



A NOTE FOR THE (p, q) -FIBONACCI AND LUCAS QUATERNION POLYNOMIALS

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ABSTRACT. In this article, we have introduced the (p, q) -Fibonacci and Lucas quaternion polynomials which are based on the (p, q) -Fibonacci and Lucas polynomials respectively. Some new identities are derived for these polynomials. The various results obtained here, include Binet formula, Catalan identity, binomial sum formula and generating function.

1. INTRODUCTION

Fibonacci, Lucas, Pell and the other special numbers and their generalizations are famous in science. Fibonacci numbers form a sequence defined by the following recurrence relation: $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. The characteristic equation of F_n is $x^2 - x - 1 = 0$ and hence the roots of it are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Also it has Binet's formula $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ for $n \geq 0$. Lucas numbers L_n are defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

In fact all of them are the special case of the second order linear recurrence $R = \{R_i\}_{i=0}^{\infty} = R(P, Q, R_0, R_1)$ if the recurrence relation for $i \geq 2$, $R_i = PR_{i-1} - QR_{i-2}$ holds for its terms, where P and Q are integers such that $D = P^2 - 4Q \neq 0$ and R_0, R_1 are fixed integers. Define the sequences

$$(1.1) \quad \begin{aligned} U_n &= U_n(P, Q) = PU_{n-1} - QU_{n-2} \\ V_n &= V_n(P, Q) = PV_{n-1} - QV_{n-2} \end{aligned}$$

for $n \geq 2$. The characteristic equation of them is $x^2 - Px + Q = 0$ and hence the roots of it are $\alpha = \frac{P+\sqrt{D}}{2}$ and $\beta = \frac{P-\sqrt{D}}{2}$. So by Binet formula, $U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$

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and $V_n = \alpha^2 + \beta^2$. Further the generating function for U_n and V_n is

$$\sum_{n=0}^{\infty} U_n x^n = \frac{x}{1 - Px + Qx^2} \quad \text{and} \quad \sum_{n=0}^{\infty} V_n x^n = \frac{2 - Px}{1 - Px + Qx^2}$$

[5, 9].

Polynomials can be defined by Fibonacci-like recursion relations are called Fibonacci polynomials. More mathematicians were involved in the study of Fibonacci polynomials, such as P.F.Byrd, M. Bicknell-Johnson and others. The $h(x)$ -Fibonacci quaternion polynomials $Q_{h,n}(x)$ are defined by the recurrence relation

$$(1.2) \quad Q_{h,n}(x) = \sum_{l=0}^3 F_{h,n+l}(x) e_l$$

where $F_{h,n}(x)$ is the $n - th$, the $h(x)$ -Fibonacci polynomial [3, 6].

In [2] they introduce the (p, q) -Fibonacci quaternion polynomials that generalized the $h(x)$ -Fibonacci quaternion polynomials. Let $p(x)$ and $q(x)$ be polynomials with real coefficients the (p, q) -Fibonacci polynomials are defined by the recurrence relation

$$(1.3) \quad F_{p,q,n+1} = p(x)F_{p,q,n} + q(x)F_{p,q,n-1}$$

with the initial conditions $F_{p,q,0} = 0, F_{p,q,1} = 1$. Also for the $p(x)$ and $q(x)$ polynomials with real coefficients the (p, q) -Lucas polynomials are defined by the recurrence relation

$$L_{p,q,n+1} = p(x)L_{p,q,n} + q(x)L_{p,q,n-1}$$

with initial conditions $L_{p,q,0} = 2, L_{p,q,1} = p(x)$ [1, 2]. Let $\alpha_1(x) = \frac{p(x) + \sqrt{p^2(x) + 4q(x)}}{2}$ and $\alpha_2(x) = \frac{p(x) - \sqrt{p^2(x) + 4q(x)}}{2}$ denote the roots of the characteristic equation

$$\alpha^2 - p(x)\alpha - q(x) = 0$$

on the recurrence relation of (1.3). Binet formulas for the (p, q) - Fibonacci and Lucas polynomials are

$$F_{p,q,n}(x) = \frac{\alpha_1^n(x) - \alpha_2^n(x)}{\alpha_1(x) - \alpha_2(x)} \quad \text{and} \quad L_{p,q,n}(x) = \alpha_1^n(x) + \alpha_2^n(x)$$

respectively.

Note that

$$(1.4) \quad \begin{aligned} \alpha_1(x) + \alpha_2(x) &= p(x) \\ \alpha_1(x) - \alpha_2(x) &= \sqrt{p^2(x) + 4q(x)} \\ \alpha_1(x) \cdot \alpha_2(x) &= -q(x) \end{aligned}$$

$$(1.5) \quad \begin{aligned} \frac{\alpha_1(x)}{\alpha_2(x)} &= \frac{-\alpha_1^2(x)}{q(x)} \\ \frac{\alpha_2(x)}{\alpha_1(x)} &= \frac{-\alpha_2^2(x)}{q(x)}. \end{aligned}$$

Division algebras are search topic of great interest, that are real numbers \mathbb{R} , complex numbers \mathbb{C} , quaternions \mathbf{H} , and octonions \mathbb{O} . The quaternions \mathbf{H} are on commutative normed division algebra over the real numbers \mathbb{R} and due the commutativity, one cannot directly extend various results on real and complex numbers to quaternions. Also studies on different types of sequences of quaternions are; Fibonacci Quaternions, Split Fibonacci Quaternions and Complex Fibonacci Quaternions.

