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Banach Uzaylarında Zenginleştirilmiş Daralmalarla İlişkilendirilen Bir İteratif Algoritma Üzerine Bazı Sonuçlar

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ÖZ

Bu çalışmada, Banach uzaylarında zenginleştirilmiş daralmalar vasıtasıyla tanımlanan, ortalama dönüşüm sınıfları ile üretilen Picard-S algoritması ele alınmıştır. Bu dönüşüm sınıfları kullanılarak Picard-S algoritmasından elde edilen iteratif dizinin, zenginleştirilmiş daralmanın sabit noktasına yakınsaklığı kontrol dizileri üzerinde herhangi ek şartlar olmaksızın elde edilmiştir. Bu dönüşüm sınıfları ile Picard-S ve CR algoritmalarından elde edilen iteratif dizilerin sabit noktaya yakınsaklıklarının denk olduğu gösterilmiş ve aynı dönüşüm sınıfları için Picard-S algoritmasının veri bağlılığı üzerine bir sonuç elde edilmiştir. Elde edilen tüm sonuçlar sonsuz boyutlu Banach uzaylarında sayısal örneklerle desteklenmiştir.

Some Results on an Iterative Algorithm Associated with Enriched Contractions in Banach Spaces

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ABSTRACT

In this study, the Picard-S algorithm, which is generated by the average mapping classes defined by the enriched contractions in Banach spaces, is considered. Using these mapping classes, the convergence of the iterative sequence obtained from the Picard-S algorithm to the fixed point of the enriched contraction has been obtained without any additional conditions on the control sequences. It has been shown that the convergence of the iterative sequences generated by Picard-S and the CR algorithms with these mapping classes to the fixed point is equivalent, and a result regarding the data dependency of the Picard-S algorithm for the same mapping classes has been obtained. All results obtained are supported by numerical examples in infinite-dimensional Banach spaces.

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INTRODUCTION AND PRELIMINARIES

Let D be a non-empty set and R be a mapping from D to D. The problem of finding of the points d that satisfy the condition $Rd = d$ is well known in the literature as the fixed point problem. Many problems, such as the problem of finding solutions of an integral or differential equation whose solutions cannot be found by applying existing analytical methods, the image restoration problems, convex optimization problems, and so on, can be considered as fixed point problems through some mappings. Therefore, finding their solutions is often the problem of finding the fixed points of some mappings. It is possible to find many iterative algorithms that converge to the fixed point of mappings under appropriate conditions and defined by various mappings in the literature. We can mentioned [1-11]. The convergence speed, convergence equivalence, stability and data dependency of these algorithms are important for the iterative algorithms to be effective and useful from each other.

Let us recall basic concepts, the terminology and notations used throughout the study. We will denote the set of non-negative integers by $\mathbb{N}$. Let E be a Banach space, D a non-empty convex subset of E and $R: D \to D$ a mapping. The set of fixed points of R will be denoted by the notation $F(R)$.

Let us recall some important iterative algorithms existing in the literature associated with the mapping R. Throughout the study, unless otherwise stated there will be control sequences $\{\alpha_n^i\}_{n=0}^{\infty}$ in $[0,1]$ for $i = 0,1,2$.

The iterative sequence $\{p_n\}_{n=1}^{\infty}$ generated by Picard iterative algorithm associated with the mapping R is defined as follows in [1]

$$p_0 \in D,$$
$$p_{n+1} = Rp_n, n \in \mathbb{N}. \tag{1.1}$$

It is well known that the sequence $\{p_n\}_{n=1}^{\infty}$ converges to the unique fixed point of R if D is a closed subset of a complete metric space and $R: D \to D$ is a mapping satisfying the contraction condition

$$d(Rx, Ry) \leq \delta d(x, y), \text{ for any } x, y \in D \tag{1.2}$$

where $\delta \in [0,1)$ (see, [12]).

The iterative sequence $\{m_n\}_{n=1}^{\infty}$ generated by Mann iterative algorithm associated with the mapping R is defined as follows in [2]

$$m_0 \in D,$$
$$m_{n+1} = (1 - \alpha_n^0)m_n + \alpha_n^0 Rm_n, n \in \mathbb{N}. \tag{1.3}$$

The iterative sequence $\{s_n\}_{n=1}^{\infty}$ generated by Ishikawa iterative algorithm associated with the mapping R is defined as follows in [3]

$$s_0 \in D,$$
$$s_{n+1} = (1 - \alpha_n^0)s_n + \alpha_n^0 Rr_n,$$
$$r_n = (1 - \alpha_n^1)s_n + \alpha_n^1 Rs_n, \ n \in \mathbb{N}. \tag{1.4}$$

The iterative sequence $\{c_n\}_{n=1}^{\infty}$ generated by CR iterative algorithm associated with the mapping R is defined as follows in [4]

$$c_0 \in D,$$
$$c_{n+1} = (1 - \alpha_n^0)q_n + \alpha_n^0 Rq_n,$$
$$q_n = (1 - \alpha_n^1)Rc_n + \alpha_n^1 Rw_n,$$
$$w_n = (1 - \alpha_n^2)c_n + \alpha_n^2 Rc_n, n \in \mathbb{N}. \tag{1.5}$$

The iterative sequence $\{x_n\}_{n=1}^{\infty}$ generated by Picard-S iterative algorithm associated with the mapping R is defined by Gürsoy and Karakaya in [5] (see also, [6]) as follows

$$x_0 \in D,$$
$$x_{n+1} = Ry_n,$$
$$y_n = (1-\alpha_n^1)Rx_n + \alpha_n^1 Rz_n,$$
$$z_n = (1-\alpha_n^2)x_n + \alpha_n^2 Rx_n, n \in \mathbb{N}. \quad (1.6)$$

The iterative algorithms mentioned above have been studied for various mapping classes, and various results such as convergence, convergence speed, convergence equivalence, stability and data dependency of these algorithms have been obtained in the literature (for example, [1-11]).

