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AFFINE TRANSLATION SURFACES IN 3-DIMENSIONAL EUCLIDEAN SPACE SATISFYING $\Delta r_i = \lambda_i r_i$

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Abstract. In this paper we study the affine translation surfaces in 3-dimensional Euclidean space $\mathbb{E}^3$ under the condition $\Delta r_i = \lambda_i r_i$, where $\lambda_i \in \mathbb{R}$ and Δ denotes the Laplace operator. We obtain the complete classification for those ones.

1. Introduction

Let $\mathbb{E}^3$ be the three-dimensional Euclidean space. An Euclidean submanifold is said to be of Chen finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ [4].

First we recall some well-known formulas for the surfaces in $\mathbb{E}^3$.

Let $r = r(u, v)$ be an isometric immersion of a surface M^2 in $\mathbb{E}^3$.

The inner product on $\mathbb{E}^3$ is

$$g(X, Y) = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

where $X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \in \mathbb{R}^3$.

The Euclidean vector product $X \wedge Y$ of X and Y is defined as follows:

$$X \wedge Y = \big(x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1\big).$$

The notion of finite type immersion of submanifolds of a Euclidean space has been widely used in classifying and characterizing well known Riemannian submanifolds [4]. B.-Y. Chen posed the problem of classifying the finite type submanifolds in the 3-dimensional Euclidean space $\mathbb{E}^3$. These can be regarded as a generalization of minimal submanifolds.

The notion of finite type immersion has played an important role in classifying and characterizing the submanifolds in Euclidean space.

Since then the theory of submanifolds of finite type has been studied by many geometers.

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A well known result due to Takahashi [19] states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition

$$\Delta r = \lambda r,\ \lambda \in \mathbb{R}.$$

In [8] Ferrandez, Garay and Lucas proved that the surfaces of $\mathbb{E}^3$ satisfying

$$\Delta H = AH,\ A \in Mat(3,3)$$

are either minimal, or an open piece of sphere or of a right circular cylinder.

In [7] F. Dillen, J. Pas and L. Verstraelen proved that the only surfaces in $\mathbb{E}^3$ satisfying

$$\Delta r = Ar + B,\ A \in Mat(3,3),\ B \in Mat(3,1),$$

are the minimal surfaces, the spheres and the circular cylinders.

In [1], the authors classified the factorable surfaces in the three-dimensional Euclidean and Lorentzian spaces, whose component functions are eigenfunctions of their Laplace operator. The authors in [2] studied the translation surfaces in the 3-dimensional Euclidean and Lorentz-Minkowski space under the condition

$$\Delta^{III} r_i = \mu_i r_i,\quad \mu_i \in \mathbb{R},$$

where Δ^{III} denotes the Laplacian of the surface with respect to the third fundamental form III.

In this paper we study the affine translation surfaces in the three-dimensional Euclidean space $\mathbb{E}^3$ under the condition

$$\Delta r_i = \lambda_i r_i, \lambda_i \in \mathbb{R}.$$

2. Preliminaries

A submanifold M^2 of a 3-dimensional Euclidean space $\mathbb{E}^3$ is said to be of finite type if each component of its position vector field r can be written as a finite sum of eigenfunctions of the Laplacian Δ of M^2, that is, if

$$r = r_0 + \sum_{i=1}^{k} r_i,$$

where r_i are $\mathbb{E}^3$ $-$valued eigenfunctions of the Laplacian of (M^2, r) [4]:

$$\Delta r_i = \lambda_i r_i,$$

where $\lambda_i \in \mathbb{R}, i = 1, 2, .., k$. If λ_i are different, then M^2 is said to be of k-type.

The coefficients of the first fundamental form and the second fundamental form are

$$\begin{aligned} E &= g(r_u, r_u),\ F = g(r_u, r_v),\ G = g(r_v, r_v), \\ L &= g(r_{uu}, \mathbf{N}),\ M = g(r_{uv}, \mathbf{N}),\ N = g(r_{vv}, \mathbf{N}), \end{aligned}$$

where $r_u = \frac{\partial r}{\partial u}$, $r_v = \frac{\partial r}{\partial v}$ and $\mathbf{N}$ is the unit normal vector to M^2.