A quaternion a is defined by

$$a = a_0 + a_1i_1 + a_2i_2 + a_3i_3$$

where $a_0, a_1, a_2, a_3 \in R$ and i_1, i_2 and i_3 are the fundamental quaternion units that $i_1^2 = i_2^2 = i_3^2 = -1$, $i_1i_2 = -i_2i_1 = i_3$, $i_2i_3 = -i_3i_2 = i_1$, $i_3i_1 = -i_1i_3 = i_2$ and $i_1i_2i_3 = -1$.

The algebra of quaternions is denoted by \mathbf{H} and it is a four-dimensional associative normed division algebra over the real numbers.

The n -th Fibonacci quaternion number of order n is defined by

$$Q_n = F_n + F_{n+1}i_1 + F_{n+2}i_2 + F_{n+3}i_3$$

where F_n is the n -th Fibonacci number and i_1, i_2, i_3 satisfy the identities stated in the previous definition and $n = 0, \pm 1, \pm 2, \dots$.

The k -Fibonacci quaternion number of order n is defined by

$$Q_{k,n} = F_{k,n} + F_{k,n+1}i_1 + F_{k,n+2}i_2 + F_{k,n+3}i_3$$

where $F_{k,n}$ is the n -th, k -Fibonacci number and i_1, i_2, i_3 and $n = 0, \pm 1, \pm 2, \dots$.

Note that,

$$Q_{k,n} = (kF_{k,n-1} + F_{k,n-2}) + F_{k,n+1}i_1 + F_{k,n+2}i_2 + F_{k,n+3}i_3,$$

another expression for the k -Fibonacci quaternion number of order n .

By definition of the generating function of some polynomials for example (1.2), the generating function associated $g_{QF}(t)$ is defined by

$$(1.6) \quad g_Q(t) = \sum_{n=0}^{\infty} Q_{h,n}(x)t^n$$

[4, 7, 8, 10].

2. MAIN THEOREMS OF (p, q) -FIBONACCI QUATERNION POLYNOMIALS.

In this section, we introduce the (p, q) -Fibonacci quaternion polynomials and derive the Binet formula, the generating function and some identities of the (p, q) -Fibonacci quaternion polynomial sequences.

The (p, q) -Fibonacci quaternion polynomials $QF_{p,q,n}(x)$ are defined by the recurrence relation

$$QF_{p,q,n}(x) = \sum_{k=0}^3 F_{p,q,n+k}(x)e_k$$

where $F_{p,q,n+k}$ is the $(n+k)$ -th, the (p, q) -Fibonacci polynomials. Note that the initial conditions of this sequence are given by

$$QF_{p,q,0}(x) = \sum_{k=0}^3 F_{p,q,k}(x)e_k = e_1 + p(x)e_2 + (p^2(x) + q(x))e_3$$

$$\begin{aligned} QF_{p,q,1}(x) &= \sum_{k=0}^3 F_{p,q,1+k}(x)e_k \\ &= e_0 + p(x)e_1 + (p^2(x) + q(x))e_2 + (p^3(x) + 2p(x)q(x))e_3. \end{aligned}$$

Also $QF_{p,q,n}(x)$ is written by a recurrence relation of order two;

$$\begin{aligned} QF_{p,q,n+1}(x) &= \sum_{k=0}^3 F_{p,q,n+k+1}(x)e_k \\ &= \sum_{k=0}^3 (p(x)F_{p,q,n+k}(x) + q(x)F_{p,q,n-1+k}(x))e_k \\ &= p(x) \sum_{k=0}^3 F_{p,q,n+k}(x)e_k + q(x) \sum_{k=0}^3 F_{p,q,n-1+k}(x)e_k \end{aligned}$$

and so on,

$$QF_{p,q,n+1}(x) = p(x)QF_{p,q,n}(x) + q(x)QF_{p,q,n-1}(x).$$

For the (p, q) -Lucas quaternion polynomials $QL_{p,q,n}(x) = \sum_{k=0}^3 L_{p,q,n+k}(x)e_k$ where $L_{p,q,n+k}(x)$ is the $(n+k)$ -th, the (p, q) -Lucas polynomials. So for $n \geq 1$,

$$QL_{p,q,n+1}(x) = p(x)QL_{p,q,n}(x) + q(x)QL_{p,q,n-1}(x).$$

with initial conditions

$$\begin{aligned} QL_{p,q,0}(x) &= \sum_{k=0}^3 L_{p,q,k}(x)e_k \\ QL_{p,q,1}(x) &= \sum_{k=0}^3 L_{p,q,k+1}(x)e_k. \end{aligned}$$

