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A study on the admissibility of fractional singular systems with variable and constant delays

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ABSTRACT

This paper deals with fractional singular systems with mixed delays and several admissibility criteria are obtained by using Lyapunov-Krasovskii functionals, model transformation, useful lemmas, zero equations and other well-known inequalities. Finally, some numerical examples are given with graps to verify and justify the admissibility of practical systems by using our proposed methods.

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1. Introduction

The concept of stability, widely used in applied sciences such as mathematics, physics, engineering, and medicine, was developed by Soviet mathematician A.M. Lyapunov in the early 1900s (see [13]). This field, which continues to expand and attract researchers, has become a broad application area for time-delayed and non-delayed systems (or equations). Numerous exciting studies have been conducted on this topic, which has been addressed by many researchers, and sufficient conditions for stability have been established. Alkhazzan et al., in article [1], discussed a new class of nonlinear fractional stochastic differential equations with fractional integrals and discussed existence, uniqueness, continuity of solutions and Ulam-Hyers stability with the help of Banach contraction theorem. They supported their work with an example. Singular and non-singular fractional systems have been studied for many types of stability, such as asymptotic stability (see [2,3,4,11,12,16,22,23,24]) and Razumikhin stability [8,17]. Using the Lyapunov-Krasovskii functional approach, the stability of fractional nonlinear Caputo-Volterra integral equations was studied in article [9]. In article [15], Tunç and Tunç investigated the qualitative behavior of the solutions of Caputo Proportional derivatives of delayed integro-differential equations. In article [19], by using Lyapunov-krasovskii functional, some sufficient conditions to guarantee robust stability and asymptotic stability for indeterminate fractional singular systems with neutral and time-varying delays in terms of the linear matrix inequality were proved. For the existence and uniqueness of solutions in singular systems, the system under consideration must have the properties of impulse-free and regular. For this reason, the authors in article [19] proved that the system they discussed in the first stage has these two properties and then has asymptotic stability and robust stability. By providing these three properties simultaneously, it was proved that the system discussed in article [19] has asymptotic admissibility and robust admissibility. Moreover, the Lyapunov stability of fractional neural networks with Riemann-Liouville delay was studied in [25]. Stability and admissibility in singular and non-singular systems with non-fractional delay have also been discussed (see [5,6,18]). In addition to these studies, there are many books that researchers can benefit from. Examples include (see [7,20,21]) for singular systems and (see [10,14]) for fractional differential equations.

This study, motivated by the above discussion and article [4] and references there in, is investigated the admissibility of a certain type of delay neutral fractional singular systems by the help of Lyapunov functionals, zero equations, model transformations and other some well-known inequalities.

Some numerical examples are presented to demonstrate the applicability of the proved results.

2. Preliminaries

In this paper motivated by the above discussion and article [4] and references there in, we define a fractional neutral singular system with constant and variable delays and nonlinear perturbation as:

$$ {}_{to}D_t^q[Sx(t)+Ax(t-\tau)] = -Bx(t)+Rx(t-\sigma)+E\int_{t-\sigma}^{t}x(s)ds $$

$$ +Tx(t-\sigma(t))+G(x(t-\sigma(t)), \qquad (1) $$

$$ x(t)=\vartheta(t), t\in[-\kappa,0], \kappa>0, \kappa\in\Re, $$

for $q\in(0,1]$, the system state $x(t)\in\Re^n$, $A,B,R,E,T\in\Re^{n\times n}$ are symmetric positive definite system matrices, the matrix $S\in\Re^{n\times n}$ is singular and satisfied $rankS=r\le n, n\ge 1$, with $\|A\|<1$, the time variable delay $\sigma(t)$ is assumed to satisfy $0\le\sigma(t)\le\sigma$ and $\dot{\sigma}(t)\le\mu$, the constant delays τ,σ are real positive numbers and $\vartheta\in C([-\kappa,0];\Re^n)$ with $\kappa=\max\{\tau,\sigma\}$. The nonlinear perturbation parameter $G(.)$ satisfying

$$ G^{\mathrm{T}}(x(t))G(x(t))\le a^2x^{\mathrm{T}}(t)x(t), \qquad (2) $$

$$ G^{\mathrm{T}}(x(t-\sigma(t)))G(x(t-\sigma(t)))\le b^2x^{\mathrm{T}}(t-\sigma(t))x(t-\sigma(t)), \qquad (3) $$

where a,b given any numbers.

Before moving on to the details of our study, we would like to remind of some useful definitions and lemmas that should be known.

Definition 2.1 ([4]). The Riemann-Liouville fractional integral and the derivation are defined as

$$ {}_{to}D_t^{-q}x(t)=\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}x(s)ds, \ (q>0), $$

$$ {}_{to}D_t^{q}x(t)=\frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_{t_0}^{t}\frac{x(s)}{(t-s)^{q+1-n}}ds, \ (n-1\le q<n). $$

Lemma 2.1 ([4]). For $x(t)\in\Re^n$ and $p>q>0$, then

$$ {}_{to}D_t^{q}({}_{to}D_t^{-p}x(t)) = {}_{to}D_t^{q-p}x(t). $$

Lemma 2.2 ([4]). Let $x(t)\in\Re^n$, be a vector of a differentiable function. For positive semi-definite matrix $\mathrm{N}\in\Re^{n\times n}$ and $\forall q\in(0,1)$, $\forall t\ge t_0$, then

$$ {}_{to}D_t^{q}(x^{\mathrm{T}}(t)\mathrm{N}x(t))\le 2x^{\mathrm{T}}(t)\mathrm{N}\,{}_{to}D_t^{q}x(t) $$

is satisfied.

3. Admissibility

We obtain sufficient conditions for the admissibility properties of the systems discussed in this section. We prove the regular, impulse-free, and stable states accepted as admissibility conditions in the first theorem. In the remaining four corollaries, we show that only the new systems defined by constructing new Lyapunov-Krasovskii functionals are stable, since regular and impulse-free states can be represented in a similar way. Therefore, we prove five sufficient conditions stating that these systems guarantee admissibility.