The Laplace-Beltrami operator of a smooth function $\varphi : M^2 \to \mathbb{R}, (u, v) \mapsto \varphi(u, v)$ with respect to the first fundamental form of the surface M^2 is the operator Δ, defined in [18] as follows:

$$\Delta\varphi = \frac{-1}{\sqrt{|EG - F^2|}} \left[\frac{\partial}{\partial u} \left(\frac{G\varphi_u - F\varphi_v}{\sqrt{|EG - F^2|}} \right) + \frac{\partial}{\partial v} \left(\frac{E\varphi_v - F\varphi_u}{\sqrt{|EG - F^2|}} \right) \right]. \tag{2.1}$$

The mean curvature H and the Gaussian curvature K_G are, respectively, defined by

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

and

$$K_G = \frac{LN - M^2}{EG - F^2}.$$

3. Affine translation surfaces in $\mathbb{E}^3$

Let M^2 be a 2-dimensional surface, of the Euclidean 3-space $\mathbb{E}^3$. Using the standard coordinate system of $\mathbb{E}^3$ we denote the parametric representation of the surface $r(u, v)$ by

$$r(u, v) = (x(u, v),\ y(u, v),\ z(u, v)).$$

In $\mathbb{E}^3$, a surface is called a translation surface if it is given by an immersion

$$r : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3 : (u, v) \mapsto (u, v, f(u) + g(v)),$$

where f and g are smooth functions on opens of $\mathbb{R}$. One of the famous examples of minimal surfaces in 3-dimensional Euclidean space $\mathbb{E}^3$ is a Scherk's minimal translation surface. In fact, Scherk showed in 1835 that except the planes, the only minimal translation surfaces are the surfaces given by

$$r(u, v) = (u,\ v, \frac{1}{\lambda}\log\cos(\lambda v) - \frac{1}{\lambda}\log\cos(\lambda u)),$$

where λ is a nonzero constant. This surface is called a Scherk's minimal translation surface.

R. López [12] studied translation surfaces in the 3-dimensional hyperbolic space $\mathbb{H}^3$ and classified minimal translation surfaces. R. López and M. I. Munteanu [13] constructed translation surfaces in Sol_3 and investigated properties of minimal one.

In a different aspect, H. Liu [10] considered the translation surfaces with constant mean curvature in 3-dimensional Euclidean space and Lorentz-Minkowski space.

Recently, K. Seo [16] gave a classification of the translation hypersurfaces with constant mean curvature or constant Gauss-Kronecker curvature in space forms.

Related works on minimal translation surfaces of $\mathbb{E}^3$ are [[10], [14], [20]].

Definition 3.1 ([11])**.** An affine translation surface in $\mathbb{E}^3$ is defined as a parameter surface M^2 in $\mathbb{E}^3$ which can be written as

$$r(u, v) = (u,\ v,\ f(u + av) + g(v)), \tag{3.1}$$

for some non zero constant a and functions $f(u + av)$ and $g(v)$.

The coefficients of the first and the second fundamental forms are:

$$E = 1 + f_u^2,\ F = f_u(af_v + g_v),\ G = 1 + (af_v + g_v)^2;$$

$$L = \frac{f_{uu}}{W},\ M = \frac{af_{uv}}{W},\ N = \frac{a^2 f_{vv} + g_{vv}}{W}.$$

The mean curvature H and the Gaussian curvature K_G of M^2 are given by

$$H = \frac{(1 + f_u^2)(a^2 f_{vv} + g_{vv}) + f_{uu}(1 + (af_v + g_v)^2) - 2af_u f_{uv}(af_v + g_v)}{2W^3} \tag{3.2}$$

and

$$K_G = \frac{f_{uu}(a^2 f_{vv} + g_{vv}) - (af_{uv})^2}{W^4}, \tag{3.3}$$

where $W = \sqrt{1 + f_u^2 + (af_v + g_v)^2}$.

By a transformation

$$\begin{cases} x = u + av \\ y = v, \end{cases} \tag{3.4}$$

and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$.