We continue with the generating function results

Theorem 2.1. *The generating function for the (p, q) -Fibonacci and Lucas quaternion polynomials $QF_{p,q,n}(x)$ and $QL_{p,q,n}(x)$ are*

$$(2.1) \quad \begin{aligned} g_{QF}(t) &= \frac{QF_{p,q,0}(x) + [QF_{p,q,1}(x) - p(x)QF_{p,q,0}(x)]t}{1 - p(x)t - q(x)t^2} \\ g_{QL}(t) &= \frac{QL_{p,q,0}(x) + [QL_{p,q,1}(x) - p(x)QL_{p,q,0}(x)]t}{1 - p(x)t - q(x)t^2}. \end{aligned}$$

Proof. The generating function $g_{QF}(t)$ for $QF_{p,q,n}(x)$ must be of the form

$$(2.2) \quad \sum_{n=0}^{\infty} QF_{p,q,n}(x)t^n = QF_{p,q,0}(x) + QF_{p,q,1}(x)t + QF_{p,q,2}(x)t^2 + \cdots + QF_{p,q,n}(x)t^n + \cdots.$$

The formal power series expansions of $-p(x)tg_{QF}(t)$, $-q(x)t^2g_{QF}(t)$ and $g_{QF}(t)$ are

$$\begin{aligned} g_{QF}(t) &= \sum_{n=0}^{\infty} QF_{p,q,n}(x)t^n = QF_{p,q,0}(x) + QF_{p,q,1}(x)t + QF_{p,q,2}(x)t^2 \\ &\quad + \cdots + QF_{p,q,n}(x)t^n + \cdots \\ -p(x)tg_{QF}(t) &= -p(x)QF_{p,q,0}(x)t - p(x)QF_{p,q,1}(x)t^2 - p(x)QF_{p,q,2}(x)t^3 \\ &\quad - \cdots - p(x)QF_{p,q,n}(x)t^{n+1} - \cdots \\ -q(x)t^2g_{QF}(t) &= -q(x)QF_{p,q,0}(x)t^2 - q(x)QF_{p,q,1}(x)t^3 - q(x)QF_{p,q,2}(x)t^4 \\ &\quad - \cdots - q(x)QF_{p,q,n}(x)t^{n+2} - \cdots \end{aligned}$$

respectively. So the expansion for $g_{QF}(t) - g_{QF}(t)p(x)t - g_{QF}(t)q(x)t^2$ is

$$\begin{aligned} g_{QF}(t)(1 - p(x)t - q(x)t^2) &= QF_{p,q,0}(x) + QF_{p,q,1}(x)t - p(x)QF_{p,q,0}(x)t \\ &\quad + [QF_{p,q,2}(x) - p(x)QF_{p,q,1}(x) - q(x)QF_{p,q,0}(x)]t^2 \\ &\quad + [QF_{p,q,3}(x) - p(x)QF_{p,q,2}(x) - q(x)QF_{p,q,1}(x)]t^3 \\ &\quad + \cdots \\ &\quad + [QF_{p,q,n}(x) - p(x)QF_{p,q,n-1}(x) - q(x)QF_{p,q,n-2}(x)]t^n \\ &\quad + \cdots \\ &= QF_{p,q,0}(x) + [QF_{p,q,1}(x) - p(x)QF_{p,q,0}(x)]t. \end{aligned}$$

Hence $QF_{p,q,0}(x) + (QF_{p,q,1}(x) - p(x)QF_{p,q,0}(x))t$ is a finite series, so we can rewrite $[1 - p(x)t - q(x)t^2]g_{QF}(t) = QF_{p,q,0}(x) + [QF_{p,q,1}(x) - p(x)QF_{p,q,0}(x)]t$ and hence

$$(2.3) \quad g_{QF}(t) = \frac{QF_{p,q,0}(x) + [QF_{p,q,1}(x) - p(x)QF_{p,q,0}(x)]t}{1 - p(x)t - q(x)t^2}$$

as we claimed.

The other assertion can be proved similarly. \square

Another way to definition of the generating function is in the following corollary.

Corollary 2.1. *The generating function for the (p, q) -Fibonacci and Lucas quaternion polynomials $QF_{p,q,n}(x)$ and $QL_{p,q,n}(x)$ are*

$$\begin{aligned} g_{QF}(t) &= \frac{e_1 + p(x)e_2 + p^2(x)e_3 + q(x)e_3 + [e_0 + q(x)e_2 + p(x)q(x)e_3]t}{1 - p(x)t - q(x)t^2} \\ g_{QL}(t) &= \frac{\left\{ \begin{array}{l} 2e_0 + p(x)e_1 + [p^2(x) + 2q(x)]e_2 + [p^3(x) - 3p(x)q(x)]e_3 \\ + (-p(x)e_0 + 2q(x)e_1 + q(x)[3p(x) - 1]e_2 + q(x)[p^2(x) + 2q(x)]e_3)t \end{array} \right\}}{1 - p(x)t - q(x)t^2}. \end{aligned}$$

Now we can give the following theorems.

