;\phi_1)$$

in Theorem 2.1, where the Meixner-Pollaczek polynomials $P_{n-pk}^{(\lambda)}(x;\phi)$ given explicitly by (1.2). Thus we give a class bilinear generating functions of Meixner-Pollaczek polynomials $P_k^{(\lambda)}(x;\phi)$.

Corollary 3.3. *If*

$$\Lambda_{\mu,p,q}(x_1;\phi_1;\tau) := \sum_{k=0}^{\infty} a_k P_{\mu+\psi k}^{(\lambda_1)}(x_1;\phi_1)\,\tau^k, \quad (a_k \neq 0,\ m\in\mathbb{N}_0, k\neq 0, \mu,\psi\in\mathbb{C})$$

and

$$\theta_{n,p}^{\mu,\psi}(x;\phi;x_1;\phi_1;\xi) = \sum_{k=0}^{\left[\frac{n}{p}\right]} a_k P_{n-pk}^{(\lambda)}(x;\phi)\, P_{\mu+\psi k}^{(\lambda_1)}(x_1;\phi_1)\,\xi^k$$

where $n, p \in \mathbb{N}$, then we have

$$\sum_{n=0}^{\infty}\theta_{n,p}^{\mu,\psi}\left(x;\phi;x_1;\phi_1;\frac{\eta}{t^p}\right)t^n = \left(1-te^{i\phi}\right)^{-\lambda+ix}\left(1-te^{-i\phi}\right)^{-\lambda-ix}\ \Lambda_{\mu,\psi}(x_1;\phi_1;\eta) \tag{3.3}$$

provided that each member of (3.3) exists.

Remark 3.4. Using the generating relation (1.2) for the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ and getting $a_k = 1$, $\mu = 0$, $\psi = 1$, we find that

$$\sum_{n=0}^{\infty}\theta_{n,p}^{0,1}\left(x;\phi;x_1;\phi_1;\frac{\eta}{t^p}\right)t^n = \left(1-te^{i\phi}\right)^{-\lambda+ix}\left(1-te^{-i\phi}\right)^{-\lambda-ix}\left(1-\eta e^{i\phi_1}\right)^{-\lambda_1+ix_1}\left(1-\mu e^{-i\phi_1}\right)^{-\lambda_1-ix_1}.$$

Remark 3.5. Using the generating relation (1.2) and (1.3) for the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ and getting $\mu = 0$, $\psi = 1$, $a_k = \frac{1}{(2\lambda_1)_k e^{ik\phi_1}}$, we find that

$$\sum_{n=0}^{\infty}\theta_{n,p}^{0,1}\left(x;\phi;x_1;\phi_1;\frac{\eta}{t^p}\right)t^n = \left(1-te^{i\phi}\right)^{-\lambda+ix}\left(1-te^{-i\phi}\right)^{-\lambda-ix} e^{\mu}\ {}_1F_1\left(\begin{array}{c}\lambda_1+ix_1\\ 2\lambda_1\end{array};\left(e^{-2i\phi_1}-1\right)\mu\right).$$

Now we set

$$s = 1 \text{ and } \Omega_{\mu+\psi k}(y_1) = P_{\mu+\psi k}^{(\lambda_1)}(x_1;\phi_1)$$

in Theorem 2.3, where the Meixner-Pollaczek polynomials $P_{n-pk}^{(\lambda)}(x;\phi)$ given explicitly by (1.5). Thus we give a class bilinear generating functions of Meixner-Pollaczek polynomials.