Gürsoy and Karakaya [5] studied with algorithm (1.6) associated with the mapping R satisfying the contraction condition (1.2). Under extra conditions on the control sequences of this algorithm, they obtained some results dealing with convergence, convergence equivalence, convergence speed and data dependenceny of the algorithm. They also showed with an example that algorithm (1.6) converges faster than the other algorithms mentioned above. Later, Ertürk and Gürsoy [7] obtained that the convergence of algorithm (1.6) for a more general class (quasi strictly contractive mappings) than the class of contraction mappings without adding any conditions on the control sequences. Also, we denote that the convergence of algorithm (1.6) for contraction mappings given in [5, Theorem 1] can be obtained without any extra conditions on the control sequences, as we can observe from the proof of Theorem 2.1 in our study and the proof of [7, Theorem 2.1].

Berinde and Păcurar [8] introduced a class of mappings, called enriched contractions. This class is significant since it is a large class of contraction mappings. An enriched contraction mapping is defined in [8] as follows.

Definition 1.1 Let E be a normed space and $R: E \to E$ be a mapping. If there are the numbers $\theta \in [0, \infty)$ and $\gamma \in [0, \theta + 1)$ satisfying

$$\|\theta(x-y) + Rx - Ry\| \leq \gamma\|x-y\|, \ \forall x, y \in E \quad (1.7)$$

then, the mapping R is called (θ, γ)-enriched contraction [8].

It is shown in [8, Example 1] that every mapping that satisfies contraction condition (1.2) satisfies also enriched contraction condition (1.7). However, there is a mapping that satisfies enriched contraction condition (1.7) but does not satisfy contraction condition (1.2). So, the class of enriched contraction mappings is a large class containing contraction mappings [8].

Let us recall the definition of the class of average mappings, which is another important mapping class.

Definition 1.2 (see, [8]) Let D be a convex subset of a vector space E and $R: D \to D$ be a mapping. For any $\omega \in (0,1)$, the averaged mapping R_ω is defined by

$$R_\omega x = (1-\omega)x + \omega Rx, \ \forall x \in D \quad (1.8)$$

and the sets of fixed points of the mappings R and R_ω coincide. That is, $F(R) = F(R_\omega)$.

Berinde and Păcurar [8, Theorem 2.4] proved that a self mapping R on a Banach space has a unique fixed point if the mapping R is an enriched contraction. They also proved the existence of an iterative algorithm associated with R_ω that converges to the fixed point of R.

Remark 1.1 Berinde and Păcurar showed in the proof of [8, Theorem 2.4] that there exists a number $\omega \in (0,1)$ such that average mapping R_ω in (1.8) is a contraction (with $\omega\gamma$) if the mapping R is a (θ, γ)-enriched contraction.

In the rest of the study, unless otherwise stated, the R_ω will be considered as the averaged mapping in (1.8) defined by an enriched contraction R.

(1.1), (1.3)-(1.6) iterative algorithms have been studied in recent years by associating with the averaged mapping R_ω. Abbas et al. [9, Theorem 4] obtained that a result dealing with the equivalence of the Mann and Ishikawa iterative algorithms associated with R_ω. Anjum et al. [10, Theorem 1] showed that the result in [9, Theorem 4] is also equivalent to Picard iterative algorithm associated with R_ω under appropriate conditions.

Picard S-iterative algorithm associated with R_ω has not been studied so far, to our knowledge. The main aim of this article is to modify the Picard-S algorithm defined by Gürsoy and Karakaya [5] to approach to the fixed points of enriched contraction mappings. Firstly, we proved that the Picard-S algorithm associated with R_ω converges to the fixed point of the enriched contraction R without any extra condition on control sequences. Secondly, we showed that the convergence of Picard-S and CR iterative algorithms associated with R_ω to the fixed point are equivalent to each other. Thirdly, we obtained a result regarding the data dependency of Picard-S algorithm associated with R_ω. Finally, we supported the results obtained with numerical examples, and examined with an example the convergence speeds of Picard-S and some algorithms associated with R_ω.

The lemma given below has an important role for our study.

Lemma 1.1 Let $\{\varrho_n^k\}_{n=0}^\infty$, $k = 1,2$ be two sequences of non-negative real numbers satisfying

$$\varrho_{n+1}^1 \leq \delta \varrho_n^1 + \varrho_n^2, \text{ for all } n \in \mathbb{N}.$$

If $\delta \in [0,1)$ is a constant and $\{\varrho_n^2\}_{n=0}^\infty$ is a sequence which converges to zero, then $\{\varrho_n^1\}_{n=0}^\infty$ is a sequence which converges to zero (see, [11]).

MAIN RESULTS

In this section, we will give the main results obtained from the study. Firstly, we denote that if D is a closed and convex subset of a Banach space E and $R: D \to D$ is a (θ,γ)-enriched contraction, then R has a unique fixed point. Infact, by Remark 1.1, we know that there exists a number $\omega \in (0,1)$ such that mapping $R_\omega: D \to D$ is a contraction. By Banach Contraction Prenciple (see, [12]), R_ω has a unique fixed point. On the other hand, by Definition 1.2, since $F(R) = F(R_\omega)$, we say that R has a unique fixed point. If $\theta = 0$, then it is clear that $R: D \to D$ is a contraction mapping. Since Picard-S algorithm associated with contraction mappings was studied by Gürsoy and Karakaya [5], we will take $\theta > 0$ in this section.

The following result is a modification of [5, Theorem 1] for enriched contraction mappings. It should be noted here that although restrictive assumption $\sum_{n=0}^\infty \alpha_n^1 \alpha_n^2 = \infty$ was made on the control sequences $\{\alpha_n^1\}$, $\{\alpha_n^2\} \subset [0,1]$ in [5, Theorem 1], our result was obtained without any restriction on the control sequences.