Lemma 2.1. *For the generating function given in Theorem 2.1 we have*

$$\begin{aligned} g_{QF}(t) &= \frac{\frac{QF_{p,q,1}(x) - \alpha_2(x)QF_{p,q,0}(x)}{1 - \alpha_1(x)t} - \frac{QF_{p,q,1}(x) - \alpha_1(x)QF_{p,q,0}(x)}{1 - \alpha_2(x)t}}{\alpha_1(x) - \alpha_2(x)} \\ g_{QL}(t) &= \frac{\frac{QL_{p,q,1}(x) - \alpha_2(x)QL_{p,q,0}(x)}{1 - \alpha_1(x)t} - \frac{QL_{p,q,1}(x) - \alpha_1(x)QL_{p,q,0}(x)}{1 - \alpha_2(x)t}}{\alpha_1(x) - \alpha_2(x)}. \end{aligned}$$

Proof. From the expression of $g_{QF}(t)$ in Teorem 2.1 and the use of (1.4) ,we have:

$$\begin{aligned}
 & \frac{QF_{p,q,0}(x) + [QF_{p,q,1}(x) - p(x)QF_{p,q,0}(x)]t}{1 - p(x)t - q(x)t^2} \\
 = & \frac{QF_{p,q,0}(x) + [QF_{p,q,1}(x) - p(x)QF_{p,q,0}(x)]t}{(1 - \alpha_1(x)t)(1 - \alpha_2(x)t)} \\
 = & \left(\frac{QF_{p,q,0}(x) + [QF_{p,q,1}(x) - (\alpha_1(x) + \alpha_2(x))QF_{p,q,0}(x)]t}{(1 - \alpha_1(x)t)(1 - \alpha_2(x)t)} \right) \times \left(\frac{\alpha_1(x) - \alpha_2(x)}{\alpha_1(x) - \alpha_2(x)} \right) \\
 = & \frac{\left\{ \begin{array}{l} \alpha_1(x)QF_{p,q,0}(x) + \alpha_1(x)QF_{p,q,1}(x)t - \alpha_1^2(x)QF_{p,q,0}(x)t \\ -\alpha_1(x)\alpha_2(x)QF_{p,q,0}(x)t - \alpha_2(x)QF_{p,q,0}(x) - \alpha_2(x)QF_{p,q,1}(x)t \\ +\alpha_1(x)\alpha_2(x)QF_{p,q,0}(x)t + \alpha_2^2(x)QF_{p,q,0}(x)t + QF_{p,q,1}(x) - QF_{p,q,1}(x) \end{array} \right\}}{(\alpha_1(x) - \alpha_2(x))(1 - \alpha_1(x)t)(1 - \alpha_2(x)t)} \\
 = & \frac{\left\{ \begin{array}{l} QF_{p,q,1}(x)(1 - \alpha_2(x)t) + \alpha_2(x)QF_{p,q,0}(x)(-1 + \alpha_2(x)t) \\ +QF_{p,q,1}(x)(-1 + \alpha_1(x)t) + \alpha_1(x)QF_{p,q,0}(x)(1 - \alpha_1(x)t) \end{array} \right\}}{(\alpha_1(x) - \alpha_2(x))(1 - \alpha_1(x)t)(1 - \alpha_2(x)t)} \\
 = & \frac{\left\{ \begin{array}{l} (1 - \alpha_2(x)t)(QF_{p,q,1}(x) - \alpha_2(x)QF_{p,q,0}(x)) \\ -(1 - \alpha_1(x)t)(QF_{p,q,1}(x) - \alpha_1(x)QF_{p,q,0}(x)) \end{array} \right\}}{(\alpha_1(x) - \alpha_2(x))(1 - \alpha_1(x)t)(1 - \alpha_2(x)t)} \\
 = & \frac{1}{\alpha_1(x) - \alpha_2(x)} \left[\frac{QF_{p,q,1}(x) - \alpha_2(x)QF_{p,q,0}(x)}{1 - \alpha_1(x)t} - \frac{QF_{p,q,1}(x) - \alpha_1(x)QF_{p,q,0}(x)}{1 - \alpha_2(x)t} \right].
 \end{aligned}$$

The other cases can be proved similarly. \square

Lemma 2.2. *Let the (p, q) -Fibonacci and Lucas polynomials are $F_{p,q,n}(x)$ and $L_{p,q,n}(x)$. We have*

(1)

$$\begin{aligned}
 F_{p,q,k+1}(x) - \alpha_2(x)F_{p,q,k}(x) &= \alpha_1^k(x) \\
 F_{p,q,k+1}(x) - \alpha_1(x)F_{p,q,k}(x) &= \alpha_2^k(x)
 \end{aligned}$$

