Sinopfbd, 2024; 9(1): 36-43 https://doi.org/10.33484/sinopfbd.1381635

Sinop Üniversitesi Fen Bilimleri Dergisi
Sinop Uni J Nat Sci

E-ISSN: 2564-7873

https://dergipark.org.tr/tr/pub/sinopfbd

Generalization of $\oplus$ −Cofinitely Radical Supplemented Modules

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How to cite: Haldız, Ş., & Eryılmaz, F. (2024). Generalization of $\oplus$ −cofinitely radical supplemented modules. *Sinop Üniversitesi Fen Bilimleri Dergisi, 9*(1), 36-43. https://doi.org/10.33484/sinopfbd.1381635

Research Article

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Received: 26.10.2023
Accepted: 12.02.2024

Abstract

In this study, we clarify $mgs^{\oplus}$ −modules that are the generalization of $\oplus$ −cofinitely radical supplemented modules and look at some of their basic characteristics. Additionally, we determine the prerequisites for the factor module of an arbitrary $mgs^{\oplus}$ −module to be a $mgs^{\oplus}$ −module and characterized semiperfect rings with the aid of this module.

Keywords: Maximal submodule, Rad-supplement, $\oplus$ −cofinitely radical supplemented module

$\oplus$ −Dual (Eş) Sonlu Radikal Tümlenmiş Modüllerin Genelleştirilmesi

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Öz

Bu makalede $\oplus$ −dual sonlu radikal tümleyen modüllerin genellemesi olarak maksimal alt modülleri $\oplus$ −radikal tümleyene sahip modüller ($mgs^{\oplus}$) tanımlandı ve bu modülün bazı temel özellikleri incelendi. Keyfi bir $mgs^{\oplus}$ −modülünün bölüm modülünün hangi şartlar altında $mgs^{\oplus}$ −modül olduğu gösterildi ve yarı mükemmel halkalar bu modül yardımıyla karakterize edildi.

Anahtar Kelimeler: Maksimal alt modül, radikal tümleyen, $\oplus$ −dual sonlu radikal tümlenmiş modül

Introduction and Preliminaries

All rings would be with identity and associative in this article. Every module is considered a unitary left module. Let W and S be a module and a ring meeting these requirements, respectively. The notation $T \leq W$ will imply that T is a submodule of W and the impression $T \leq_{\oplus} W$ means that T is a direct summand of W. A submodule T of W is referred to as the small module in W, if $W \neq T + T_1$ for any proper submodule T_1 of W and indicated by $T \ll W$. The sum of its small submodules will be shown by $Rad(W)$. A submodule T of W is called as *supplement* of P in W, if it is a minimal element of the set $\{Y \leq W | W = P + Y\}$ which is equivalent to $W = P + T$ and $P \cap T \ll T$. If each submodule of W has a supplement in W, then W is named *supplemented*, [1]. Let W be a module and $T \leq W$. T is called a *cofinite submodule* of W, if $W/_T$ is finitely generated. Cofinite submodules are one of the interesting concept of module theory, and they have various properties and applications in the study of algebraic structures. There are many different studies related with these modules in the literature [2, 3]. If each submodule of W has a supplement that is a direct summand of W, then W is named $\oplus$ *−supplemented* [4]. Otherwise, if each cofinite submodule of W has a supplement which is a direct summand of W, then W is called *cofinitely* $\oplus$ *−supplemented* [5]. According to [6], a module W is called *radical supplemented* (Rad-supplemented) when each submodule of W has a Rad-supplement in W. In other words, for any submodule T of W, a submodule P of W is named a Rad-supplemented of the submodule T in W if $W = P + T$ and $P \cap T \subseteq Rad(P)$. In reference [7]; radical supplement and radical supplemented modules are called as *generalized supplement* and *generalized supplemented* modules, respectively. By generalizing this definition, cofinite radical supplemented modules are defined. In [8], a module W is called *cofinitely radical supplemented (cofinitely Rad-supplemented)*, if each cofinite submodule of W has a Rad-supplement in W. Besides these, $\oplus$ −radical supplemented modules ($\oplus$ −Rad-supplemented) are studied and defined in [9, 10]. Meanwhile, $\oplus$ −cofinitely radical supplemented modules introduced and examined in [11]. According to this, if each (cofinite) submodule of a module has a Rad-supplement which is a direct summand of itself, then it is called $\oplus$ −(cofinitely) radical supplemented. This definition is given as *generalized* $\oplus$ *−cofinitely supplemented* in [12]. In [11], $cgs^{\oplus}$ −module notation is used briefly instead of $\oplus$ −cofinitely radical supplemented modules and basic fundamental aspects of these modules are examined in there. In this article, we studied another version of $cgs^{\oplus}$ −module by using the concept of "maximal submodule" instead of "cofinite submodule". A maximal submodule of W is a submodule T where there are no other submodules of W that properly contains T, except for W itself. In other words, if T is maximal, there are no larger submodules contained in W that properly extend T. Equivalently, for T, being a maximal submodule of W implies that for any submodule K of W,either K is equal to T or K is equal to W. Also, it is well known that each maximal submodule is cofinite. A module is called a $mgs^{\oplus}$ *−module*, if each maximal submodule of it contains a Rad-supplement that is a direct summand of itself. Since each maximal submodule is a cofinite submodule, this study will be the most general study about this subject in the literature. It will be shown that $mgs^{\oplus}$ −modules and $\oplus$ −cofinite supplemented modules coincide in coatomic modules. It will be later proved that direct sum of $mgs^{\oplus}$ −modules brings out a $mgs^{\oplus}$ −module. Nevertheless, we will prove that the factor module created by the fully invariant submodule of $mgs^{\oplus}$ −module is also the $mgs^{\oplus}$ −module. We will show for a ring that each free S −module is a $mgs^{\oplus}$ −module if and only if S is semiperfect.