Theorem 2.1 Let D be a closed and convex subset of a Banach space E and $R: D \to D$ be a (θ,γ)-enriched contraction, $\{\alpha_n^1\}$, $\{\alpha_n^2\} \subset [0,1]$ be any sequences and $x_0^* \in D$ be any initial point. Then, there is a number $\omega \in (0,1)$ such that the sequence $\{x_n^*\}$ generated by Picard-S algorithm associated with R_ω given by

$$\begin{aligned} x_{n+1}^* &= R_\omega y_n^*, \\ y_n^* &= (1-\alpha_n^1) R_\omega x_n^* + \alpha_n^1 R_\omega z_n^*, \\ z_n^* &= (1-\alpha_n^2) x_n^* + \alpha_n^2 R_\omega x_n^*, \ n \in \mathbb{N} \end{aligned} \tag{1.9}$$

converges to the unique fixed point p^* of R.

Proof Since $\theta > 0$, if $\omega := 1/(\theta+1)$ is taken, then $\omega \in (0,1)$. By the definition of (θ,γ)-enriched contraction, for all $x, y \in D$, we have

$$\|R_\omega x - R_\omega y\| \leq \omega\gamma \|x - y\| \tag{1.10}$$

where, we denote that $\omega\gamma \in (0,1)$. That is, $R_\omega: D \to D$ is a contraction mapping with the fixed $\omega\gamma$ ([8, Theorem 2.4]). Let $\{\alpha_n^1\}$, $\{\alpha_n^2\}$ be any sequences in $[0,1]$, and we consider the sequence $\{x_n\}$ generated by (1.9). By standard methods in [5, Theorem 1], we will prove that $x_n^* \xrightarrow{n\to\infty} p^*$. Using (1.9), (1.10), $\{\alpha_n^1\}$ and $\{\alpha_n^2\}$ in $[0,1]$, we get

$$\begin{aligned}\|z_n^* - p^*\| &= \|(1-\alpha_n^2)x_n^* + \alpha_n^2 R_\omega x_n^* - p^*\| \\ &\le (1-\alpha_n^2)\|x_n^* - p^*\| + \alpha_n^2\omega\gamma\|x_n^* - p^*\| \\ &= [1-\alpha_n^2(1-\omega\gamma)]\|x_n^* - p^*\| \end{aligned} \tag{1.11}$$

and

$$\begin{aligned}\|y_n^* - p^*\| &= \|(1-\alpha_n^1)R_\omega x_n^* + \alpha_n^1 R_\omega z_n^* - p^*\| \\ &\le (1-\alpha_n^1)\omega\gamma\|x_n^* - p^*\| + \alpha_n^1\omega\gamma\|z_n^* - p^*\|. \end{aligned} \tag{1.12}$$

If (1.11) is written in (1.12), then the inequality given below is obtained

$$\begin{aligned}\|y_n^* - p^*\| &\le (1-\alpha_n^1)\omega\gamma\|x_n^* - p^*\| + \alpha_n^1\omega\gamma[1-\alpha_n^2(1-\omega\gamma)]\|x_n^* - p^*\| \\ &= \omega\gamma[1-\alpha_n^1\alpha_n^2(1-\omega\gamma)]\|x_n^* - p^*\|. \end{aligned} \tag{1.13}$$

Thus, by (1.9) and (1.13), we get

$$\begin{aligned}\|x_{n+1}^* - p^*\| &= \|R_\omega y_n^* - p^*\| \\ &\le \omega\gamma\|y_n^* - p^*\| \le (\omega\gamma)^2[1-\alpha_n^1\alpha_n^2(1-\omega\gamma)]\|x_n^* - p^*\|. \end{aligned} \tag{1.14}$$

On the other hand, since $0 < 1-\alpha_n^1\alpha_n^2(1-\omega\gamma) \le 1$, for all $n \in \mathbb{N}$, by (1.14), we obtain

$$\|x_{n+1}^* - p^*\| \le (\omega\gamma)^2\|x_n^* - p^*\|.$$

Since $\omega\gamma < 1$, by Lemma 1.1, we get $x_n^* \xrightarrow{n\to\infty} p^*$. □

The result given below is a modification of [5, Theorem 2] for enriched contraction mappings. The result shows that the convergence of Picard-S and CR algorithms associated with R_ω to the fixed point is equivalent. We denote that the result is obtained on the control sequences $\{\alpha_n^0\}$, $\{\alpha_n^1\}$, $\{\alpha_n^2\} \subset [0,1]$ without any extra conditions.

Theorem 2.2 Let D, E and R be as in Theorem 2.1, $\{\alpha_n^0\}$, $\{\alpha_n^1\}$, $\{\alpha_n^2\} \subset [0,1]$ be any sequences and x_0^*, $c_0^* \in D$ be any initial points. Then, there is a number $\omega \in (0,1)$ such that the sequence $\{x_n^*\}$ generated by (1.9) and the sequence $\{c_n^*\}$ generated by CR algorithm associated with R_ω given by

$$\begin{aligned} c_{n+1}^* &= (1-\alpha_n^0)q_n^* + \alpha_n^0 R_\omega q_n^*, \\ q_n^* &= (1-\alpha_n^1)R_\omega c_n^* + \alpha_n^1 R_\omega w_n^*, \\ w_n^* &= (1-\alpha_n^2)c_n^* + \alpha_n^2 R_\omega c_n^*,\ n \in \mathbb{N} \end{aligned} \tag{1.15}$$

are equivalent in the case of convergence to the fixed point p^* of R. That is, $x_n^* \xrightarrow{n\to\infty} p^*$ if and only if $c_n^* \xrightarrow{n\to\infty} p^*$.