If we define a new operator like $\Omega(t)=Sx(t)+Ax(t-\tau)$, then the system (1) can be rewritten as in the form below:

$$ {}_{to}D_t^{q}\Omega(t)=-Bx(t)+Rx(t-\sigma)+E\int_{t-\sigma}^{t}x(s)ds $$

$$ +Tx(t-\sigma(t))+G(x(t-\sigma(t)). \qquad (4) $$

Theorem 3.1. We suppose that the following hypothesis is met:

(H1) Let a,b be any numbers and $\tau>0,\sigma>0$, if there are symmetric matrices $K_k=K_k^{\mathrm{T}}>0, (k=1,...,7)$, $W_i=\mathrm{W}_i^{\mathrm{T}}>0, (i=1,4,5)$, and any suitable dimensions matrices W_i $(i=2,3)$ such that the following relationship is satisfied:

$$\Theta=\begin{bmatrix}\Theta_{11} & \Theta_{12} & \Theta_{13} & K_1R & K_1T & K_1 & K_1E & 0\\ * & \Theta_{22} & \Theta_{23} & W_5R & W_5T & W_5 & W_5E & \Theta_{28}\\ * & * & \Theta_{33} & 0 & 0 & 0 & 0 & 0\\ * & * & * & -K_3 & 0 & 0 & 0 & RW_4^{\mathrm{T}}R\\ * & * & * & * & \Theta_{55} & 0 & 0 & TW_4^{\mathrm{T}}R\\ * & * & * & * & * & \Theta_{66} & 0 & W_4^{\mathrm{T}}R\\ * & * & * & * & * & * & -K_7 & E^{\mathrm{T}}W_4^{\mathrm{T}}R\\ * & * & * & * & * & * & * & \Theta_{88}\end{bmatrix}<0, \quad (5)$$

where

$$\Theta_{11}=-W_1-W_1^{\mathrm{T}},\ \Theta_{12}=-K_1B+W_1S-W_2^{\mathrm{T}},$$
$$\Theta_{13}=W_1A-W_3^{\mathrm{T}},$$
$$\Theta_{22}=W_2S+S^{\mathrm{T}}W_2^{\mathrm{T}}-W_5B-BW_5^{\mathrm{T}}+K_2+K_3+\tau K_4+K_5+a^2K_6+\sigma^2K_7,$$
$$\Theta_{23}=W_2A+S^{\mathrm{T}}W_3^{\mathrm{T}},\ \Theta_{28}=-BW_4^{\mathrm{T}}R-W_5,$$
$$\Theta_{33}=W_3A+A^{\mathrm{T}}W_3^{\mathrm{T}}-K_2,$$
$$\Theta_{55}=-(1-\mu)K_5+\varepsilon b^2\mathrm{I},$$
$$\Theta_{66}=-(1-\mu)K_6-\varepsilon\mathrm{I},\ \Theta_{88}=-RW_4-W_4^{\mathrm{T}}R.$$

where ε is a positive number and I is identity matrix with appropriate dimension.

Then the zero solution of system (1) is admissible.

Proof. The proof of Theorem 3.1 is divided into two steps. The first step deals with the properties impulse free and regularity. The second step is related to the stability property. Firstly, we can write from (5) that

$$\Theta_{22}=W_2S+S^{\mathrm{T}}W_2^{\mathrm{T}}-W_5B-BW_5^{\mathrm{T}}+K_2+K_3+\tau K_4+K_5+a^2K_6+\sigma^2K_7<0.$$

Since

$$W_2S+S^{\mathrm{T}}W_2^{\mathrm{T}}+K_2+K_3+\tau K_4+K_5+a^2K_6+\sigma^2K_7>0,$$

we can write

$$\nabla=-W_5B-BW_5<0. \quad (6)$$

Because of W_5, B positive definite symmetric matrices and $rankS=r\le n, n\ge 1,$ there exist two regular matrices U and L such that

$$\bar S=USL=\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix},\ \bar B=UBL=\begin{bmatrix}\bar B_1 & \bar B_2\\ \bar B_3 & \bar B_4\end{bmatrix},\ \bar W_5=U^{-T}W_5U^{-1}=\begin{bmatrix}\bar W_{51} & \bar W_{52}\\ \bar W_{53} & \bar W_{54}\end{bmatrix}.$$

Pre- and post- multiplying (6) by L^T and L respectively, then we obtain

$$\bar\nabla=L^T\nabla L=-\bar W_5\bar B-\bar B\bar W_5=\begin{bmatrix}\bar\nabla_1 & \bar\nabla_2\\ \bar\nabla_3 & \bar\nabla_4\end{bmatrix}<0,$$

here, $\bar\nabla_4=-\bar W_{54}\bar B_4-\bar B_4\bar W_{54}$. Because of $\bar\nabla_1,\bar\nabla_2$ and $\bar\nabla_3$ are unrelated to the discussion in proof. , their real expressions are omitted here. From here, we obtain $-\bar W_{54}\bar B_4-\bar B_4\bar W_{54}<0.$Therefore, $\bar B_4$ is regular. From this reason, the pair (S,B) is regular and impulse free (see [21]). According to Dai [7] and Q. Wu et al., [19] the system (1) is regular and impulse free.

Secondly, we prove that the system (1) is stable. For symmetric matrices $K_k=K_k^{\mathrm{T}}>0,(k=1,...,7),$ $W_i=\mathrm{W}_i^{\mathrm{T}}>0,(i=1,4,5)$ and any suitable dimensions matrices W_i $(i=2,3)$. Let us define a new positive definite functional as:

$$V(t)={}_{to}D_t^{q-1}(\Omega^{\mathrm{T}}(t)K_1\Omega(t))+\int_{t-\tau}^{t}x^{\mathrm{T}}(s)K_2x(s)ds+\int_{t-\sigma}^{t}x^{\mathrm{T}}(s)K_3x(s)ds$$
$$+\int_{t-\tau}^{t}(\tau-t+s)x^{\mathrm{T}}(s)K_4x(s)ds+\int_{t-\sigma(t)}^{t}x^{\mathrm{T}}(s)K_5x(s)ds$$
$$+\int_{t-\sigma(t)}^{t}G^{\mathrm{T}}(x(s))K_6G(x(s))ds+\sigma\int_{-\sigma}^{0}\int_{t+\beta}^{t}x^{\mathrm{T}}(\xi)K_7x(\xi)d\xi d\eta. \quad (7)$$