Corollary 3.6. *If*

$$\Lambda_{\mu,\psi}(x_1;\phi_1;\tau) = \sum_{k=0}^{\infty} a_k P_{\mu+\psi k}^{(\lambda_1)}(x_1;\phi_1)\,\tau^k, \quad (a_k \neq 0,\ m\in\mathbb{N}_0, k\neq 0, \mu,\psi\in\mathbb{C})$$

and

$$\theta_{n,p}^{\mu,\psi}(x;\phi;x_1;\phi_1;\xi) = \sum_{k=0}^{[n/p]} a_k \frac{1}{e^{i(n-pk)\phi}} P_{n-pk}^{(\lambda)}(x;\phi)\, P_{\mu+\psi k}^{(\lambda_1)}(x_1;\phi_1)\,\xi^k,$$

where $n, p \in \mathbb{N}$, then we have

$$\sum_{n=0}^{\infty} \theta_{n,p}^{\mu,\psi}\left(x; \phi; x_1; \phi_1; \frac{\eta}{t^p}\right) t^n = (1-t)^{-\lambda+ix}\left(1-te^{-2i\phi}\right)^{-\lambda-ix} \Lambda_{\mu,\psi}(x_1; \phi_1; \eta) \tag{3.4}$$

provided that each member of (3.4) exists.

Remark 3.7. Using the generating relation (1.5) for the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x; \phi)$ and getting $\mu = 0$, $\psi = 1$, $a_k = \frac{1}{e^{ik\phi_1}}$, we find that

$$\sum_{n=0}^{\infty} \theta_{n,p}^{0,1}\left(x; \phi; x_1; \phi_1; \frac{\eta}{t^p}\right) t^n = (1-t)^{-\lambda+ix}\left(1-te^{-2i\phi}\right)^{-\lambda-ix} (1-\eta)^{-\lambda_1+ix_1}\left(1-\mu e^{-2i\phi_1}\right)^{-\lambda_1-ix_1}.$$

Remark 3.8. Using the generating relation (1.4) for the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x; \phi)$ and getting $\mu = 0$, $\psi = 1$, $a_k = \frac{(z)_k}{(2\lambda)_k e^{ik\phi}}$, we find that

$$\sum_{n=0}^{\infty} \theta_{n,p}^{0,1}\left(x; \phi; x_1; \phi_1; \frac{\eta}{t^p}\right) t^n = (1-t)^{-\lambda+ix}\left(1-te^{-2i\phi}\right)^{-\lambda-ix} (1-\eta)^{-z} {}_2F_1\left(\begin{matrix} z, \lambda_1+ix_1 \\ 2\lambda_1 \end{matrix}; \frac{\left(1-e^{-2i\phi_1}\right)\eta}{\eta-1}\right).$$

Now we set

$$s = 1 \text{ and } \Omega_{\mu+\psi k}(y_1) = P_{\mu+\psi k}^{(\lambda_1)}(x_1; \phi_1)$$

in Theorem 2.3, where the Meixner-Pollaczek polynomials $P_{n-pk}^{(\lambda)}(x; \phi)$ given explicitly by (1.3). Thus we give a class bilinear generating functions of Meixner-Pollaczek polynomials.

Corollary 3.9. *If*

$$\Lambda_{\mu,\psi}(x_1; \phi_1; \tau) = \sum_{k=0}^{\infty} a_k P_{\mu+\psi k}^{(\lambda_1)}(x_1; \phi_1) \tau^k, \quad (a_k \neq 0,\ m \in \mathbb{N}_0, k \neq 0, \mu, \psi \in \mathbb{C})$$

and

$$\theta_{n,p}^{\mu,\psi}(x; \phi; x_1; \phi_1; \xi) = \sum_{k=0}^{\left[\frac{n}{p}\right]} a_k \frac{P_{n-pk}^{(\lambda)}(x; \phi)}{(2\lambda)_{n-pk}\ e^{i(n-pk)\phi}} P_{\mu+\psi k}^{(\lambda_1)}(x_1; \phi_1) \xi^k,$$

where $n, p \in \mathbb{N}$, then we have

$$\sum_{n=0}^{\infty} \theta_{n,p}^{\mu,\psi}\left(x; \phi; x_1; \phi_1; \frac{\eta}{t^p}\right) t^n = e^t\ {}_1F_1\left(\begin{matrix} \lambda+ix \\ 2\lambda \end{matrix}; \left(e^{-2i\phi}-1\right)t\right) \Lambda_{\mu,\psi}(x_1; \phi_1; \eta) \tag{3.5}$$

provided that each member of (3.5) exists.