Proof If the mapping R is a (θ,γ)-enriched contraction, then by the proof of [8, Theorem 2.4], we know that there is a number $\omega \in (0,1)$ such that R_ω is a contraction mapping with $\omega\gamma$. We will prove this theorem by standard methods in [5, Theorem 2]. Let us consider algorithms (1.9) and (1.15) associated with R_ω.

We assume that $x_n^* \xrightarrow{n\to\infty} p^*$. By (1.9) and (1.15), we get, for all $n \in \mathbb{N}$

$$\begin{aligned}\|x_{n+1}^* - c_{n+1}^*\| &= \|R_\omega y_n^* - (1-\alpha_n^0)q_n^* - \alpha_n^0 R_\omega q_n^*\| \\ &\le (1-\alpha_n^0)\|R_\omega y_n^* - q_n^*\| + \alpha_n^0\|R_\omega y_n^* - R_\omega q_n^*\| \end{aligned}$$

$$\leq (1-\alpha_n^0)\|R_\omega y_n^* - y_n^*\| + (1-\alpha_n^0)\|y_n^* - q_n^*\| + \alpha_n^0\omega\gamma\|y_n^* - q_n^*\|$$

$$= (1-\alpha_n^0+\alpha_n^0\omega\gamma)\|y_n^* - q_n^*\| + (1-\alpha_n^0)\|R_\omega y_n^* - y_n^*\|. \quad (1.16)$$

On the other hand, by (1.9) and (1.15), we get the inequalities given below

$$\|y_n^* - q_n^*\| = \|(1-\alpha_n^1)R_\omega x_n^* + \alpha_n^1 R_\omega z_n^* - (1-\alpha_n^1)R_\omega c_n^* - \alpha_n^1 R_\omega w_n^*\|$$

$$\leq (1-\alpha_n^1)\|R_\omega x_n^* - R_\omega c_n^*\| + \alpha_n^1\|R_\omega z_n^* - R_\omega w_n^*\|$$

$$\leq (1-\alpha_n^1)\omega\gamma\|x_n^* - c_n^*\| + \alpha_n^1\omega\gamma\|z_n^* - w_n^*\| \quad (1.17)$$

and

$$\|z_n^* - w_n^*\| = \|(1-\alpha_n^2)x_n^* + \alpha_n^2 R_\omega x_n^* - (1-\alpha_n^2)c_n^* - \alpha_n^2 R_\omega c_n^*\|$$

$$\leq (1-\alpha_n^2)\|x_n^* - c_n^*\| + \alpha_n^2\omega\gamma\|x_n^* - c_n^*\|$$

$$= [1-\alpha_n^2(1-\omega\gamma)]\|x_n^* - c_n^*\|. \quad (1.18)$$

If (1.18) is written in (1.17), then the inequality given below is obtained

$$\|y_n^* - q_n^*\| \leq (1-\alpha_n^1)\omega\gamma\|x_n^* - c_n^*\| + \alpha_n^1\omega\gamma[1-\alpha_n^2(1-\omega\gamma)]\|x_n^* - c_n^*\|$$

$$= \omega\gamma[1-\alpha_n^1\alpha_n^2(1-\omega\gamma)]\|x_n^* - c_n^*\|. \quad (1.19)$$

So, inequality (1.16) turns into the following inequality

$$\|x_{n+1}^* - c_{n+1}^*\| \leq (1-\alpha_n^0+\alpha_n^0\omega\gamma)\omega\gamma[1-\alpha_n^1\alpha_n^2(1-\omega\gamma)]\|x_n^* - c_n^*\| + (1-\alpha_n^0)\|R_\omega y_n^* - y_n^*\|.$$

Using $1-\alpha_n^0+\alpha_n^0\omega\gamma \leq 1$, $1-\alpha_n^1\alpha_n^2(1-\omega\gamma) \leq 1$ and $1-\alpha_n^0 \leq 1$, for all $n \in \mathbb{N}$, we obtain

$$\|x_{n+1}^* - c_{n+1}^*\| \leq \omega\gamma\|x_n^* - c_n^*\| + \|R_\omega y_n^* - y_n^*\|. \quad (1.20)$$

If the continuity of the mapping R_ω, algorithm (1.9), $\{\alpha_n^1\}$, $\{\alpha_n^2\} \subset [0,1]$ and $x_n^* \xrightarrow{n\to\infty} p^*$ are used, then $\|R_\omega y_n^* - y_n^*\| \xrightarrow{n\to\infty} 0$ is obtained. Also, since $\omega\gamma < 1$, using Lemma 1.1 in (1.20), $\|x_n^* - c_n^*\| \xrightarrow{n\to\infty} 0$ is obtained. By hypothesis, this denotes that $\|c_n^* - p^*\| \xrightarrow{n\to\infty} 0$.

Conversely, assume that $c_n^* \xrightarrow{n\to\infty} p^*$. By (1.9), (1.15) and (1.19), we have the inequality given below, for all $n \in \mathbb{N}$

$$\|c_{n+1}^* - x_{n+1}^*\| = \|(1-\alpha_n^0)q_n^* + \alpha_n^0 R_\omega q_n^* - R_\omega y_n^*\|$$

$$\leq \|q_n^* - R_\omega y_n^*\| + \alpha_n^0\|q_n^* - R_\omega q_n^*\|$$

$$\leq \|R_\omega q_n^* - q_n^*\| + \|R_\omega q_n^* - R_\omega y_n^*\| + \alpha_n^0\|q_n^* - R_\omega q_n^*\|$$

$$\leq (\alpha_n^0+1)\|R_\omega q_n^* - q_n^*\| + \omega\gamma\|q_n^* - y_n^*\|$$

$$\leq (\alpha_n^0+1)\|R_\omega q_n^* - q_n^*\| + (\omega\gamma)^2[1-\alpha_n^1\alpha_n^2(1-\omega\gamma)]\|x_n^* - c_n^*\|.$$