In view of the fact that Lemma 2.1, Lemma 2.2 and Jhensen inequality Lemma (see [18]), by the time-derivative of $V(t)$ on the solution of system (1), we can get the following inequality as:

$$\dot V(t,x)\le 2\Omega^{\mathrm{T}}(t)K_1({}_{to}D_t^q\Omega(t))+x^{\mathrm{T}}(t)K_2x(t)-x^{\mathrm{T}}(t-\tau)K_2x(t-\tau)$$

$$+x^{\mathrm{T}}(t)K_3x(t)-x^{\mathrm{T}}(t-\sigma)K_3x(t-\sigma)+\tau x^{\mathrm{T}}(t)K_4x(t)$$
$$+x^{\mathrm{T}}(t)K_5x(t)-(1-\dot{\sigma}(t))x^{\mathrm{T}}(t-\sigma(t))K_5x(t-\sigma(t))$$
$$+G^{\mathrm{T}}(x(t))K_6G(x(t))-(1-\dot{\sigma}(t))G^{\mathrm{T}}(x(t-\sigma(t)))K_6G(x(t-\sigma(t)))$$
$$+\sigma^2x(t)K_7x(t)-\sigma\int_{t-\sigma}^{t}x^{\mathrm{T}}(s)K_7x(s)ds$$
$$\le 2\Omega^{\mathrm{T}}(t)K_1[-Bx(t)+Rx(t-\sigma)+E\int_{t-\sigma}^{t}x(s)ds+Tx(t-\sigma(t))+G(x(t-\sigma(t)))]$$
$$+2\Omega^{\mathrm{T}}(t)W_1[-\Omega(t)+Sx(t)+Ax(t-\tau)]+x^{\mathrm{T}}(t)K_2x(t)$$
$$+2x^{\mathrm{T}}(t)W_2[-\Omega(t)+Sx(t)+Ax(t-\tau)]-x^{\mathrm{T}}(t-\tau)K_2x(t-\tau)$$
$$+2x^{\mathrm{T}}(t-\tau)W_3[-\Omega(t)+Sx(t)+Ax(t-\tau)]+x^{\mathrm{T}}(t)K_3x(t)$$
$$-x^{\mathrm{T}}(t-\sigma)K_3x(t-\sigma)+\tau x^{\mathrm{T}}(t)K_4x(t)+x^{\mathrm{T}}(t)K_5x(t)$$
$$-(1-\mu)x^{\mathrm{T}}(t-\sigma(t))K_5x(t-\sigma(t))+a^2x^{\mathrm{T}}(t)K_6x(t)$$
$$-(1-\mu)G^{\mathrm{T}}(x(t-\sigma(t)))K_6G(x(t-\sigma(t)))+\sigma^2x(t)K_7x(t)$$
$$-\left(\int_{t-\sigma}^{t}x(s)ds\right)^{\mathrm{T}}K_7\left(\int_{t-\sigma}^{t}x(s)ds\right). \quad (8)$$

We noting that

$$0=-{}_{t_0}D_t^q\Omega(t)-Bx(t)+Rx(t-\sigma)+E\int_{t-\sigma}x(s)ds+Tx(t-\sigma(t))+G(x(t-\sigma(t))).$$

From here, we can obtain

$$0=2\,{}_{t_0}D_t^q\Omega^{\mathrm{T}}(t)BW_4(-{}_{t_0}D_t^q\Omega(t)-Bx(t)+Rx(t-\sigma)+E\int_{t-\sigma}^{t}x(s)ds+Tx(t-\sigma(t))+G(x(t-\sigma(t))))$$
$$+2x^{\mathrm{T}}(t)W_5(-{}_{t_0}D_t^q\Omega(t)-Bx(t)+Rx(t-\sigma)+E\int_{t-\sigma}^{t}x(s)ds+Tx(t-\sigma(t))+G(x(t-\sigma(t)))). \quad (9)$$

Additionally, from the nonlinear parameter $G(.)$ condition given with (3), we get

$$0\le \varepsilon b^2x^{\mathrm{T}}(t-\sigma(t))x(t-\sigma(t))-\varepsilon G^{\mathrm{T}}(x(t-\sigma(t)))G(x(t-\sigma(t))). \quad (10)$$

where $\varepsilon>0$.

Combining (8)-(10), we can have the below inequality as:

$$\dot{V}(t,x)\le\chi^{\mathrm{T}}(t)\Theta\chi(t),$$

here the matrix Θ is defined with (5) and

$$\chi^{\mathrm{T}}(t)=[\Omega^{\mathrm{T}}(t)\ \ x^{\mathrm{T}}(t)\ \ x^{\mathrm{T}}(t-\tau)\ \ x^{\mathrm{T}}(t-\sigma)\ \ x^{\mathrm{T}}(t-\sigma(t))\ \ G^{\mathrm{T}}(x(t-\sigma(t)))\ \ \int_{t-\sigma}^{t}x(s)ds\ \ ({}_{t_0}D_t^q\Omega(t))^{\mathrm{T}}].$$

Because of matrix inequality (5) and impulse free and regularity criteria are satisfied and $\chi^{\mathrm{T}}(t)\ne 0$, then the neutral fractional singular system (1) is admissible. □

Further, we define the following fractional neutral singular system (1) with $E\int_{t-\sigma}^{t}x(s)ds=0$,

$${}_{t_0}D_t^q[Sx(t)+Ax(t-\tau)]=-Bx(t)+Rx(t-\sigma)+Tx(t-\sigma(t))+G(x(t-\sigma(t))),$$
$$x(t)=\vartheta(t),t\in[-\kappa,0],\kappa>0,\kappa\in\Re, \quad (11)$$

for $q\in(0,1]$, the system state $x(t)\in\Re^n$, $A,B,R,T\in\Re^{n\times n}$ are symmetric positive definite system matrices, the matrix $S\in\Re^{n\times n}$ is singular and satisfied $rankS=r\le n, n\ge 1$, with $\|A\|<1$, the time variable delay $\sigma(t)$ is assumed to satisfy $0\le\sigma(t)\le\sigma$ and $\dot{\sigma}(t)\le\mu$, the constant delays τ,σ are real positive numbers and $\vartheta\in C([-\kappa,0];\Re^n)$ with $\kappa=\max\{\tau,\sigma\}$ and the nonlinear perturbation parameter $G(.)$ satisfying

$$G^{\mathrm{T}}(x(t))G(x(t))\le a^2x^{\mathrm{T}}(t)x(t),$$

$$G^{\mathrm{T}}(x(t-\sigma(t)))G(x(t-\sigma(t)))\le b^2x^{\mathrm{T}}(t-\sigma(t))x(t-\sigma(t)),$$

where a,b are given any numbers.