$Mgs^{\oplus}$ −Modules

Definition 1. If each maximal submodule of a module has a Rad-supplement which is a direct summand of it, then it is called a $mgs^{\oplus}-$ module.

Lemma 2. Every $cgs^{\oplus}$ −module is a $mgs^{\oplus}$ −module.

Proof. Let W be a $cgs^{\oplus}$ −module and the submodule X be maximal in W. Since every maximal submodule is cofinite submodule, the rest is easy.

Recall from [8] that, $w-$*local module* is a module which has a unique maximal submodule.

Proposition 3. If a $mgs^{\oplus}$ −module W satisfies the condition $Rad(W) \ll W$, then it is $\oplus$ −cofinitely supplemented.

Proof. Consider the submodule X as a maximal of W. Based on the assumption, there are submodules $T$and T_1 of $W$where$W = X + T, X \cap T \subseteq Rad(T)$ and $W = T \oplus T_1$. Hence, we have $X \cap T \subseteq Rad(T) \subseteq Rad(W) \ll W$ and $T \leq_{\oplus} W$. If we consider [1, 19.3(5)], then we can write $X \cap T \ll T$. Therefore W is $\oplus$ −cofinitely supplemented.

Recall from [1] that, if each proper submodule of W is included in a maximal submodule of W and every coatomic module has a small radical, then $W$is said to be *coatomic*. Thus, the following can be given without its proof.

Corollary 4. Let W be a coatomic module. W is $\oplus$ −cofinitely supplemented if and only if it is a $mgs^{\oplus}-$ module.

Proposition 5. Any $w-$local module is a $mgs^{\oplus}-$ module.

Proof. It is easily obtained by combining Proposition 2.3 in [11] and Lemma 2.

For any prime p, the $\mathbb{Z}-$module $\mathbb{Q} \oplus \mathbb{Z}_p$ is $w-$local because $Rad(\mathbb{Q} \oplus \mathbb{Z}_p) \cong \mathbb{Q}$ is a unique submodule of $\mathbb{Q} \oplus \mathbb{Z}_p$. So, $\mathbb{Q} \oplus \mathbb{Z}_p$ is a $mgs^{\oplus}-$ module.

Theorem 6. Any arbitrary $mgs^{\oplus}$ −module with a maximal submodule includes a $w-$local direct summand.

Proof. Let W be a $mgs^{\oplus}-$ module and X be a maximal submodule of it. Then, there are submodules Y, Y_1 of W such that $W = X + Y$, $X \cap Y \subseteq Rad(Y)$ and $W = Y \oplus Y_1$. Also, it can be said that $Y$is a Rad-supplement of X in W. If we consider Lemma 3.3 of [8], then we get Y is $w-$local. Therefore $Y$is a $w-$local direct summand of W.