Since $1-\alpha_n^1\alpha_n^2(1-\omega\gamma) \leq 1$ and $\alpha_n^0 + 1 \leq 2$, for all $n \in \mathbb{N}$, the inequality can be written as follows

$$\|c_{n+1}^* - x_{n+1}^*\| \leq (\omega\gamma)^2\|c_n^* - x_n^*\| + 2\|R_\omega q_n^* - q_n^*\|. \quad (1.21)$$

If the continuity of R_ω, algorithm (1.15), $\{\alpha_n^1\}$, $\{\alpha_n^2\} \subset [0,1]$ and $c_n^* \xrightarrow{n\to\infty} p^*$ are used, then $\|R_\omega q_n^* - q_n^*\| \xrightarrow{n\to\infty} 0$ is obtained. Also, since $\omega\gamma < 1$, using Lemma 1.1 in (1.21), it is obtained $\|c_n^* - x_n^*\| \xrightarrow{n\to\infty} 0$. By hypothesis, this denotes that $\|x_n^* - p^*\| \xrightarrow{n\to\infty} 0$. Thus, the proof is completed. □

Finally, we will give a result on data dependency for Picard-S algorithm associated with R_ω. We denote that the result is a modification of [5, Theorem 4] for Picard-S algorithm associated with R_ω and it is obtained without any extra conditions on control sequences.

Theorem 2.3 Let D, E, R, x_0^*, $\{\alpha_n^1\}$ and $\{\alpha_n^2\}$ be as in Theorem 2.1. Let $\tilde{R}: D \to D$ be a mapping such that

$\|Rx-\tilde{R}x\|<\varepsilon$, for all $x\in D$ (where ε is an enough small number) and $\tilde{p}^*$ be a fixed point of $\tilde{R}$. Then, there is a number $\omega\in(0,1)$, and if the sequence $\{\tilde{x}_n^*\}$ generated by $\tilde{R}_\omega$

$$\begin{aligned}&\tilde{x}_0^*\in D,\\&\tilde{x}_{n+1}^*=\tilde{R}_\omega\tilde{y}_n^*,\\&\tilde{y}_n^*=(1-\alpha_n^1)\tilde{R}_\omega\tilde{x}_n^*+\alpha_n^1\tilde{R}_\omega\tilde{z}_n^*,\\&\tilde{z}_n^*=(1-\alpha_n^2)\tilde{x}_n^*+\alpha_n^2\tilde{R}_\omega\tilde{x}_n^*,\ n\in\mathbb{N}\end{aligned}\tag{1.22}$$

converges to $\tilde{p}^*$ (where $\tilde{R}_\omega$ is the averaged mapping associated with $\tilde{R}$), then

$$\|p^*-\tilde{p}^*\|\le\frac{\omega\varepsilon(\vartheta^2+\vartheta+1)}{1-\vartheta^2}\tag{1.23}$$

in which $\vartheta=\omega\gamma$.

Proof By the proof of [8, Theorem 2.4], we know that there is a number $\omega\in(0,1)$ such that the mapping R_ω is a contraction with the number $\omega\gamma$. We will prove this theorem by standard methods in [5, Theorem 4]. Using (1.9), (1.22), (1.8), the contraction condition of R_ω and the choice of $\tilde{R}$, we get the inequalities given below, for all $n\in\mathbb{N}$

$$\begin{aligned}\|x_{n+1}^*-\tilde{x}_{n+1}^*\|&=\|R_\omega y_n^*-R_\omega\tilde{y}_n^*+R_\omega\tilde{y}_n^*-\tilde{R}_\omega\tilde{y}_n^*\|\\&\le\omega\gamma\|y_n^*-\tilde{y}_n^*\|+\omega\varepsilon\end{aligned}\tag{1.24}$$

and

$$\begin{aligned}\|y_n^*-\tilde{y}_n^*\|&=\|(1-\alpha_n^1)R_\omega x_n^*+\alpha_n^1R_\omega z_n^*-(1-\alpha_n^1)\tilde{R}_\omega\tilde{x}_n^*-\alpha_n^1\tilde{R}_\omega\tilde{z}_n^*\|\\&\le(1-\alpha_n^1)\|R_\omega x_n^*-R_\omega\tilde{x}_n^*+R_\omega\tilde{x}_n^*-\tilde{R}_\omega\tilde{x}_n^*\|+\alpha_n^1\|R_\omega z_n^*-R_\omega\tilde{z}_n^*+R_\omega\tilde{z}_n^*-\tilde{R}_\omega\tilde{z}_n^*\|\\&\le(1-\alpha_n^1)\left[\|R_\omega x_n^*-R_\omega\tilde{x}_n^*\|+\|R_\omega\tilde{x}_n^*-\tilde{R}_\omega\tilde{x}_n^*\|\right]\\&\quad+\alpha_n^1\left[\|R_\omega z_n^*-R_\omega\tilde{z}_n^*\|+\|R_\omega\tilde{z}_n^*-\tilde{R}_\omega\tilde{z}_n^*\|\right]\\&\le(1-\alpha_n^1)[\omega\gamma\|x_n^*-\tilde{x}_n^*\|+\omega\varepsilon]+\alpha_n^1[\omega\gamma\|z_n^*-\tilde{z}_n^*\|+\omega\varepsilon]\\&=(1-\alpha_n^1)\omega\gamma\|x_n^*-\tilde{x}_n^*\|+\alpha_n^1\omega\gamma\|z_n^*-\tilde{z}_n^*\|+\omega\varepsilon\end{aligned}\tag{1.25}$$

and

$$\begin{aligned}\|z_n^*-\tilde{z}_n^*\|&=\|(1-\alpha_n^2)x_n^*+\alpha_n^2R_\omega x_n^*-(1-\alpha_n^2)\tilde{x}_n^*-\alpha_n^2\tilde{R}_\omega\tilde{x}_n^*\|\\&\le(1-\alpha_n^2)\|x_n^*-\tilde{x}_n^*\|+\alpha_n^2\|R_\omega x_n^*-R_\omega\tilde{x}_n^*+R_\omega\tilde{x}_n^*-\tilde{R}_\omega\tilde{x}_n^*\|\\&\le(1-\alpha_n^2)\|x_n^*-\tilde{x}_n^*\|+\alpha_n^2[\omega\gamma\|x_n^*-\tilde{x}_n^*\|+\omega\varepsilon]\\&=[1-\alpha_n^2(1-\omega\gamma)]\|x_n^*-\tilde{x}_n^*\|+\alpha_n^2\omega\varepsilon.\end{aligned}\tag{1.26}$$