Corollary 3.1. We suppose that the following hypothesis is met:

(H2) Let a,b be any numbers and $\tau>0,\sigma>0$, if there are symmetric matrices $K_k=K_k^{\mathrm{T}}>0, (k=1,...,6)$, $W_i=W_i^{\mathrm{T}}>0, (i=1,4,5)$ and any suitable dimensions matrices $W_i\ (i=2,3)$ such that the following relationship is satisfied:

$$\begin{bmatrix}\Lambda_{11} & \Lambda_{12} & \Lambda_{13} & K_1R & K_1T & K_1 & 0\\ * & \Lambda_{22} & \Lambda_{23} & W_5R & W_5T & W_5 & \Lambda_{27}\\ * & * & \Lambda_{33} & 0 & 0 & 0 & 0\\ * & * & * & -K_3 & 0 & 0 & RW_4^{\mathrm{T}}R\\ * & * & * & * & \Lambda_{55} & 0 & TW_4^{\mathrm{T}}R\\ * & * & * & * & * & \Lambda_{66} & W_4^{\mathrm{T}}R\\ * & * & * & * & * & * & \Lambda_{77}\end{bmatrix}<0, \quad (12)$$

where

$\Lambda_{11} = -W_1 - W_1^{\mathrm{T}}$, $\Lambda_{12} = -K_1B + W_1S - W_2^{\mathrm{T}}$, $\Lambda_{13} = W_1A - W_3^{\mathrm{T}}$,
$\Lambda_{22} = W_2S + S^{\mathrm{T}}W_2^{\mathrm{T}} - W_5B - BW_5^{\mathrm{T}} + K_2 + K_3 + \tau K_4 + K_5 + a^2K$
$\Lambda_{23} = W_2A + S^{\mathrm{T}}W_3^{\mathrm{T}}$, $\Lambda_{27} = -BW_4^{\mathrm{T}}R - W_5$,
$\Lambda_{33} = W_3A + A^{\mathrm{T}}W_3^{\mathrm{T}} - K_2$,
$\Lambda_{55} = -(1-\mu)K_5 + \varepsilon b^2\mathrm{I}$,
$\Lambda_{66} = -(1-\mu)K_6 - \varepsilon\mathrm{I}$,
$\Lambda_{77} = -RW_4 - W_4^{\mathrm{T}}R$.
where ε is a positive number and I is identity matrix with appropriate dimension.

Then the zero solution of system (11) is admissible.

Proof. For symmetric matrices $K_k = K_k^{\mathrm{T}} > 0, (k = 1,...,6)$, $W_i = \mathrm{W}_i^{\mathrm{T}} > 0, (i = 1,4,5)$ and any suitable dimensions matrices $W_i\ (i = 2,3)$. Let us define a new positive definite functional as:

$$V(t) = {}_{to}D_t^{q-1}(\Omega^{\mathrm{T}}(t)K_1\Omega(t)) + \int_{t-\tau}^{t} x^{\mathrm{T}}(s)K_2x(s)ds + \int_{t-\sigma}^{t} x^{\mathrm{T}}(s)K_3x(s)ds$$
$$+ \int_{t-\tau}^{t}(\tau - t + s)x^{\mathrm{T}}(s)K_4x(s)ds + \int_{t-\sigma(t)}^{t} x^{\mathrm{T}}(s)K_5x(s)ds$$
$$+ \int_{t-\sigma(t)}^{t} G^{\mathrm{T}}(x(s))K_6G(x(s))ds.$$

In view of Theorem 3.1, we show that the admissibility condition (12) of system given with (11). □

Additionally, we define the following fractional neutral singular system (1) with $E\int_{t-\sigma}^{t} x(s)ds = 0$ and $G(x(t-\sigma(t))) = 0$,

$$ {}_{to}D_t^q[Sx(t) + Ax(t-\tau)] = -Bx(t) + Rx(t-\sigma) + Tx(t-\sigma(t)), \quad (13)$$
$$x(t) = \vartheta(t), t \in [-\kappa, 0], \kappa > 0, \kappa \in \Re,$$

for $q \in (0,1]$, the system state $x(t) \in \Re^n$, $A, B, R, T \in \Re^{n\times n}$ are symmetric positive definite system matrices, the matrix $S \in \Re^{n\times n}$ is singular and satisfied $rankS = r \le n, n \ge 1$, with $\| A \| < 1$, the time variable delay $\sigma(t)$ is assumed to satisfy $0 \le \sigma(t) \le \sigma$ and $\dot{\sigma}(t) \le \mu$, the constant delays τ, σ are real positive numbers and $\vartheta \in C([-\kappa, 0]; \Re^n)$ with $\kappa = \max\{\tau, \sigma\}$.

Corollary 3.2. We suppose that the following hypothesis is met:

(H3) Let $\tau > 0, \sigma > 0$ be numbers, if there are symmetric matrices $K_k = K_k^{\mathrm{T}} > 0, (k = 1,...,5)$, $W_i = \mathrm{W}_i^{\mathrm{T}} > 0, (i = 1,4,5)$ and any suitable matrices $W_i\ (i = 2,3)$ such that the following relationship is satisfied:

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & K_1R & K_1T & 0 \\ * & \Psi_{22} & \Psi_{23} & W_5R & W_5T & \Psi_{26} \\ * & * & \Psi_{33} & 0 & 0 & 0 \\ * & * & * & -K_3 & 0 & RW_4R \\ * & * & * & * & \Psi_{55} & TW_4R \\ * & * & * & * & * & \Psi_{66} \end{bmatrix} < 0, \quad (14)$$

Where

$\Psi_{11} = -W_1 - W_1^{\mathrm{T}}$, $\Psi_{12} = -K_1B + W_1S - W_2^{\mathrm{T}}$, $\Psi_{13} = W_1A - W_3^{\mathrm{T}}$,
$\Psi_{22} = W_2S + S^{\mathrm{T}}W_2^{\mathrm{T}} - W_5B - BW_5^{\mathrm{T}} + K_2 + K_3 + \tau K_4 + K_5$,
$\Psi_{23} = W_2A + S^{\mathrm{T}}W_3^{\mathrm{T}}$, $\Psi_{26} = -BW_4^{\mathrm{T}}R - W_5$,
$\Psi_{33} = W_3A + A^{\mathrm{T}}W_3^{\mathrm{T}} - K_2$,
$\Psi_{55} = -(1-\mu)K_5$, $\Psi_{66} = -RW_4 - W_4^{\mathrm{T}}R$.

Then, the zero solution of system (13) is admissible.