We point out the sum of whole $w-$local direct summands of W by $wLoc^{\oplus}W$ and the sum of whole $mgs^{\oplus}-$ submodules of W by $Mgs^{\oplus}W$.

Lemma 7. $wLoc^{\oplus}W \leq Mgs^{\oplus}W$, for any module W.

Proof. Let L represent a $w-$local submodule of W where $L \leq_{\oplus} W$. By using Proposition 5, we can say that L is a $mgs^{\oplus}$ −module. Then we get $L < Mgs^{\oplus}W$ and so $wLoc^{\oplus}W \leq Mgs^{\oplus}W$.

Theorem 8. Let any $mgs^{\oplus}-$ submodule of W be a direct summand of W. In that case, $W/_{Mgs^{\oplus}W}$ does not include a maximal submodule if and only if $W$is a $mgs^{\oplus}-$ module.

Proof. (⇐) Suppose that $L/_{Mgs^{\oplus}W}$ is a maximal submodule of $W/_{Mgs^{\oplus}W}$. Then L is the maximal submodule of W . Based on the hypothesis, there are submodules L_1, K of W such that

$$W = L + L_1, \qquad L \cap L_1 \subseteq Rad(L_1) \text{ and } W = L_1 \oplus K.$$

By Lemma 3.3 in [8], $L_1$is a $w-$local module and L_1 is a $mgs^{\oplus}-$ module by Proposition 5.
From here, we can say that $L_1 \subseteq Mgs^{\oplus}W$and so we can write that

$$\begin{aligned} W/_{Mgs^{\oplus}W} &= \left(L/_{Mgs^{\oplus}W}\right) + \left[(L_1 + Mgs^{\oplus}W)/_{Mgs^{\oplus}W}\right] \\ &= \left(L/_{Mgs^{\oplus}W}\right) + \left(Mgs^{\oplus}W/_{Mgs^{\oplus}W}\right). \end{aligned}$$

Consequently, we obtain $W/_{Mgs^{\oplus}W} = L/_{Mgs^{\oplus}W}$ and so $W = L$ which is a contradiction. Hence $W/_{Mgs^{\oplus}W}$ does not include a maximal submodule.

(⇒) Let L be a maximal submodule of W. If L includes the $mgs^{\oplus}-$ modules, then $Mgs^{\oplus}W < L$ can be obtained. Thus $L/_{Mgs^{\oplus}W}$ would be maximal submodule of $W/_{Mgs^{\oplus}W}$, which contradicts the hypothesis. In that case, L does not contain $Mgs^{\oplus}W$ and there is a $mgs^{\oplus}-$submodule X of W where $X \not\subset L$ and $W = X + L$. Remember that $W/_L \cong X/_{X \cap L}$. From here $X \cap L$ is a maximal submodule of X. As X is a $mgs^{\oplus}-$module, there are submodules Y and Y_1 of W where $X = (X \cap L) + Y$, $(X \cap L) \cap Y \subseteq Rad(Y)$ and $X = Y \oplus Y_1$. Therefore, we can obtain that
$W = X + L = (X \cap L) + Y + L = L + Y, L \cap Y = L \cap (X \cap Y) = (L \cap X) \cap Y \subseteq Rad(Y)$.
Since $X \leq_{\oplus} W$,there is a submodule X_1 of $W$with $W = X \oplus X_1$. Therefore
$W = X \oplus X_1 = (Y \oplus Y_1) \oplus X_1 = Y \oplus (Y_1 \oplus X_1)$ and so W is a $mgs^{\oplus}-$ module.

Theorem 9. Any direct sum of $mgs^{\oplus}-$ modules is a $mgs^{\oplus}-$ module.