By inequalities (1.24)-(1.26),

$$\|x_{n+1}^*-\tilde{x}_{n+1}^*\|\le(\omega\gamma)^2[1-\alpha_n^1\alpha_n^2(1-\omega\gamma)]\|x_n^*-\tilde{x}_n^*\|+\alpha_n^1\alpha_n^2(\omega\gamma)^2\omega\varepsilon+\omega^2\gamma\varepsilon+\omega\varepsilon\tag{1.27}$$

is obtained. Since $1-\alpha_n^1\alpha_n^2(1-\omega\gamma)\le1$ and $\alpha_n^1\alpha_n^2\le1$, for all $n\in\mathbb{N}$, we get

$$\|x_{n+1}^*-\tilde{x}_{n+1}^*\|\le(\omega\gamma)^2\|x_n^*-\tilde{x}_n^*\|+(\omega\gamma)^2\omega\varepsilon+\omega^2\gamma\varepsilon+\omega\varepsilon.\tag{1.28}$$

Considering that $x_n^*\xrightarrow{n\to\infty}p^*$ and $\tilde{x}_n^*\xrightarrow{n\to\infty}\tilde{p}^*$, if the limits of both sides are taken in inequality (1.28), then the inequality given below is obtained.

$$\begin{aligned}\|p^*-\tilde{p}^*\|&\le(\omega\gamma)^2\|p^*-\tilde{p}^*\|+(\omega\gamma)^2\omega\varepsilon+\omega^2\gamma\varepsilon+\omega\varepsilon\\&=(\omega\gamma)^2\|p^*-\tilde{p}^*\|+\omega\varepsilon[(\omega\gamma)^2+\omega\gamma+1].\end{aligned}$$

Thus, an upper bound on difference between fixed points p^* and $\tilde{p}^*$ is obtained as

$$\|p^* - \tilde{p}^*\| \le \frac{\omega\varepsilon[(\omega\gamma)^2 + \omega\gamma + 1]}{1 - (\omega\gamma)^2}.$$

Remark 2.1 If we take $\lim_{n\to\infty} \alpha_n^1 \alpha_n^2 = 0$ or $\lim_{n\to\infty} \alpha_n^1 \alpha_n^2 = 1$ additionally in the hypotheses of Theorem 2.3, then, by inequality (1.27), we get $\|p^* - \tilde{p}^*\| \le \frac{\omega\varepsilon}{1-\omega\gamma}$.

NUMERICAL EXAMPLES

In this section, we will give some examples to support the theoretical results we obtained.

The first example given below demonstrates the accuracy and validity of the results in Theorem 2.1 and Theorem 2.2, as well as the convergence speeds of Picard-S, CR, Picard, Mann and Ishikawa algorithms associated with the averaged mapping R_ω.

Example 3.1 Let $E = l_1 = \{\{\alpha_n\}_{n=0}^\infty : \sum_{n=0}^\infty |\alpha_n| < \infty\}$ with the norm $\|\{\alpha_n\}_{n=0}^\infty\| = \sum_{n=0}^\infty |\alpha_n|$ and $D = \{\{\alpha_n\}_{n=0}^\infty \in E : \|\{\alpha_n\}_{n=0}^\infty\| \le 1\}$. It is well known that E is a Banach space and D is a convex and closed subset of E (see, [13]). We define the mapping $R: D \to D$ as follows

$$R(\{\alpha_n\}_{n=0}^\infty) = \{\mu_n\}_{n=0}^\infty, \quad \mu_n = \begin{cases} -\frac{\alpha_0}{2}, & n = 0 \\ \frac{\alpha_{n-1} - \alpha_n}{2}, & n \ge 1. \end{cases}$$

It can easily be seen that R is well defined. For every $\alpha = \{\alpha_n\}_{n=0}^\infty$, $\beta = \{\beta_n\}_{n=0}^\infty \in D$

$$\|R\alpha - R\beta\| \le \left|\frac{\alpha_0 - \beta_0}{2}\right| + \left|\frac{\alpha_0 - \beta_0}{2}\right| + \left|\frac{\alpha_1 - \beta_1}{2}\right| + \left|\frac{\alpha_1 - \beta_1}{2}\right| + \cdots$$

$$= \sum_{k=0}^\infty |\alpha_k - \beta_k| = \|\alpha - \beta\|.$$

So, R is a nonexpansive mapping. However, R is not a contraction mapping. Indeed, for $\alpha = \{0,1,0,0,0,\dots\}$ and $\beta = \{0,0,1,0,0,0,\dots\}$

$$\|R\alpha - R\beta\| = \left\|\left\{0, -\frac{1}{2}, 1, -\frac{1}{2}, 0,0,0,\dots\right\}\right\| = 2 \text{ and } \|\alpha - \beta\| = 2.$$