Proof. For symmetric matrices $K_k = K_k^{\mathrm{T}} > 0, (k = 1,...,5)$, $W_i = \mathrm{W}_i^{\mathrm{T}} > 0, (i = 1,4,5)$, and any suitable dimensions matrices $W_i\ (i = 2,3)$. Let us define a new positive definite functional as:

$$V(t) = {}_{to}D_t^{q-1}(\Omega^{\mathrm{T}}(t)K_1\Omega(t)) + \int_{t-\tau}^{t} x^{\mathrm{T}}(s)K_2x(s)ds + \int_{t-\sigma}^{t} x^{\mathrm{T}}(s)K_3x(s)ds$$
$$+ \int_{t-\tau}^{t}(\tau - t + s)x^{\mathrm{T}}(s)K_4x(s)ds + \int_{t-\sigma(t)}^{t} x^{\mathrm{T}}(s)K_5x(s)ds.$$

In view of Theorem 3.1, we show that the admissibility condition (14) of system given with (13). □

Next, we define the following fractional neutral singular system (1) with $E\int_{t-\sigma}^{t} x(s)ds = 0,\ Tx(t-\sigma(t)) = 0$ and $G(x(t-\sigma(t))) = 0,$

$$ {}_{to}D_t^q[Sx(t) + Ax(t-\tau)] = -Bx(t) + Rx(t-\sigma), \qquad (15)$$
$$x(t) = \vartheta(t), t \in [-\kappa, 0], \kappa > 0, \kappa \in \Re,$$

for $q \in (0,1]$, the system state $x(t) \in \Re^n$, $A, B, R \in \Re^{n\times n}$ are symmetric positive definite system matrices, the matrix $S \in \Re^{n\times n}$ is singular and satisfied $rankS = r \le n, n \ge 1,$ with $\| A \| < 1,$ the constant delays τ, σ are real positive numbers and $\vartheta \in C([-\kappa, 0]; \Re^n)$ with $\kappa = \max\{\tau, \sigma\}$.

Corollary 3.3. We suppose that the following hypothesis is met:

(H4) Let $\tau > 0, \sigma > 0$ be numbers, if there are symmetric matrices $K_k = K_k^{\mathrm{T}} > 0, (k = 1,...,4),$ $W_i = \mathrm{W}_i^{\mathrm{T}} > 0, (i = 1,4,5),$ and any suitable dimensions matrices W_i $(i = 2,3)$ such that the following relationship is satisfied:

$$\begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & K_1R & 0 \\ * & \Delta_{22} & \Delta_{23} & W_5R & \Delta_{25} \\ * & * & \Delta_{33} & 0 & 0 \\ * & * & * & -K_3 & BW_4^{\mathrm{T}}R \\ * & * & * & * & \Delta_{55} \end{bmatrix} < 0, \qquad (16)$$

where
$\Delta_{11} = -W_1 - W_1^{\mathrm{T}},\ \Delta_{12} = -K_1B + W_1S - W_2^{\mathrm{T}},\ \Delta_{13} = W_1A - W_3^{\mathrm{T}},$
$\Delta_{22} = W_2S + S^{\mathrm{T}}W_2^{\mathrm{T}} - W_5B - BW_5^{\mathrm{T}} + K_2 + K_3 + \tau K_4,$
$\Delta_{23} = W_2A + S^{\mathrm{T}}W_3^{\mathrm{T}},\ \Delta_{25} = -BW_4^{\mathrm{T}}R - W_5,$
$\Delta_{33} = W_3A + A^{\mathrm{T}}W_3^{\mathrm{T}} - K_2,\ \Delta_{55} = -RW_4 - W_4^{\mathrm{T}}R.$

Then the system (15) is admissible.

Proof. For symmetric matrices $K_k = K_k^{\mathrm{T}} > 0, (k = 1,...,4),$ $W_i = \mathrm{W}_i^{\mathrm{T}} > 0, (i = 1,4,5),$ and any suitable dimensions matrices W_i $(i = 2,3)$. Let us define a new positive definite functional as:

$$V(t) = {}_{to}D_t^{q-1}(\Omega^{\mathrm{T}}(t)K_1\Omega(t)) + \int_{t-\tau}^{t} x^{\mathrm{T}}(s)K_2x(s)ds + \int_{t-\sigma}^{t} x^{\mathrm{T}}(s)K_3x(s)ds$$
$$+ \int_{t-\tau}^{t} (\tau - t + s)x^{\mathrm{T}}(s)K_4x(s)ds.$$

In view of Theorem 3.1, we show that the admissibility condition (16) of system given with (15). □

Next, we define the following fractional neutral singular system (1) with $E\int_{t-\sigma}^{t} x(s)ds = 0,$ $Tx(t-\sigma(t)) = 0,$ $G(x(t-\sigma(t))) = 0$ and $\sigma = \tau,$

$$ {}_{to}D_t^q[Sx(t) + Ax(t-\tau)] = -Bx(t) + Rx(t-\tau), \qquad (17)$$
$$x(t) = \vartheta(t), t \in [-\kappa, 0], \kappa > 0, \kappa \in \Re,$$

for $q \in (0,1]$, the system state $x(t) \in \Re^n$, $A, B, R \in \Re^{n\times n}$ are symmetric positive definite system matrices, the matrix $S \in \Re^{n\times n}$ is singular and satisfied $rankS = r \le n, n \ge 1,$ with $\| A \| < 1,$ the constant delay τ is real positive number and $\vartheta \in C([-\kappa, 0]; \Re^n)$.

Corollary 3.4. We suppose that the following hypothesis is met:
(H5) Let $\tau > 0$ be number, if there are symmetric matrices $K_k = K_k^{\mathrm{T}} > 0, (k = 1,2,3),$ $W_i = \mathrm{W}_i^{\mathrm{T}} > 0, (i = 1,4,5)$ and any suitable dimensions matrices W_i $(i = 2,3)$ such that the following relationship is satisfied:

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & 0 \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ * & * & \Pi_{33} & 0 \\ * & * & * & \Pi_{44} \end{bmatrix} < 0, \qquad (18)$$

where

$$\Pi_{11}=-W_1-W_1^{\mathrm{T}},\ \Pi_{12}=-K_1B+W_1S-W_2^{\mathrm{T}},\ \Pi_{13}=W_1A-W_3^{\mathrm{T}}+K_1R,$$
$$\Pi_{22}=W_2S+S^{\mathrm{T}}W_2^{\mathrm{T}}-W_5B-BW_5^{\mathrm{T}}+K_2+\tau K_3,$$
$$\Pi_{23}=W_2A+S^{\mathrm{T}}W_3^{\mathrm{T}},\ \Pi_{24}=-BW_4^{\mathrm{T}}R-W_5,$$
$$\Pi_{33}=W_3A+A^{\mathrm{T}}W_3^{\mathrm{T}}-K_2,\ \Pi_{44}=-RW_4-W_4^{\mathrm{T}}R.$$

Then the system (17) is admissible.