From (3.4) we have

$$E = 1 + f_x^2,\ F = -a + f_x g_y,\ G = 1 + a^2 + g_y^2;$$

$$L = \frac{f_{xx}}{W},\ M = 0,\ N = \frac{g_{yy}}{W}.$$

From (3.2) and (3.3) we get

$$H = \frac{(1 + f_x^2)g_{yy} + (1 + a^2 + g_y^2)f_{xx}}{2W^3} \tag{3.5}$$

and

$$K_G = \frac{f_{xx} g_{yy}}{W^4}, \tag{3.6}$$

where $W = \sqrt{1 + f_x^2 + (af_x + g_y)^2}$.

Theorem 3.1 ([11])**.** *Let $r(x, y) = (x,\ y,\ z(x, y) = f(x) + g(ax + y))$ be a minimal affine translation surface. Then either $z(u, v)$ is linear or can be written as*

$$z(u, v) = \frac{1}{c} \log \left| \frac{\cos(c\sqrt{1 + a^2} x)}{\cos[c(ax + y)]} \right|. \tag{3.7}$$

Remark 3.1. If $a = 0$, the minimal affine translation surface given by (3.7) is the classical Scherk surface.

Definition 3.2 ([11])**.** The minimal affine translation surface (3.7) is called generalized Scherk surface or affine Scherk surface in Euclidean 3 - space.

4. Affine translation surfaces satisfying $\Delta r_i = \lambda_i r_i$ in $\mathbb{E}^3$

In this part we explore the classification of the affine translation surfaces M^2 of $\mathbb{E}^3$ satisfying the condition

$$\Delta r_i = \lambda_i r_i. \tag{4.1}$$

The Laplacian Δ of M^2 can be expressed as follows:

$$\Delta\varphi = \frac{-1}{W^3}\left[W(G\varphi_{xx} + E\varphi_{yy} - 2F\varphi_{xy}) + Q(x, y)\varphi_x + P(x, y)\varphi_y\right], \tag{4.2}$$

where

$$Q(x, y) = -H_1(f_x + a(af_x + g_y)),\quad P(x, y) = -H_1(af_x + g_y),\quad H_1 = EN + GL - 2FM.$$

Applying (4.2) on the coordinate functions $x - ay$, y and $z(x, y) = f(x) + g(y)$ of the position vector r we find

$$\begin{cases} \Delta(f+g) = \frac{-2H}{W} \\ \Delta(x - ay) = \frac{2Hf_x}{W} \\ \Delta(y) = \frac{2H(af_x+g_y)}{W}. \end{cases} \tag{4.3}$$

By using (4.1) and (4.3) we have the following equations

$$\frac{-2H}{W} = \lambda_3(f+g) \tag{4.4}$$

$$\frac{2Hf_x}{W} = \lambda_1(x - ay) \tag{4.5}$$

$$\frac{2H(af_x + g_y)}{W} = \lambda_2 y. \tag{4.6}$$

Therefore, the problem of classifying the affine translation surfaces M^2 satisfying (4.1) is reduced to the integration of this system of ordinary differential equations. Next we study it according to the constants λ_1, λ_2, λ_3.

Case 1. Let $\lambda_3 = 0$.

Then, the equation (4.4) gives rise to $H = 0$, which means that the surfaces are minimal. We get also, by the equations (4.5) and (4.6), $\lambda_2 = \lambda_3 = 0$.

Case 2. Let $\lambda_3 \neq 0$.

In this case we have four possibilities:

a) If $\lambda_1 = 0$ and $\lambda_2 \neq 0$ equations (4.5) and (4.6) imply that

$$\frac{2Hf_x}{W} = 0$$

$$\frac{2H(af_x + g_y)}{W} = \lambda_2 y.$$

It follows that $f(x) = \alpha \in \mathbb{R}$ and g_y is not the constant function.