(2)

$$\begin{aligned}
 \frac{L_{p,q,k+1}(x) - \alpha_2(x)L_{p,q,k}(x)}{\alpha_1(x) - \alpha_2(x)} &= \alpha_1^k(x) \\
 \frac{\alpha_1(x)L_{p,q,k}(x) - L_{p,q,k+1}(x)}{\alpha_1(x) - \alpha_2(x)} &= \alpha_2^k(x).
 \end{aligned}$$

Proof. We prove it by induction. Let $k = 1$, then

$$F_{p,q,2}(x) - \alpha_2(x)F_{p,q,1}(x) = p(x) - \alpha_2(x) = \alpha_1(x).$$

Let us assume that the equation is $F_{p,q,n}(x) - \alpha_2(x)F_{p,q,n-1}(x) = \alpha_1^{n-1}(x)$ for $k = n - 1$. For $k = n$ it becomes

$$\begin{aligned}
\alpha_1^n(x) &= \alpha_1^{n-1}(x)\alpha_1(x) \\
&= (F_{p,q,n}(x) - \alpha_2(x)F_{p,q,n-1}(x))\alpha_1(x) \\
&= \alpha_1(x)F_{p,q,n}(x) - \alpha_1(x)\alpha_2(x)F_{p,q,n-1}(x) \\
&= (p(x) - \alpha_2(x))F_{p,q,n}(x) - (-q(x))F_{p,q,n-1}(x) \\
&= p(x)F_{p,q,n}(x) + q(x)F_{p,q,n-1}(x) - \alpha_2(x)F_{p,q,n}(x) \\
&= F_{p,q,n+1}(x) - \alpha_2(x)F_{p,q,n}(x).
\end{aligned}$$

So we get the desired result. (2) can be proved similarly. \square

Now we want to derive the Binet formulas for $QF_{p,q,n}(x)$ and $QL_{p,q,n}(x)$. To get this we can give the following theorems.

Theorem 2.2. *For $n \geq 0$, the (p, q) -Fibonacci and Lucas quaternion polynomials $QF_{p,q,n}(x)$ and $QL_{p,q,n}(x)$ are*

$$\begin{aligned}
QF_{p,q,n}(x) &= \frac{\alpha_1^*(x)\alpha_1^n(x) - \alpha_2^*(x)\alpha_2^n(x)}{\alpha_1(x) - \alpha_2(x)} \\
QL_{p,q,n}(x) &= \alpha_1^*(x)\alpha_1^n(x) + \alpha_2^*(x)\alpha_2^n(x)
\end{aligned}$$

$$\text{where } \alpha_1^*(x) = \sum_{k=0}^3 \alpha_1^k(x)e_k \quad \text{and} \quad \alpha_2^*(x) = \sum_{k=0}^3 \alpha_2^k(x)e_k.$$

Proof. Using the Lemma 2.2, we easily get

$$\begin{aligned}
g_{QF}(t) &= \frac{1}{\alpha_1(x) - \alpha_2(x)} \left[(QF_{p,q,1}(x) - \alpha_2(x)QF_{p,q,0}(x)) \sum_{n=0}^{\infty} \alpha_1^n(x)t^n \right. \\
&\quad \left. - (QF_{p,q,1}(x) - \alpha_1(x)QF_{p,q,0}(x)) \sum_{n=0}^{\infty} \alpha_2^n(x)t^n \right] \\
&= \frac{\left\{ \begin{array}{l} \sum_{k=0}^3 (F_{p,q,1+k}(x) - \alpha_2(x)F_{p,q,k}(x))e_k \sum_{n=0}^{\infty} \alpha_1^n(x)t^n \\ - \sum_{k=0}^3 (F_{p,q,1+k}(x) - \alpha_1(x)F_{p,q,k}(x))e_k \sum_{n=0}^{\infty} \alpha_1^n(x)t^n \end{array} \right\}}{\alpha_1(x) - \alpha_2(x)} \\
&= \frac{1}{\alpha_1(x) - \alpha_2(x)} \left[\sum_{k=0}^3 \alpha_1^k(x)e_k \sum_{n=0}^{\infty} \alpha_1^n(x)t^n - \sum_{k=0}^3 \alpha_2^k(x)e_k \sum_{n=0}^{\infty} \alpha_2^n(x)t^n \right] \\
&= \sum_{n=0}^{\infty} \frac{\alpha_1^*(x)\alpha_1^n(x) - \alpha_2^*(x)\alpha_2^n(x)}{\alpha_1(x) - \alpha_2(x)} t^n
\end{aligned}$$

and by the identity (1.6). This completes the proof. \square

Now we can also formulate the Catalan identity and Cassini identity by using Binet formulas.