Proof. For symmetric matrices $K_k=K_k^{\mathrm{T}}>0,\ (k=1,2,3)$, $W_i=W_i^{\mathrm{T}}>0,\ (i=1,4,5)$ and any suitable dimensions matrices $W_i\ (i=2,3)$. Let us define a new positive definite functional as:

$$V(t)={}_{t0}D_t^{q-1}(\Omega^{\mathrm{T}}(t)K_1\Omega(t))+\int_{t-\tau}^{t}x^{\mathrm{T}}(s)K_2x(s)ds+\int_{t-\tau}^{t}(\tau-t+s)x^{\mathrm{T}}(s)K_3x(s)ds.$$

In view of Theorem 3.1, we show that the admissibility condition (18) of system given with (17). □

Remark 3.1. As mentioned in the first section, which is introduction; this study was mainly inspired by some studies cited references. However, since the systems discussed in this study are singular, it is a new study different from some of the studies cited in the references (see [4,11,18,22]) mentioned in the references, and it is clear that it is more difficult to achieve the goal. To overcome this difficulty, we benefited suitable Lyapunov-Krasovskii functionals and zero equations. Moreover, the Lyapunov-Krasovskii functionals used in the Study are new and obviously more comprehensive than their counterparts used in the literature (see [4,6,11,18,19,22]).

4. Numerical applications

In this section, we bring to the attention of the readers some examples, with their annotated solutions and graphics, showing that the theoretically sufficient conditions obtained in the previous section are applicable in practice. When the solutions of these examples are examined, it is clearly seen that the zero solutions of the systems in question are stable after a certain time interval. In addition, each example is support with the corresponding simulation result obtained with the help of MATLAB-Simulink.

Example 4.1. We define a fractional nonlinear neutral singular system as:

$$ {}_{t0}D_t^q[Sx(t)+Ax(t-\tau)]=-Bx(t)+Rx(t-\sigma)+E\int_{t-\sigma}^{t}x(s)ds + Tx(t-\sigma(t))+G(x(t-\sigma(t)), \quad (19)$$

for

$$q\in(0,1],\ x(t)=[x_1(t)\quad x_2(t)]^{\mathrm{T}},$$
$$G(x(t-\sigma(t)))=[\sin(x_1(t-\sigma(t))\quad 1-\cos(x_2(t-\sigma(t)))]^{\mathrm{T}}.$$

Solving the inequality (5), with the help of MATLAB software, when

$$S=\begin{bmatrix}1&0\\0&0\end{bmatrix},\ B=\begin{bmatrix}25&0\\0&26\end{bmatrix},\ R=\begin{bmatrix}0.43&0\\0&0.1225\end{bmatrix},$$
$$A=\begin{bmatrix}0.28&0\\0&0.1725\end{bmatrix},\ E=\begin{bmatrix}0.015&0\\0&0.02\end{bmatrix},\ T=\begin{bmatrix}0.02&0\\0&0.01\end{bmatrix},$$

we have a set of parameters that provides admissibility of the system (19) that

$$a=0.002,$$
$$b=0.005,$$
$$\tau=0.2,$$
$$0\le\sigma(t)=0.25+0.25\sin t\le 0.5=\sigma,$$
$$\dot\sigma(t)=0.25\cos t\le 0.25=\mu<1,$$
$$\varepsilon=1$$

as follows:

$$W_1=\begin{bmatrix}6{,}95&0\\0&17.5\end{bmatrix},\ W_2=\begin{bmatrix}-7.98&0\\0&-0.02\end{bmatrix},\ W_3=\begin{bmatrix}-2&0\\0&-3\end{bmatrix},$$
$$W_4=\begin{bmatrix}0.02&0\\0&0.2802\end{bmatrix},\ W_5=\begin{bmatrix}0.1028&0\\0&0.878\end{bmatrix},\ K_1=\begin{bmatrix}0.1&0\\0&0.00035\end{bmatrix},$$
$$K_2=\begin{bmatrix}0.15&0\\0&0.00012\end{bmatrix},\ K_3=\begin{bmatrix}0.0065&0\\0&0.0008\end{bmatrix},\ K_4=\begin{bmatrix}0.0005&0\\0&0.0001\end{bmatrix},$$
$$K_5=\begin{bmatrix}0.01&0\\0&0.00012\end{bmatrix},\ K_6=\begin{bmatrix}0.03&0\\0&8\end{bmatrix},\ K_7=\begin{bmatrix}0.01&0\\0&0.001\end{bmatrix}.$$

It is clear that all conditions of admissibility criteria for system (19) are satisfied. Therefore, the system (19) is admissible. Also, the graph showing the orbital behavior of the system (19) is as follows.

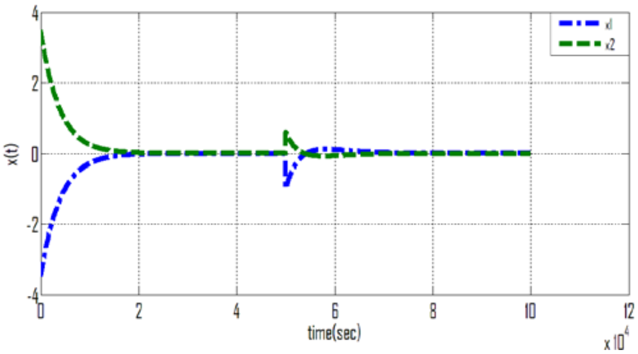

Figure 1. *Orbital behavior of the system (19).*

Example 4.2. We define a fractional nonlinear neutral singular system as:

$$ {}_{t_0}D_t^q[Sx(t)+Ax(t-\tau)] = -Bx(t)+Rx(t-\sigma)+Tx(t-\sigma(t))+G(x(t-\sigma(t)), \quad (20)$$

for $q\in(0,1],\ x(t)=[x_1(t)\quad x_2(t)\quad x_3(t)]^{\mathrm{T}}$,

$G(x(t-\sigma(t)))=[\sin(x_1(t-\sigma(t)))\quad \sin(x_1(t-\sigma(t)))+x_2(t-\sigma(t)))\quad \sin(x_2(t-\sigma(t)))]^{\mathrm{T}}$.