Remark 3.10. Using the generating relation (1.3) for the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x; \phi)$ and getting $\mu = 0$, $\psi = 1$, $a_k = \frac{1}{(2\lambda_1)_k\ e^{ik\phi_1}}$, we find that

$$\sum_{n=0}^{\infty} \theta_{n,p}^{0,1}\left(x; \phi; x_1; \phi_1; \frac{\eta}{t^p}\right) t^n = e^t\ {}_1F_1\left(\begin{matrix} \lambda+ix \\ 2\lambda \end{matrix}; \left(e^{-2i\phi}-1\right)t\right) e^{\mu}\ {}_1F_1\left(\begin{matrix} \lambda_1+ix_1 \\ 2\lambda_1 \end{matrix}; \left(e^{-2i\phi_1}-1\right)\mu\right).$$

Now we set

$$s = 1 \text{ and } \Omega_{\mu+\psi k}(y_1) = P_{\mu+\psi k}^{(\lambda_3)}(x_3; \phi)$$

in Theorem 2.3, where the Meixner-Pollaczek polynomials $P_{n-pk}^{(\lambda)}(x; \phi)$ given explicitly by (1.1). Thus we give a class bilinear generating functions of Meixner-Pollaczek polynomials.

Corollary 3.11. *If*

$$\Lambda_{\mu,\psi,\lambda_1,\lambda_2}^{n,p}(x_1+x_2; \phi; x_3; \phi; z) = \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(\lambda_1+\lambda_2)}(x_1+x_2; \phi) P_{\mu+\psi k}^{(\lambda_3)}(x_3; \phi) z^k, \quad (a_k \neq 0,\ m \in \mathbb{N}_0,\ k \neq 0, \mu, \psi \in \mathbb{C})$$

and

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_\ell P_{n-k}^{(\lambda_1)}(x_1;\phi)\, P_{k-pl}^{(\lambda_2)}(x_2;\phi)\, P_{\mu+\psi l}^{(\lambda_3)}(x_3;\phi) z^l = \Lambda_{\mu,\psi,\lambda_1,\lambda_2}^{n,p}(x_1+x_{2;}\phi;x_3;\phi;z)\,.$$

Remark 3.12. Using (1.6) and for the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ and taking $p=1,\ a_l=1, z^l=1,\ \mu=0,\ \psi=1$, we have

$$\sum_{k=0}^{n}\sum_{l=0}^{k} P_{n-k}^{(\lambda_1)}(x_1;\phi)\, P_{k-l}^{(\lambda_2)}(x_2;\phi)\, P_{l}^{(\lambda_3)}(x_3;\phi) = P_n^{(\lambda_1+\lambda_2+\lambda_3)}(x_1+x_2+x_3;\phi)\,.$$

Now we set

$$s=1 \text{ and } \Omega_{\mu+\psi k}(y_1) = P_{\mu+\psi k}^{(\lambda_1)}(x_1;\phi_1)$$

in Theorem 2.4, where the Meixner-Pollaczek polynomials $P_{n-pk}^{(\lambda)}(x;\phi)$ given explicitly by (1.4). Thus we give a class bilinear generating functions of Meixner-Pollaczek polynomials.