Thus, there is not a number $\delta \in [0,1)$ satisfying the condition $\|R\alpha - R\beta\| \le \delta\|\alpha - \beta\|$. Now, we will show that R is an enriched contraction mapping. For every $\alpha = \{\alpha_n\}_{n=0}^\infty$, $\beta = \{\beta_n\}_{n=0}^\infty \in D$

$$\|\theta(\alpha - \beta) + R\alpha - R\beta\|$$

$$= \left\|\{\theta(\alpha_0 - \beta_0), \theta(\alpha_1 - \beta_1), \theta(\alpha_2 - \beta_2), \dots\} + \left\{-\frac{\alpha_0 - \beta_0}{2}, \frac{\alpha_0 - \beta_0 - (\alpha_1 - \beta_1)}{2}, \frac{\alpha_1 - \beta_1 - (\alpha_2 - \beta_2)}{2}, \dots\right\}\right\|$$

$$= \left\|\left\{\left(\theta - \frac{1}{2}\right)(\alpha_0 - \beta_0), \left(\theta - \frac{1}{2}\right)(\alpha_1 - \beta_1) + \frac{1}{2}(\alpha_0 - \beta_0), \left(\theta - \frac{1}{2}\right)(\alpha_2 - \beta_2) + \frac{1}{2}(\alpha_1 - \beta_1), \dots\right\}\right\|$$

$$\le \left(\left|\theta - \frac{1}{2}\right| + \frac{1}{2}\right)\sum_{k=0}^\infty |\alpha_k - \beta_k| = \left(\left|\theta - \frac{1}{2}\right| + \frac{1}{2}\right)\|\alpha - \beta\|.$$

If $0 < \theta < \frac{1}{2}$, then we can take $\gamma = 1 - \theta$. In this case, R is a $(\theta, 1 - \theta)$-enriched contraction mapping with $0 < \theta < \frac{1}{2}$. If $\theta \ge \frac{1}{2}$, then we can take $\gamma = \theta$. In this case, R is a (θ, θ)-enriched contraction mapping with $\theta \ge \frac{1}{2}$. Thus, E, D and R satisfy all conditions of Theorem 2.1. Also, R has a unique fixed point $p^* = \{0,0,0,\dots\}$.

Let $\alpha_n^0 = \alpha_n^1 = \alpha_n^2 = \frac{1}{n+1}$, for all $n \in \mathbb{N}$. For these chosen control sequences, Abbas et al. [9, Theorem 4] showed that Mann and Ishikawa algorithms associated with R_ω converge to p^*, and Anjum et al. [10, Corollary 1] showed that Picard algorithm associated with R_ω converges to p^*. Let the initial points of all algorithms be $\left\{\frac{1}{10^{n+1}}\right\}_{n=0}^{\infty}$.

For the values of $\theta = 2/5$ ($\omega = 5/7$), $\theta = 1/2$ ($\omega = 2/3$), $\theta = 2$ ($\omega = 1/3$) and $\theta = 1/10$ ($\omega = 10/11$), we show in Figure 3.1 that the convergence states of the sequences $\{\sigma_n^*\}$ generated by Picard-S, CR, Picard, Mann and Ishikawa algorithms associated with R_ω to the point p^*, respectively.