Solving the inequality (12), with the help of MATLAB software, when

$$S=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}, B=\begin{bmatrix}25&0&0\\0&15&0\\0&0&26\end{bmatrix}, R=\begin{bmatrix}0.43&0&0\\0&0.44&0\\0&0&0.1225\end{bmatrix},$$

$$A=\begin{bmatrix}0.28&0&0\\0&0.25&0\\0&0&0.1725\end{bmatrix}, T=\begin{bmatrix}0.02&0&0\\0&0.03&0\\0&0&0.01\end{bmatrix},$$

we have a set of parameters that provides admissibility of the system (20) that $a=0.002$, $b=0.005,\ \tau=0.2,\ 0\le\sigma(t)=0.25+0.25\sin t\le 0.5=\sigma,\ \dot{\sigma}(t)=0.25\cos t\le 0.25=\mu<1,\ \varepsilon=1$

as follows:

$$W_1=\begin{bmatrix}6.95&0&0\\0&6.98&0\\0&0&17.5\end{bmatrix}, W_2=\begin{bmatrix}-7.98&0&0\\0&-8&0\\0&0&-0.02\end{bmatrix}, W_3=\begin{bmatrix}-2&0&0\\0&-0.095&0\\0&0&-3\end{bmatrix},$$

$$W_4=\begin{bmatrix}0.02&0&0\\0&0.03&0\\0&0&0.2802\end{bmatrix}, W_5=\begin{bmatrix}0.1028&0&0\\0&0.15&0\\0&0&0.878\end{bmatrix}, K_1=\begin{bmatrix}0.1&0&0\\0&0.14&0\\0&0&0.00035\end{bmatrix},$$

$$K_2=\begin{bmatrix}0.15&0&0\\0&0.36&0\\0&0&0.00012\end{bmatrix}, K_3=\begin{bmatrix}0.0065&0&0\\0&0.0063&0\\0&0&0.0008\end{bmatrix}, K_4=\begin{bmatrix}0.0005&0&0\\0&0.0008&0\\0&0&0.0001\end{bmatrix},$$

$$K_5=\begin{bmatrix}0.01&0&0\\0&0.015&0\\0&0&0.00012\end{bmatrix}, K_6=\begin{bmatrix}0.03&0&0\\0&0.05&0\\0&0&8\end{bmatrix}.$$

It is clear that all conditions of admissibility criteria for system (20) are satisfied. Therefore, the system (20) is admissible. Also, the graph showing the orbital behavior of the system (20) is as follows.

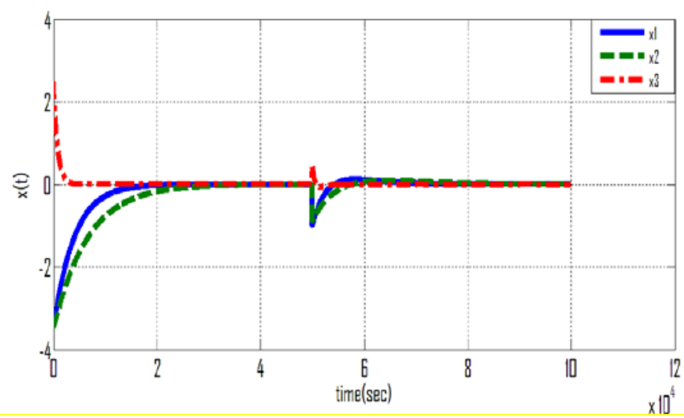

Figure 2. *Orbital behavior of the system (20).*

Example 4.3. We define a fractional neutral singular system as:

$$ {}_{t_0}D_t^q[Sx(t)+Ax(t-\tau)] = -Bx(t)+Rx(t-\sigma)+Tx(t-\sigma(t)), \quad (21)$$

for $q\in(0,1],\ x(t)=[x_1(t)\quad x_2(t)\quad x_3(t)\quad x_4(t)]^{\mathrm{T}}$,

Solving the inequality (14), with the help of MATLAB software, when

$$S=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}, B=\begin{bmatrix}25&0&0&0\\0&15&0&0\\0&0&8&0\\0&0&0&26\end{bmatrix}, R=\begin{bmatrix}0.43&0&0&0\\0&0.44&0&0\\0&0&0.45&0\\0&0&0&0.1225\end{bmatrix},$$

$$A=\begin{bmatrix}0.28&0&0&0\\0&0.25&0&0\\0&0&0.02&0\\0&0&0&0.1725\end{bmatrix}, T=\begin{bmatrix}0.02&0&0&0\\0&0.03&0&0\\0&0&0.025&0\\0&0&0&0.01\end{bmatrix},$$

we have a set of parameters that provides admissibility of the system (21) that

$\tau = 0.2,\ 0 \le \sigma(t) = 0.25 + 0.25\sin t \le 0.5 = \sigma,\ \dot{\sigma}(t) = 0.25\cos t \le 0.25 = \mu < 1$

as follows:

$$W_1 = \begin{bmatrix} 6.95 & 0 & 0 & 0 \\ 0 & 6.98 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 17.5 \end{bmatrix}, W_2 = \begin{bmatrix} -7.98 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & -7.8 & 0 \\ 0 & 0 & 0 & -0.02 \end{bmatrix}, W_3 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -0.095 & 0 & 0 \\ 0 & 0 & 0.85 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 0.02 & 0 & 0 & 0 \\ 0 & 0.03 & 0 & 0 \\ 0 & 0 & 0.025 & 0 \\ 0 & 0 & 0 & 0.2802 \end{bmatrix}, W_5 = \begin{bmatrix} 0.1028 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0 \\ 0 & 0 & 0.14 & 0 \\ 0 & 0 & 0 & 0.878 \end{bmatrix}, K_1 = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.14 & 0 & 0 \\ 0 & 0 & 0.18 & 0 \\ 0 & 0 & 0 & 0.00035 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.15 & 0 & 0 & 0 \\ 0 & 0.36 & 0 & 0 \\ 0 & 0 & 0.28 & 0 \\ 0 & 0 & 0 & 0.00012 \end{bmatrix}, K_3 = \begin{bmatrix} 0.0065 & 0 & 0 & 0 \\ 0 & 0.0063 & 0 & 0 \\ 0 & 0 & 0.006 & 0 \\ 0 & 0 & 0 & 0.0008 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} 0.0005 & 0 & 0 & 0 \\ 0 & 0.0008 & 0 & 0 \\ 0 & 0 & 0.0006 & 0 \\ 0 & 0 & 0 & 0.0001 \end{bmatrix}, K_5 = \begin{bmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.015 & 0 & 0 \\ 0 & 0 & 0.025 & 0 \\ 0 & 0 & 0 & 0.00012 \end{bmatrix}.$$

It is clear that all conditions of admissibility criteria for system (21) are satisfied. Therefore, the system (21) is admissible. Also, the graph showing the orbital behavior of the system (21) is as follows.