Theorem 2.3. *(Catalan identity) Let the generating function for the (p, q) -Fibonacci and Lucas quaternion polynomials are $QF_{p,q,n}(x)$ and $QL_{p,q,n}(x)$. For n and α , nonnegative integer numbers, such that $\alpha \leq n$, we have*

$$\begin{aligned}
 QF_{p,q,n+r}(x)QF_{p,q,n-r}(x) - QF_{p,q,n}^2(x) &= \frac{(-1)^{r+n+1}\alpha_1^*(x)\alpha_2^*(x)q^{n-r}(x)(\alpha_1(x) - \alpha_2(x))^r}{(\alpha_1(x) - \alpha_2(x))^2} \\
 QL_{p,q,n+r}(x)QL_{p,q,n-r}(x) - QL_{p,q,n}^2(x) &= (-1)^{r+n}\alpha_1^*(x)\alpha_2^*(x)q^{n-r}(x)(\alpha_1(x) - \alpha_2(x))^r.
 \end{aligned}$$

Proof. Using the Binet formula of Theorem 2.3 and Lemma 2.2 also some identity of (1.4) involving the roots $\alpha_1(x)$ and $\alpha_2(x)$, we have

$$\begin{aligned}
 & QF_{p,q,n+r}(x)QF_{p,q,n-r}(x) - QF_{p,q,n}^2(x) \\
 = & \frac{\alpha_1^*(x)\alpha_1^{n+r}(x) - \alpha_2^*(x)\alpha_2^{n+r}(x)}{\alpha_1(x) - \alpha_2(x)} \times \frac{\alpha_1^*(x)\alpha_1^{n-r}(x) - \alpha_2^*(x)\alpha_2^{n-r}(x)}{\alpha_1(x) - \alpha_2(x)} \\
 & - \left(\frac{\alpha_1^*(x)\alpha_1^n(x) - \alpha_2^*(x)\alpha_2^n(x)}{\alpha_1(x) - \alpha_2(x)} \right)^2 \\
 = & \frac{\left\{ \begin{array}{l} \alpha_1^*(x)^2\alpha_1^{2n}(x) - \alpha_1^*(x)\alpha_2^*(x)\alpha_1^{n-r}(x)\alpha_2^{n+r}(x) \\ -\alpha_1^*(x)\alpha_2^*(x)\alpha_1^{n+r}(x)\alpha_2^{n-r}(x) + \alpha_2^*(x)^2\alpha_2^{2n}(x) \\ -\alpha_1^*(x)^2\alpha_1^{2n}(x) + 2\alpha_1^*(x)\alpha_2^*(x)\alpha_1^n(x)\alpha_2^n(x) \\ -\alpha_2^*(x)^2\alpha_2^{2n}(x) \end{array} \right\}}{(\alpha_1(x) - \alpha_2(x))^2} \\
 = & \frac{\left\{ \begin{array}{l} -\alpha_1^*(x)\alpha_2^*(x)\alpha_1^{n-r}(x)\alpha_2^{n+r}(x) \\ -\alpha_1^*(x)\alpha_2^*(x)\alpha_1^{n+r}(x)\alpha_2^{n-r}(x) \\ +2\alpha_1^*(x)\alpha_2^*(x)\alpha_1^n(x)\alpha_2^n(x) \end{array} \right\}}{(\alpha_1(x) - \alpha_2(x))^2} \\
 = & \frac{-\alpha_1^*(x)\alpha_2^*(x)(\alpha_1^{n-r}(x)\alpha_2^{n+r}(x) + \alpha_1^{n+r}(x)\alpha_2^{n-r}(x) - 2\alpha_1^n(x)\alpha_2^n(x))}{(\alpha_1(x) - \alpha_2(x))^2} \\
 = & \frac{-\alpha_1^*(x)\alpha_2^*(x)\alpha_1^n(x)\alpha_2^n(x) \left[\left(-\frac{\alpha_2^2(x)}{q(x)} \right)^r + \left(-\frac{\alpha_1^2(x)}{q(x)} \right)^r - 2\frac{(\alpha_1(x)\alpha_2(x))^r}{q^r(x)} \right]}{(\alpha_1(x) - \alpha_2(x))^2} \\
 = & \frac{(-1)^{r+n+1}\alpha_1^*(x)\alpha_2^*(x)q^{n-r}(x)(\alpha_1(x) - \alpha_2(x))^r}{(\alpha_1(x) - \alpha_2(x))^2}.
 \end{aligned}$$

The other case can be proved similarly. \square

Theorem 2.4. (*Cassini identity*) For the (p, q) -Fibonacci quaternion polynomials $QF_{p,q,n}(x)$ and the (p, q) -Lucas quaternion polynomials are $QL_{p,q,n}(x)$, we have

$$QF_{p,q,n+1}(x)QF_{p,q,n-1}(x) - QF_{p,q,1}^2(x) = \frac{(-1)^n\alpha_1^*(x)\alpha_2^*(x)q^{n-1}(x)}{\alpha_1(x) - \alpha_2(x)}$$

$$QL_{p,q,n+r}(x)QL_{p,q,n-r}(x) - QL_{p,q,n}^2(x) = (-1)^{1+n}\alpha_1^*(x)\alpha_2^*(x)q^{n-1}(x)(\alpha_1(x) - \alpha_2(x))$$

for any natural number n .