Therefore, this system of equations is equivalently reduced to

$$\frac{-g_{yy}}{(1+g_y^2)^2} = \lambda_3(\alpha + g) \tag{4.7}$$

$$\frac{g_{yy}g_y}{(1+g_y^2)^2} = \lambda_2 y. \tag{4.8}$$

Equation (4.8) gives rise to

$$g_y^2 = \frac{-1}{\lambda_2 y^2 + c} - 1,$$

where c is a constant such that $-1 < \lambda_2 y^2 + c < 0$.

We find

$$g_{yy} = \frac{\varepsilon \lambda_2 y}{(-\lambda_2 y^2 - c)^{\frac{3}{2}} \sqrt{\lambda_2 y^2 + c + 1}}, \quad -1 < \lambda_2 y^2 + c < 0.$$

Using equation (4.7) we get

$$g(y) = \frac{-\varepsilon \lambda_2 y \sqrt{-\lambda_2 y^2 - c}}{\lambda_1 \sqrt{\lambda_2 y^2 + c + 1}} - \alpha.$$

So

$$\begin{cases} g(v) = \frac{-\varepsilon\lambda_2 v\sqrt{-\lambda_2 v^2 - c}}{\lambda_1\sqrt{\lambda_2 v^2+c+1}} - \alpha \\ f(u+av) = \alpha. \end{cases}$$

Substituting these functions in (3.1), we obtain

$$r(u,v) = \left(u,\ v, \frac{-\varepsilon\lambda_2 v\sqrt{-\lambda_3 v^2 - c}}{\lambda_1\sqrt{\lambda_2 v^2 + c + 1}}\right), \quad -1 < \lambda_2 y^2 + c < 0.$$

b) If $\lambda_1 \neq 0$ and $\lambda_2 = 0$ equations (4.4), (4.5) and (4.6) imply that

$$\frac{-2H}{W} = \lambda_3(f+g) \tag{4.9}$$

$$\frac{2Hf_x}{W} = \lambda_1(x - ay) \tag{4.10}$$

$$\frac{2H(af_x + g_y)}{W} = 0. \tag{4.11}$$

It follows that $af_x + g_y = 0$. On differentiating $af_x + g_y = 0$ we find $f_{xx} = 0$ and $g_{yy} = 0$, which together with (3.5) leads to $H = 0$, a contradiction. So, in this case there are no affine translation surfaces in this case satisfying (4.1).

c) If $\lambda_2 = 0$ and $\lambda_1 = 0$ equations (4.5) and (4.6) imply that

$$\begin{aligned} \frac{-2H}{W} &= \lambda_3(f+g) \\ \frac{2Hf_x}{W} &= 0 \\ \frac{2H(af_x + g_y)}{W} &= 0. \end{aligned}$$

It follows that $af_x + g_y = 0$. On differentiating $af_x + g_y = 0$ we find $f_{xx} = 0$ and $g_{yy} = 0$, which together with (3.5) leads to $H = 0$, a contradiction. So, in this case there are no affine translation surfaces in this case satisfying (4.1).

d) If $\lambda_2 \neq 0$ and $\lambda_1 \neq 0$ equations (4.4) and (4.5) imply that

$$\lambda_3(f+g)f_x = -\lambda_1(x - ay). \tag{4.12}$$

On differentiating (4.12) we find $f_{xx} = 0$ and $g_{yy} = 0$, which together with (3.5) leads to $H = 0$. We deduce that $\lambda_2 = \lambda_1 = 0$, which is clearly a contradiction. So, in this case there are no affine translation surfaces in this case satisfying (4.1).

Consequently, we have:

Theorem 4.1. *Let M^2 be a affine translation surface given by (3.1) in $\mathbb{E}^3$. Then M^2 satisfies the equation $\Delta r_i = \lambda_i r_i$ $(i = 1, 2, 3)$ if and only if the following statement is true:*

1) M^2 has zero mean curvature everywhere.

2) M^2 is parametrized as

$$r(u,v) = \left(u,\ v, \frac{-\varepsilon\lambda_2 v\sqrt{-\lambda_3 v^2 - c}}{\lambda_1\sqrt{\lambda_2 v^2 + c + 1}}\right), \quad -1 < \lambda_2 y^2 + c < 0.$$

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