Corollary 3.13. *If*

$$\Lambda_{\mu,\psi}(x_1;\phi_1;\tau) = \sum_{k=0}^{\infty} a_k P_{\mu+\psi k}^{(\lambda_1)}(x_1;\phi_1)\,\tau^k,\ \ (a_k\neq 0,\ m\in\mathbb{N}_0, k\neq 0,\mu,\psi\in\mathbb{C})$$

and

$$\theta_{n,p}^{\mu,\psi}(x;\phi;x_1;\phi_1;\xi) = \sum_{k=0}^{\left[\frac{n}{p}\right]} a_k \frac{(z)_{n-pk}\, P_{n-pk}^{(\lambda)}(x;\phi)}{(2\lambda)_{n-pk}\, e^{i(n-pk)\phi}} P_{\mu+\psi k}^{(\lambda_1)}(x_1;\phi_1)\,\xi^k,$$

where $n,p\in\mathbb{N}$, then we have

$$\sum_{n=0}^{\infty}\theta_{n,p}^{\mu,\psi}\left(x;\phi;x_1;\phi_1;\frac{\eta}{t^p}\right)t^n = (1-t)^{-z}\ {}_2F_1\left(\begin{array}{c} z,\ \lambda+ix \\ 2\lambda\end{array};\frac{\left(1-e^{-2i\phi}\right)t}{t-1}\right)\Lambda_{\mu,\psi}(x_1;\phi_1;\eta) \tag{3.6}$$

provided that each member of (3.6) exists.

Remark 3.14. Using the generating relation (1.4) for the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ and getting $\mu=0\ ,\ \ \psi=1\ ,\ a_k=\frac{(z)_k}{(2\lambda_1)_k e^{ik\phi_1}}$, we find that

$$\sum_{n=0}^{\infty}\theta_{n,p}^{0,1}\left(x;\phi;x_1;\phi_1;\frac{\eta}{t^p}\right)t^n = (1-t)^{-z}\ {}_2F_1\left(\begin{array}{c} z,\ \lambda+ix \\ 2\lambda\end{array};\frac{\left(1-e^{-2i\phi}\right)t}{t-1}\right)(1-\eta)^{-z}\ {}_2F_1\left(\begin{array}{c} z,\ \lambda_1+ix_1 \\ 2\lambda_1\end{array};\frac{\left(1-e^{-2i\phi_1}\right)\eta}{\eta-1}\right).$$

Remark 3.15. Using the generating relation (1.4) for the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ and getting $\mu=0\ ,\ \ \psi=1\ ,\ a_k=\frac{1}{(2\lambda_1)_k e^{ik\phi_1}}$, we find that

$$\sum_{n=0}^{\infty}\theta_{n,p}^{0,1}\left(x;\phi;x_1;\phi_1;\frac{\eta}{t^p}\right)t^n = (1-t)^{-z}\ {}_2F_1\left(\begin{array}{c} z,\ \lambda+ix \\ 2\lambda\end{array};\frac{\left(1-e^{-2i\phi}\right)t}{t-1}\right)e^{\eta}\ {}_1F_1\left(\begin{array}{c} \lambda_1+ix_1 \\ 2\lambda_1\end{array};\left(e^{-2i\phi_1}-1\right)\eta\right).$$

Furthermore, for every suitable choice of the coefficients a_k $(k\in\mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $(r\in\mathbb{N})$ and $P_{n-pk}(x,u)$ are expressed as an appropriate product of several simpler functions, the assertions of Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4, can be applied in order to derive various families of multilinear and multilateral generating functions for the family of Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ given explicitly by (1.1).

4. Miscellaneous Properties

In this section, we give some properties for the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ given by (1.1).

Theorem 4.1. *The Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ have the following integral representation:*

$$P_n^{(\lambda)}(x;\phi)=\frac{1}{\Gamma(\lambda-ix)\Gamma(\lambda+ix)}\int_0^\infty\int_0^\infty e^{-u_1-u_2}\frac{\left(u_1e^{i\phi}+u_2e^{-i\phi}\right)^n}{n!}u_1^{\lambda-ix-1}u_2^{\lambda+ix-1}du_1du_2,$$

where $Re(\lambda)>0$.