Theorem 2.5. The (p, q) -Fibonacci and Lucas quaternion polynomials are $QF_{p,q,n}(x)$ and $QL_{p,q,n}(x)$, for $n \geq 0$, we have

(1)

$$q(x)(QF_{p,q,n}(x))^2 + (QF_{p,q,n+1}(x))^2 = \frac{(\alpha_1^*)^2(x)\alpha_1^{2n+1}(x) - (\alpha_2^*)^2(x)\alpha_2^{2n+1}(x)}{\alpha_1(x) - \alpha_2(x)}$$

$$q(x)(QL_{p,q,n}(x))^2 + (QL_{p,q,n+1}(x))^2 = \left\{ \begin{array}{l} (\alpha_1(x) - \alpha_2(x))(\alpha_1^*)^2(x)\alpha_1^{2n+1}(x) \\ -(\alpha_2^*)^2(x)\alpha_2^{2n+1}(x) \end{array} \right\}$$

(2)

$$QF_{p,q,1}(x) - \alpha_1(x)QF_{p,q,0}(x) = \alpha_2^*(x)$$

$$QF_{p,q,1}(x) - \alpha_2(x)QF_{p,q,0}(x) = \alpha_1^*(x)$$

and

$$QL_{p,q,1}(x) - \alpha_1(x)QL_{p,q,0}(x) = (\alpha_1(x) - \alpha_2(x))\alpha_2^*(x)$$

$$QL_{p,q,1}(x) - \alpha_2(x)QL_{p,q,0}(x) = (\alpha_1(x) - \alpha_2(x))\alpha_1^*(x).$$

Proof. In order to prove the identity (1), we use the formulas to get

$$q(x)(QF_{p,q,n}(x))^2 + (QF_{p,q,n+1}(x))^2$$

$$= q(x) \left(\frac{\alpha_1^*(x)\alpha_1^n(x) - \alpha_2^*(x)\alpha_2^n(x)}{\alpha_1(x) - \alpha_2(x)} \right)^2 + \left(\frac{\alpha_1^*(x)\alpha_1^{n+1}(x) - \alpha_2^*(x)\alpha_2^{n+1}(x)}{\alpha_1(x) - \alpha_2(x)} \right)^2$$

$$= \frac{\left\{ \begin{array}{l} q(x)(\alpha_1^*)^2(x)\alpha_1^{2n}(x) - 2q(x)\alpha_1^*(x)\alpha_1^n(x)\alpha_2^*(x)\alpha_2^n(x) \\ + q(x)(\alpha_2^*)^2(x)\alpha_2^{2n}(x) + (\alpha_1^*)^2(x)\alpha_1^{2n+2}(x) \\ - 2\alpha_1^*(x)\alpha_1^{n+1}(x)\alpha_2^*(x)\alpha_2^{n+1}(x) + (\alpha_2^*)^2(x)\alpha_2^{2n+2}(x) \end{array} \right\}}{(\alpha_1(x) - \alpha_2(x))^2}$$

$$= \frac{\left\{ \begin{array}{l} (\alpha_1^*)^2(x)\alpha_1^{2n}(x) [q(x) + \alpha_1^2(x)] \\ + (\alpha_2^*)^2(x)\alpha_2^{2n}(x) [q(x) + \alpha_2^2(x)] \\ - 2\alpha_1^*(x)\alpha_1^n(x)\alpha_2^*(x)\alpha_2^n(x) [q(x) + \alpha_1(x)\alpha_2(x)] \end{array} \right\}}{(\alpha_1(x) - \alpha_2(x))^2}$$

$$= \frac{(\alpha_1^*)^2(x)\alpha_1^{2n}(x) \left(q(x) - q(x)\frac{\alpha_1(x)}{\alpha_2(x)} \right) + (\alpha_2^*)^2(x)\alpha_2^{2n}(x) \left(q(x) + q(x)\frac{\alpha_2(x)}{\alpha_1(x)} \right)}{(\alpha_1(x) - \alpha_2(x))^2}$$

$$= \frac{(\alpha_1^*)^2(x)\alpha_1^{2n+1}(x) - (\alpha_2^*)^2(x)\alpha_2^{2n+1}(x)}{\alpha_1(x) - \alpha_2(x)}$$

and the result follows. The other cases can be done similarly. Also the proof of the identity for the (p, q) -Lucas quaternion polynomials (2) is similar to (1). \square

Then we can give the following theorem relative to binomial sum.