Proof. Assume that $\{W_i\}_{i\in I}$ is a family of $mgs^{\oplus}-$ modules such that $W = \bigoplus_{i\in I} W_i$ and L is maximal submodule of W. Then we can write $W = L + W_{i_0}$ for $W_{i_0} \subset L$, $i_0 \in I$.
Since $W/_L \cong W_{i_0}/_{L \cap W_{i_0}}$ and $W/_L$ is a simple module, $W_{i_0}/_{L \cap W_{i_0}}$is simple and so $L \cap W_{i_0}$ is a maximal submodule of W_{i_0}.Then, there are submodules X, $X_1$of W_{i_0} where $W_{i_0} = (L \cap W_{i_0}) + X$, $X \cap (L \cap W_{i_0}) \subseteq Rad(X)$ and $W_{i_0} = X \oplus X_1$ because W_{i_0} is a $mgs^{\oplus}$-module. From here, we can obtain that $W = L + W_{i_0} = L + (L \cap W_{i_0}) + X = L + X$ and $L \cap X \subseteq Rad(X)$. Nevertheless, we get $L \leq_{\oplus} W$ since W_{i_0} and X are direct summands of $W$and W_{i_0}, respectively. As a result, $W$is a $mgs^{\oplus}-$ module. Recall from [13] that, a module W has the *summand sum property (SSP)* if the sum of two

direct summands of W is again a direct summand of W. Also, W has the property (D_3), if $X, Y \leq_{\oplus} W$ with $W = X + Y$, then $X \cap Y \leq_{\oplus} W$ [13].

Theorem 10. Let W be a $mgs^{\oplus}$ $-$module which has the property $(D_3)$and (SSP). Then, every maximal direct summand of W is a $mgs^{\oplus}$ $-$ module.

Proof. Let K be a maximal direct summand of W. Then, there is a submodule K_1 of W where $W = K \oplus K_1$ and K_1 is finitely generated. Suppose that T is a maximal submodule of K. Since ${W}/{T} = \left({K}/{T}\right) \oplus K_1$ is finitely generated, T is a maximal submodule of W. Therefore $W$is a $cgs^{\oplus}$ $-$module by Theorem 2.2 in [11] and there are submodules K_1 and $K_2$of W with $W = T + K_1$, $T \cap K_1 \subseteq Rad(K_1)$ and $W = K_1 \oplus K_2$. Note that $W = T + K_1 = K + K_1$. Since $W = K \oplus K_1, W = K_1 \oplus K_2$, $W = K + K_1$ and W has the property (D_3), it can be written that $K \cap K_1 \leq_{\oplus} W$. Therefore, we can write $W = (K \cap K_1) \oplus X$ for a submodule X of W. Hence one can easily get the equality $K = K \cap W = K \cap (T + K_1) = T + (K \cap K_1)$. Besides these, $T \cap (K \cap K_1) = T \cap K_1 \subseteq Rad(K_1) \subseteq Rad(W)$. If one uses the chapter 19.3 in [1], then $T \cap (K \cap K_1) \subseteq Rad(K \cap K_1)$ can be obtained due to $K \cap K_1 \leq_{\oplus} W$. As a result, by taking the intersection of both sides of the equation $W = (K \cap K_1) \oplus X$ with K, we can obtain that $W = (K \cap K_1) \oplus (K \cap X)$ and so K is a $mgs^{\oplus}$ $-$ module.

Corollary 11. Let W be a $mgs^{\oplus}$ $-$ module and $End_S(W)$ has the (SSP). Then, every maximal direct summand of W is a $mgs^{\oplus}$ $-$ module.

Proof. By Theorem 2.3 in [14], Whas (SIP) and (SSP). It well down that any module having (SIP) satisfies the (D_3) condition. Now the proof follows by Theorem 10.

In [15], a submodule L of $W$is called as *fully invariant* if $f(L)$ is included in L for each endomorphism f of W. It is known that $Rad(W)$ and $\tau(W)$ are fully invariant submodules of W.

Theorem 12. Let W be a $mgs^{\oplus}$ $-$ module and $L \leq W$. If L is a fully invariant submodule of W, then ${W}/{L}$ is a $mgs^{\oplus}$ $-$ module.

Proof. Assume that ${T}/{L}$ is a maximal submodule of ${W}/{L}$. Then, T is a maximal submodule of W and so we have submodules X, X_1 of $W$where $W = T + X$, $T \cap X \subseteq Rad(X)$ and $W = X \oplus X_1$ by the hypothesis. Since L is a fully invariant submodule of W, we can write $L = (L \cap X) \oplus (L \cap X_1)$ by Lemma 2.1 in [15]. Moreover, ${(X + L)}/{L}$ is a Rad-supplement of ${T}/{L}$ in ${W}/{L}$ according to Proposition 2.6 in [7]. Then ${W}/{L} = \left[{(X + L)}/{L}\right] \oplus \left[{(X_1 + L)}/{L}\right]$. Consequently, ${(X + L)}/{L}$ is a Rad-supplement of ${T}/{L}$ such that $\left({(X + L)}/{L}\right) \leq_{\oplus} {W}/{L}$ and ${W}/{L}$ is a $mgs^{\oplus}$ $-$ module.