Proof. If we use the identity

$$a^{-\nu}=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-at}t^{\nu-1}dt,\qquad (Re(\nu)>0)$$

on the left-hand side of the generating function (1.2), we have

$$\begin{aligned}
\sum_{n=0}^\infty P_n^{(\lambda)}(x;\phi)t^n &= \left(1-te^{i\phi}\right)^{-\lambda+ix}\left(1-te^{-i\phi}\right)^{-\lambda-ix}\\
&= \left(\frac{1}{\Gamma(\lambda-ix)}\int_0^\infty e^{\left(te^{i\phi}-1\right)u_1}u_1^{\lambda-ix-1}du_1\right)\left(\frac{1}{\Gamma(\lambda+ix)}\int_0^\infty e^{\left(te^{-i\phi}-1\right)u_2}u_2^{\lambda+ix-1}du_2\right)\\
&= \frac{1}{\Gamma(\lambda-ix)}\frac{1}{\Gamma(\lambda+ix)}\int_0^\infty\int_0^\infty e^{u_1te^{i\phi}-u_1}u_1^{\lambda-ix-1}e^{u_2te^{-i\phi}-u_2}u_2^{\lambda+ix-1}du_1du_2\\
&= \frac{1}{\Gamma(\lambda-ix)}\frac{1}{\Gamma(\lambda+ix)}\int_0^\infty\int_0^\infty e^{-u_1-u_2}e^{t\left(u_1e^{i\phi}+u_2e^{-i\phi}\right)}u_1^{\lambda-ix-1}u_2^{\lambda+ix-1}du_1du_2\\
&= \frac{1}{\Gamma(\lambda-ix)}\frac{1}{\Gamma(\lambda+ix)}\int_0^\infty\int_0^\infty e^{-u_1-u_2}\sum_{n=0}^\infty t^n\frac{\left(u_1e^{i\phi}+u_2e^{-i\phi}\right)^n}{n!}u_1^{\lambda-ix-1}u_2^{\lambda+ix-1}du_1du_2.
\end{aligned}$$

From the coefficents of t^n on the both sides of the last equality, one can get the desired result. □

We now discuss some miscellaneous recurrence relations of the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ given by (1.1). By differentiating each member of the generating function relation (1.2) with respect to x and using

$$\sum_{n=0}^\infty\sum_{k=0}^\infty A(k,n)=\sum_{n=0}^\infty\sum_{k=0}^n A(k,n-k),$$

we arrive at the following (differential) recurrence relation for the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ given explicitly by (1.2):

$$\frac{\partial}{\partial x}P_n^{(\lambda)}(x;\phi)=i\left[\sum_{p=0}^{n-1}\frac{1}{p+1}\left(e^{-i\phi}\right)^{p+1}P_{n-p-1}^{(\lambda)}(x;\phi)-\sum_{m=0}^{n-1}\frac{1}{m+1}\left(e^{i\phi}\right)^{m+1}P_{n-m-1}^{(\lambda)}(x;\phi)\right].$$

Writing p instead of m, we may write that

$$\frac{\partial}{\partial x}\frac{1}{e^{in\phi}}P_n^{(\lambda)}(x;\phi)=i\sum_{p=0}^{n-1}\frac{1}{p+1}P_{n-p-1}^{(\lambda)}(x;\phi)\left[(e^{-i\phi})^{p+1}-(e^{i\phi})^{p+1}\right].$$

Besides, by differentiating each member of the generating function relation (1.2) with respect to t, we have

$$(n+1)P_{n+1}^{(\lambda)}(x;\phi)=(\lambda-ix)\sum_{m=0}^n P_{n-m}^{(\lambda)}(x;\phi)\left(e^{i\phi}\right)^{m+1}+(\lambda+ix)\sum_{p=0}^n P_{n-p}^{(\lambda)}(x;\phi)\left(e^{-i\phi}\right)^{p+1}.$$

Writing p instead of m, we may write that

$$(n+1)\,P_{n+1}^{(\lambda)}(x;\phi)=\sum_{p=0}^{n}P_{n-p}^{(\lambda)}(x;\phi)\left[(\lambda-ix)\left(e^{i\phi}\right)^{p+1}+(\lambda+ix)\left(e^{-i\phi}\right)^{p+1}\right].$$