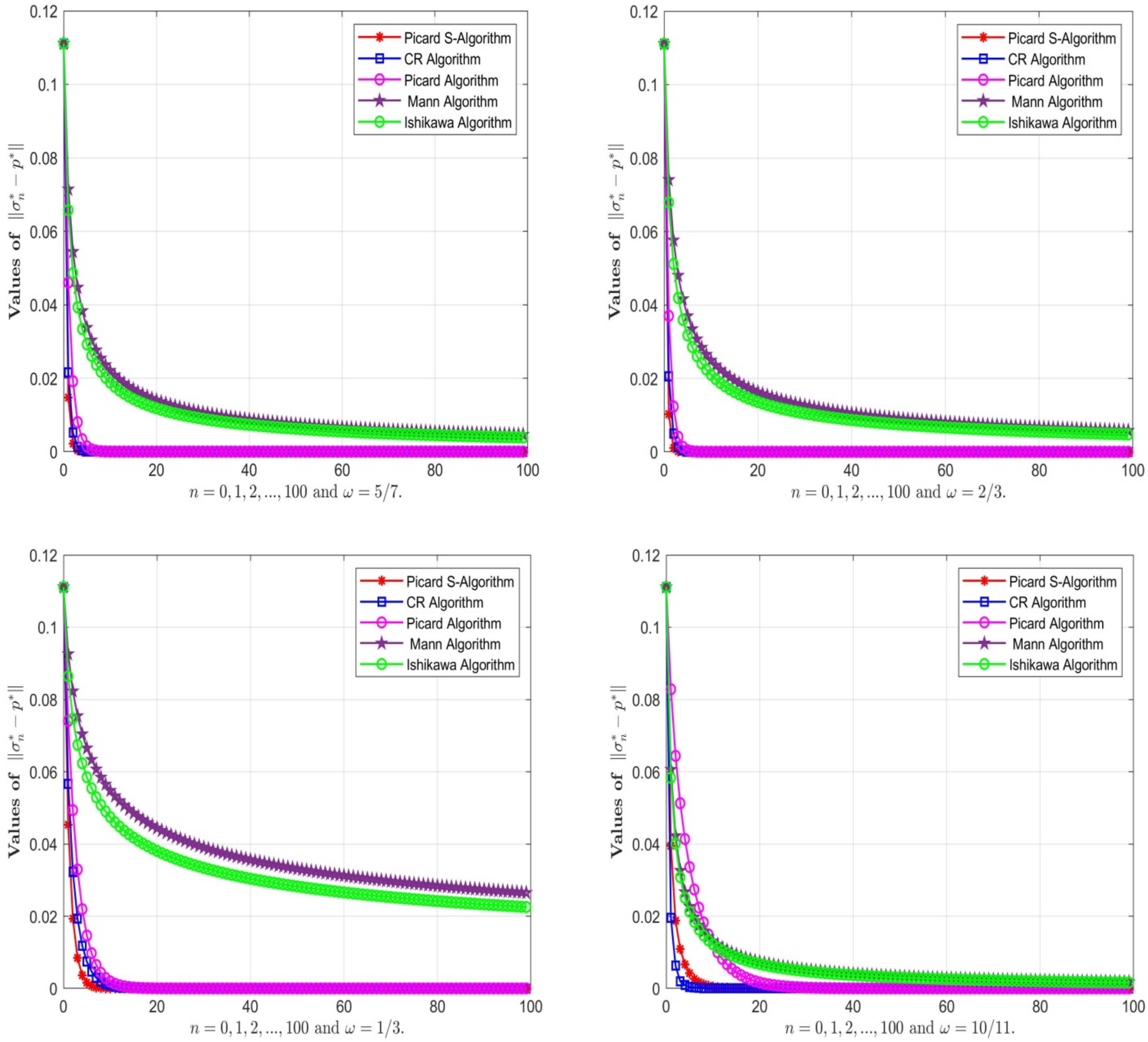

Figure 3.1 *Graphs show the convergence state of algorithms associated with R_ω.*

The following example supporting Theorem 2.3 shows that we can find an upper bound for $\|p^* - \tilde{p}^*\|$ without knowing the values of p^*.

Example 3.2 Let E, D and R be in Example 3.1. We define the mapping $\widetilde{R} : D \to D$ as follows

$$\widetilde{R}\left(\{\alpha_n\}_{n=0}^{\infty}\right) = \{\mu_n\}_{n=0}^{\infty}, \quad \mu_n = \begin{cases} \frac{1}{10}, & n = 0 \\ \frac{\alpha_{n-1}}{2}, & n \geq 1. \end{cases}$$

For all $\alpha = \{\alpha_n\}_{n=0}^{\infty} \in D$, we get

$$\|R\alpha - \widetilde{R}\,\alpha\| = \left|-\frac{\alpha_0}{2} - \frac{1}{10}\right| + \left|-\frac{\alpha_1}{2}\right| + \left|-\frac{\alpha_2}{2}\right| + \left|-\frac{\alpha_3}{2}\right| + \cdots$$

$$\leq \frac{1}{10} + \frac{1}{2}\sum_{k=0}^{\infty}|\alpha_k| \leq \frac{1}{10} + \frac{1}{2} = 0.6$$

Thus, $\varepsilon = 0.6$, and $\tilde{p}^* = \left\{\frac{1}{10}\frac{1}{2^n}\right\}_{n=0}^{\infty}$ is the unique fixed point of $\widetilde{R}$. Let the control sequences be as in Example 3.1. In Figure 3.2, we show that the sequence $\{\tilde{x}_n^*\}$ generated by (1.22) for initial point $\tilde{x}_0^* = x_0^* = \left\{\frac{1}{10^{n+1}}\right\}_{n=0}^{\infty}$ and $\theta = 2/5$ $(\omega = 5/7)$ converges to $\tilde{p}^*$. By (1.23), we have below upper bound for p^* and $\tilde{p}^*$

$$\|p^* - \tilde{p}^*\| = 0.2 \leq \frac{\omega\varepsilon[(\omega\gamma)^2 + \omega\gamma + 1]}{1 - (\omega\gamma)^2} = 0.8464.$$

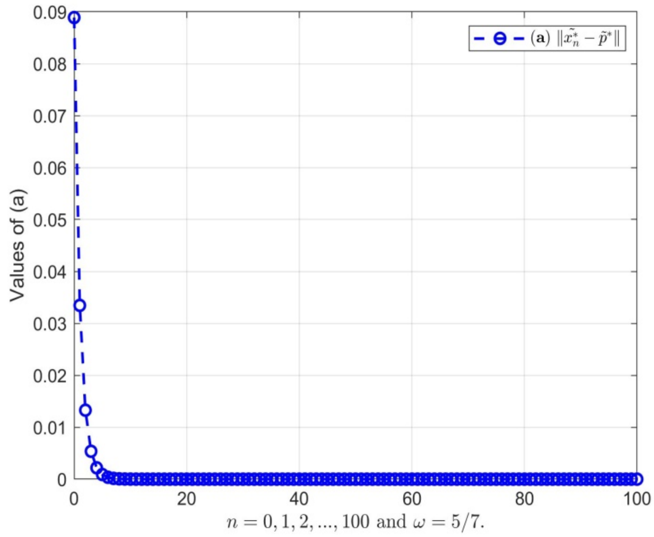

Figure 3.2 *Graph shows the values of* $\|\tilde{x}_n^* - \tilde{p}^*\|$ *for* $n = 0,1,2,\ldots,100$ *and* $\omega = 5/7$.

For the other values of θ in Example 3.1, we get that if $\theta = 1/2$ $(\omega = 2/3)$, then the value of upper bound is 0.65. If $\theta = 2$ $(\omega = 1/3)$, then the value of upper bound is 0.76. If $\theta = 1/10$ $(\omega = 10/11)$, then the value of upper bound is 4.104. That is, for the all values of θ in Example 3.1, inequality (1.23) is satisfied.

DISCUSSION AND CONCLUSIONS

In this study, we obtained some results on the convergence, data dependency and convergence equivalence of Picard-S algorithm associated with the average mapping of an enriched contraction. All results were obtained without any extra conditions except being at $[0,1]$ of the control sequences. The results obtained were supported by numerical examples. Gürsoy and Karakaya [5] obtained a result showing that the Picard-S algorithm converges faster than the CR algorithm for contraction mappings under some conditions. For enriched contraction mappings, as can be seen from the graphs given in Figure 3.1, this situation changes depending on the choice of ω. In other words, the convergence speeds of Picard-S and CR algorithms associated with R_ω can change depending on the choice of ω. Therefore, a general result comparing the convergence speeds for these algorithms associated with R_ω could not be obtained regardless of the choice of ω. This part is open to researchers interested in the study.

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