Corollary 13. Let W be a $mgs^{\oplus}$ $-$module. Then ${W}/{Rad(W)}$ and ${W}/{\tau(W)}$ are $mgs^{\oplus}$ $-$ modules.

Proposition 14. Let W be a $cgs^{\oplus}$ $-$module and L be a fully invariant submodule of W. If L is a maximal direct summand of W, then L is a $mgs^{\oplus}$ $-$module.

Proof. Assume that L is a maximal direct summand of W. Then, there is a submodule L_1 of W satisfying $W = L \oplus L_1$. Let T be a maximal submodule of L. As every maximal submodule is cofinite, evidently L/T is finitely generated. Since $W/L \cong L_1$ is simple and so $W/L \cong L_1$ is finitely generated. Hence T is a cofinite submodule of W. Because

$$W/T = (L \oplus L_1)/T = (L/T) \oplus (L_1 \oplus T)/T \cong (L/T) \oplus L_1$$

is finitely generated. By using the hypothesis, one can write $W = T + K,\ T \cap K \subseteq Rad(K)$ and $W = K \oplus K_1$ where $K, K_1 \leq W$. Since L is a fully invariant submodule of W,we can write $L = (L \cap K) \oplus (L \cap K_1)$ by Lemma 2.1 in [15]. By taking the intersection of both sides of the equation $W = T + K$ with L, we can obtain the following equality $L = L \cap W = L \cap (T + K) = T + (L \cap K)$. In addition to these, it can be written that $T \cap (L \cap K) = T \cap K \subseteq Rad(K) \subseteq Rad(W)$. Since $L \cap K \leq_{\oplus} L$ and $L \leq_{\oplus} W$, we can get $L \cap K \leq_{\oplus} W$. By using chapter 2.2.(6) in [6], $T \cap (L \cap K) = T \cap K \subseteq Rad(L \cap K)$ can be written. This implies that L is a $mgs^{\oplus}-$ module.

Theorem 15. Let Wbe a module and $W_1, W_2 \leq W$ such that $W = W_1 \oplus W_2$. Then W_2 is a $mgs^{\oplus}-$module if and only if there is a submodule T of W_2 such that $T \leq_{\oplus} W$, $W = L + T$ and $L \cap T \subseteq Rad(T)$ for each maximal submodule L/W_1 of W/W_1.

Proof. ($\Rightarrow$)Assume that L/W_1 is a maximal submodule of W/W_1. It is well known that $\left(W/W_1\right)/\left(L/W_1\right) \cong W/L$ is simple. Since the following equality

$$W/L = (W_1 + W_2)/L = \big(W_1 + W_2 + (L \cap W_2)\big)/L = (L + W_2)/L \cong W_2/(L \cap W_2)$$

can be written, we get that $L \cap W_2$ is a maximal submodule of W_2. From the hypothesis, we have submodules, T_1 of W_2 such that $W_2 = (L \cap W_2) + T,\ (L \cap W_2) \cap T \subseteq Rad(T)$ and $W_2 = T \oplus T_1$. From here, $W = L + T$ and $L \cap T \subseteq Rad(T)$ can be obtained. Hence $T \leq_{\oplus} W$ because of $T \leq_{\oplus} W_2$.
($\Leftarrow$)Let S be a maximal submodule of W_2 . If one consider the following equality

$$\left(W/W_1\right)/\left((S + W_1)/W_1\right) \cong W/(S + W_1) = (W_1 + W_2)/(S + W_1) = (S + W_1 + W_2)/(S + W_1)$$
$$\cong W_2/[W_2 \cap (S + W_1)] = W_2/[S + (W_1 \cap W_2)] = W_2/S.$$

It can be written that $(S + W_1)/W_1$ is a maximal submodule of W/W_1. By the assumption, since there is a submodule T of W_2 where $T \leq_{\oplus} W_2,\ \ W = T + S + W_1,\ \ (S + W_1) \cap T \subseteq Rad(T)$, it is easy to see that $W_2 = T + S$, $W_2 = T \oplus (W_2 \cap T_1)$ and $S \cap T \subseteq (S + W_1) \cap T \subseteq Rad(T)$. Hence, W_2 is a $mgs^{\oplus} -$ module.