Besides, by differentiating each member of the generating function relation (1.2) with respect to ϕ, we have

$$\frac{\partial}{\partial\phi}P_{n}^{(\lambda)}(x;\phi)=i\sum_{m=0}^{n-1}(\lambda-ix)\left(e^{i\phi}\right)^{m+1}P_{n-m-1}^{(\lambda)}(x;\phi)-i\sum_{p=0}^{n-1}(\lambda+ix)\left(e^{-i\phi}\right)^{p+1}P_{n-p-1}^{(\lambda)}(x;\phi).$$

Writing p instead of m, we may write that

$$\frac{\partial}{\partial\phi}P_{n}^{(\lambda)}(x;\phi)=i\sum_{p=0}^{n-1}P_{n-p-1}^{(\lambda)}(x;\phi)\left[(\lambda-ix)\left(e^{i\phi}\right)^{p+1}-(\lambda+ix)\left(e^{-i\phi}\right)^{p+1}\right].$$

By differentiating each member of the generating function relation (1.5) with respect to x and using

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k),$$

we arrive at the following (differential) recurrence relation for the Meixner-Pollaczek polynomials $P_{n}^{(\lambda)}(x;\phi)$ given explicitly by (1.5):

$$\frac{\partial}{\partial x}\frac{1}{e^{in\phi}}P_{n}^{(\lambda)}(x;\phi)=-i\sum_{m=0}^{n-1}\frac{1}{m+1}\frac{1}{e^{i(n-m-1)\phi}}P_{n-m-1}^{(\lambda)}(x;\phi)+i\sum_{p=0}^{n-1}\frac{1}{p+1}\left(e^{-2i\phi}\right)^{p+1}\frac{1}{e^{i(n-p-1)\phi}}P_{n-p-1}^{(\lambda)}(x;\phi).$$

Writing p instead of m, we may write that

$$\frac{\partial}{\partial x}\frac{1}{e^{in\phi}}P_{n}^{(\lambda)}(x;\phi)=i\sum_{p=0}^{n-1}\frac{1}{p+1}\frac{1}{e^{i(n-p-1)\phi}}P_{n-p-1}^{(\lambda)}(x;\phi)\left(\left(e^{-2i\phi}\right)^{p+1}-1\right).$$

Besides, by differentiating each member of the generating function relation (1.5) with respect to t, we have

$$(n+1)\,\frac{1}{e^{i(n+1)\phi}}P_{n+1}^{(\lambda)}(x;\phi)=(\lambda-ix)\sum_{m=0}^{n}\frac{1}{e^{i(n-m)\phi}}P_{n-m}^{(\lambda)}(x;\phi)+(\lambda+ix)\sum_{p=0}^{n}\left(e^{-2i\phi}\right)^{p+1}\frac{1}{e^{i(n-p)\phi}}P_{n-p}^{(\lambda)}(x;\phi).$$

Writing p instead of m, we may write that

$$(n+1)\,\frac{1}{e^{i(n+1)\phi}}P_{n+1}^{(\lambda)}(x;\phi)=\sum_{p=0}^{n}\frac{1}{e^{i(n-p)\phi}}P_{n-p}^{(\lambda)}(x;\phi)\left[(\lambda-ix)+(\lambda+ix)\left(e^{-2i\phi}\right)^{p+1}\right].$$

Besides, by differentiating each member of the generating function relation (1.5) with respect to ϕ, we have

$$\frac{\partial}{\partial\phi}P_{n}^{(\lambda)}(x;\phi)=i\left[nP_{n}^{(\lambda)}(x;\phi)-2(\lambda+ix)\sum_{m=0}^{n-1}\frac{1}{e^{i(m+1)\phi}}P_{n-m-1}^{(\lambda)}(x;\phi)\right].$$

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