Theorem 2.6. *For the (p, q) -Fibonacci and Lucas quaternion polynomials are $QF_{p,q,n}(x)$ and $QL_{p,q,n}(x)$, $n \geq 0$ we have following binomial sum formula for odd and even terms,*

(1)

$$QF_{p,q,2n}(x) = \sum_{m=0}^n \binom{n}{m} q(x)^{n-m} p(x)^m QF_{p,q,m}(x)$$

$$QF_{p,q,2n+1}(x) = \sum_{m=0}^n \binom{n}{m} q(x)^{n-m} p(x)^m QF_{p,q,m+1}(x)$$

(2)

$$\begin{aligned} QL_{p,q,2n}(x) &= \sum_{m=0}^n \binom{n}{m} q(x)^{n-m} p(x)^m QL_{p,q,m}(x) \\ QL_{p,q,2n+1}(x) &= \sum_{m=0}^n \binom{n}{m} q(x)^{n-m} p(x)^m QL_{p,q,m+1}(x). \end{aligned}$$

Proof. For (1) from (1.4) and Binet formulas, we get

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} q(x)^{n-m} p(x)^m QF_{p,q,m}(x) \\ &= \sum_{m=0}^n \binom{n}{m} q(x)^{n-m} p(x)^m \frac{\alpha_1^*(x)\alpha_1^m(x) - \alpha_2^*(x)\alpha_2^m(x)}{\alpha_1(x) - \alpha_2(x)} \\ &= \frac{\alpha_1^*(x)}{\alpha_1(x) - \alpha_2(x)} \sum_{m=0}^n \binom{n}{m} q(x)^{n-m} (p(x)\alpha_1(x))^m \\ & \quad - \frac{\alpha_2^*(x)}{\alpha_1(x) - \alpha_2(x)} \sum_{m=0}^n \binom{n}{m} q(x)^{n-m} (p(x)\alpha_2(x))^m \\ &= \frac{\alpha_1^*(x)}{\alpha_1(x) - \alpha_2(x)} (q(x) + p(x)\alpha_1(x))^n - \frac{\alpha_2^*(x)}{\alpha_1(x) - \alpha_2(x)} (q(x) - p(x)\alpha_2(x))^n \\ &= \frac{\alpha_1^*(x)\alpha_1^{2n}(x) - \alpha_2^*(x)\alpha_2^{2n}(x)}{\alpha_1(x) - \alpha_2(x)} \\ &= QF_{p,q,2n}(x). \end{aligned}$$

The other cases (2) can be done similarly. \square

We formulate the sum of the first n terms of these sequences of (p, q) -Fibonacci quaternion polynomials.

Theorem 2.7. *The sum of the first n -terms of the sequence $QF_{p,q,n}(x)$ and $QL_{p,q,n}(x)$ is given by*

$$\begin{aligned} \sum_{m=0}^n QF_{p,q,m}(x) &= \frac{\left\{ \begin{array}{l} -q(x)QF_{p,q,n}(x) - QF_{p,q,n+1}(x) \\ +QF_{p,q,0}(x) - \frac{\alpha_1^*(x)\alpha_2(x) - \alpha_2^*(x)\alpha_1(x)}{\alpha_1(x) - \alpha_2(x)} \end{array} \right\}}{(\alpha_1(x) - 1)(\alpha_2(x) - 1)} \\ \sum_{m=0}^n QL_{p,q,m}(x) &= \frac{\left\{ \begin{array}{l} -q(x)QL_{p,q,n}(x) - QL_{p,q,n+1}(x) + QL_{p,q,0}(x) \\ - [\alpha_1^*(x)\alpha_2(x) + \alpha_2^*(x)\alpha_1(x)] \end{array} \right\}}{(\alpha_1(x) - 1)(\alpha_2(x) - 1)}. \end{aligned}$$

Proof. Note that using Binet formula and some identities related with the roots $\alpha_1(x)$ and $\alpha_2(x)$, we get

$$\begin{aligned}
 \sum_{m=0}^n QF_{p,q,n}(x) &= \frac{\alpha_1^*(x)\alpha_1^s(x) - \alpha_2^*(x)\alpha_2^s(x)}{\alpha_1(x) - \alpha_2(x)} \\
 &= \frac{1}{\alpha_1(x) - \alpha_2(x)} \sum_{m=0}^n (\alpha_1^*(x)\alpha_1^s(x) - \alpha_2^*(x)\alpha_2^s(x)) \\
 &= \frac{1}{\alpha_1(x) - \alpha_2(x)} (\alpha_1^*(x) \sum_{m=0}^n \alpha_1^s(x) - \alpha_2^*(x) \sum_{m=0}^n \alpha_2^s(x)) \\
 &= \frac{1}{\alpha_1(x) - \alpha_2(x)} (\alpha_1^*(x) \frac{\alpha_1^{n+1}(x) - 1}{\alpha_1(x) - 1} - \alpha_2^*(x) \frac{\alpha_2^{n+1}(x) - 1}{\alpha_2(x) - 1}) \\
 &= \frac{\alpha_1^*(x)(\alpha_1^{n+1}(x) - 1)(\alpha_2(x) - 1) - \alpha_2^*(x)(\alpha_2^{n+1}(x) - 1)(\alpha_1(x) - 1)}{(\alpha_1(x) - \alpha_2(x))(\alpha_1(x) - 1)(\alpha_2(x) - 1)}.
 \end{aligned}$$

The other cases can be done similarly. \square

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