Theorem 16. An arbitrary ring S is semiperfect if and only if every free $S-$module is a $mgs^{\oplus}$ $-$module.

Proof. Firstly, assume that W is an arbitrary free $S-$module. By using Theorem 2.4 in [11], ${}_SS$ is a $cgs^{\oplus} -$ module and so ${}_SS$ is a $mgs^{\oplus} -$ module. Conversely, let a free $S-$module be a $mgs^{\oplus} -$ module. Then ${}_SS$ is a $mgs^{\oplus}$ $-$module. ${}_SS$ is (cofinitely) $\oplus$-supplemented, i.e. S is semiperfect, due to Proposition 3.

Acknowledgment The authors would like to thank the referees for their helpful comments and valuable suggestions for improving the manuscript.

Funding/Financial Disclosure The authors have no received any financial support for the research, authorship, or publication of this study.

Ethics Committee Approval and Permissions The work does not require ethics committee approval and any private permission.

Conflicts of Interest The authors stated that there are no conflict of interest in this article.

Authors Contribution Authors contributed equally to the study.

References

[1] Wisbauer, R. (1991). *Foundations of Module and Ring Theory*. Gordon and Breach Science Publishing, Philadelphia.

[2] Eryılmaz, F. & Sözen, E. Ö. (2023). On a generalization of $\oplus$ $-$co-coatomically supplemented modules. *Honam Mathematical Journal*, *45*(1), 146-159. https://doi:10.5831/hmj.2023.45.1.146

[3] Sözen E. Ö. & Eren, Ş. (2018). Modules that have a generalized δ $-$supplement in every cofinite extension. *JP Journal of Algebra, Number Theory and Applications*, *40*(3), 241-254.

[4] Harmancı, A., Keskin, D. & Smith, P. F. (1999). On $\oplus$ $-$supplemented modules. *Acta Mathematica Hungarica*, *83*(1–2), 161–169.

[5] Çalışıcı, H. & Pancar, A. (2004).$\oplus$ $-$cofinitely supplemented modules. *Czechoslovak Mathematical Journal*, *54*(129), 1083–1088.

[6] Clark, J., Lomp, C., Vajana, N. & Wisbauer, R. (2006). *Lifting Modules*. Birkhauser, Basel.

[7] Wang, Y. & Ding, N. (2006). Generalized supplemented modules. *Taiwanese Journal of Mathematics*, *10*(6), 1589-1601.

[8] Büyükaşık, E. & Lomp,C. (2008). On a recent generalization of semiperfect rings. *Bulletin of Australian Mathematical Society*, *78*, 317-325.

[9] Talebi, Y., Hamzekolaei, A. R. & Tütüncü, D. K. (2009). On Rad-$\oplus$-supplemented modules. *Hadronic Journal*, *32*(5), 505-512.

[10] Talebi, Y. & Mahmoudi, A. R. (2011). On Rad-⊕-supplemented modules. *Thai Journal of Mathematics*, *9*(2), 373–381.

[11] Nişancı, B. & Pancar, A. (2010). On generalization of ⊕ −cofinitely supplemented modules. *Ukrainian Mathematical Journal*, *62*(2), 203-209, https://doi.org/10.1007/s11253-010-0344-4

[12] Koşan, M. T. (2009). Generalized cofinitely semiperfect modules. *International Electronic Journal of Algebra*, *5*(5), 58-69.

[13] Mohamed, S. H & Müller, B. J. (1990). *Continuous and Discrete Modules*. Cambridge University Press, Cambridge.

[14] Garcia, J. L. (1989). Properties of direct summands of modules. *Communications in Algebra*, *17*(1), 73-92, https://doi.org/10.1080/00927878908823714

[15] Özcan, A. Ç., Harmancı, A. & Smith, P. F. (2006). Duo modules. *Glasgow Mathematical Journal*, *48*, 533-545, https://doi.org/10.1017/S0017089506003260