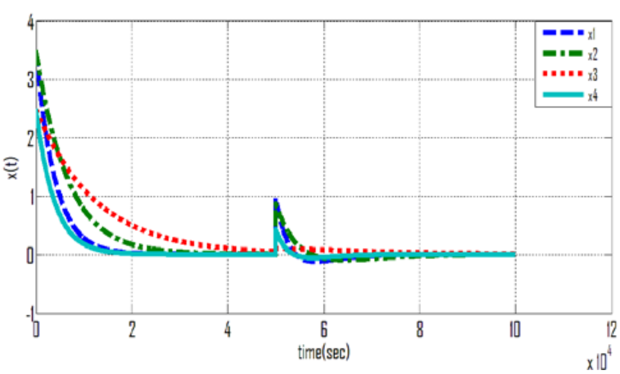

Figure 3. *Orbital behavior of the system (21).*

Example 4.4. We define a fractional neutral singular system as:

$$_{t_0}D_t^q[Sx(t) + Ax(t-\tau)] = -Bx(t) + Rx(t-\sigma), \qquad (22)$$

for $q \in (0,1],\ x(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t) \quad x_5(t)]^T,$

Solving the inequality (16), with the help of MATLAB software, when $S = diag(1,1,1,1,0)$,

$B = diag(25,15,8,10,26),\ R = diag(0.43,0.44,0.45,,0.48,0.1225),$
$A = diag(0.28,0.25,0.02,0.03,0.1725),$

we have a set of parameters that provides admissibility of the system (22) that $\tau = 0.2,\ \sigma = 0.5$ as follows:

$$W_1 = diag(6.95,6.98,7,6.5,17.5),$$
$$W_2 = diag(-7.98,-8,-7.8,-8.02,-0.02),$$
$$W_3 = diag(-2,-0.095,0.85,0.5,-3),$$
$$W_4 = diag(0.02,0.03,0.025,0.028,0.2802),$$
$$W_5 = diag(0.1028,0.15,0.14,0.2,0.878),$$
$$K_1 = diag(0.1,0.14,0.18,0.23,0.00035),$$
$$K_2 = diag(0.15,0.36,0.28,0.18,0.00012),$$
$$K_3 = diag(0.0065,0.0063,0.006,0.008,0.0008),$$
$$K_4 = diag(0.0005,0.0008,0.0006,0.0012,0.0001).$$

It is clear that all conditions of admissibility criteria for system (22) are satisfied. Therefore, the system (22) is admissible. Also, the graph showing the orbital behavior of the system (22) is as follows.

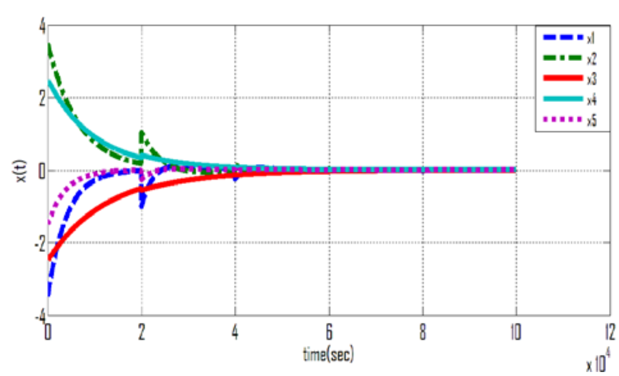

Figure 4. *Orbital behavior of the system (22).*

Example 4.5. We define a fractional neutral singular system as:

$$ {}_{t_0}D_t^q[Sx(t)+Ax(t-\tau)] = -Bx(t)+Rx(t-\tau), \quad (23)$$

for

$q \in (0,1]$, $x(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t) \quad x_5(t) \quad x_6(t)]^T$,

Solving the inequality (18), with the help of MATLAB software, when $S = diag(1,1,1,1,1,0)$,

$$B = diag(25,15,8,10,12,26),$$
$$R = diag(0.43,0.44,0.45,,0.48,0.5,0.1225),$$
$$A = diag(0.28,0.25,0.02,0.03,0.04,0.1725),$$

we have a set of parameters that provides admissibility of the system (23) that $\tau = 0.2$ as follows:

$$W_1 = diag(6.95,6.98,7,6.5,6.8,17.5),$$
$$W_2 = diag(-7.98,-8,-7.8,-8.02,-8.05,-0.02),$$
$$W_3 = diag(-2,-0.095,0.85,0.5,0.75,-3),$$
$$W_4 = diag(0.02,0.03,0.025,0.028,0.035,0.2802),$$
$$W_5 = diag(0.1028,0.15,0.14,0.2,0.18,0.878),$$
$$K_1 = diag(0.1,0.14,0.18,0.23,0.25,0.00035),$$
$$K_2 = diag(0.15,0.36,0.28,0.18,0.23,0.00012),$$
$$K_3 = diag(0.0065,0.0008,0.006,0.0012,,00009,0.0001).$$

It is clear that all conditions of admissibility criteria for system (23) are satisfied. Therefore, the system (23) is admissible. Also, the graph showing the orbital behavior of the system (23) is as follows.

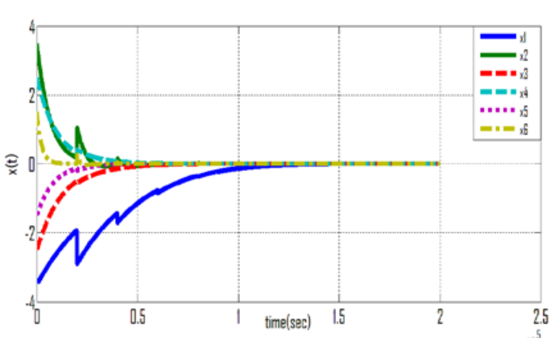

Figure 5. *Orbital behavior of the system (23).*

5. Conclusions

In this paper, we proposed new delay-dependent criteria for the admissibility of linear and nonlinear fractional singular systems with variable and constant delays. We used some useful lemmas and Lyapunov-Krasovskii functionals to obtain these proposed criteria. The numerical examples we present with graps at the end of the paper reveal the advantages and applicability of our results.

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