Konuralp Journal of Mathematics
Volume 5 No. 2 pp. 119–130 (2017) ©KJM

UNIQUENESS OF DIFFERENCE-DIFFERENTIAL POLYNOMIALS OF ENTIRE FUNCTIONS SHARING ONE SMALL FUNCTION

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Abstract. In this paper, we investigate the uniqueness problem of difference-differential polynomials sharing a small function with finite weight. The results of the paper improve and generalize the recent results due to Pulak Sahoo and the present author [Applied Mathematics E-Notes 16(2016), 33-44]

1. Introduction, Definitions and Results

In the paper, by a meromorphic function we shall always mean a meromorphic function in the whole complex plane. We assume that the reader is familiar with the basic notions of Nevanlinna value distribution theory [see [7, 10, 15]]. Let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}(r \to \infty, r \notin E)$. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$, by $S(r)$ any quantity satisfying $S(r) = o\{T(r)\}$ $(r \to \infty, r \notin E)$.

Let f and g be two nonconstant meromorphic functions. We say that f and g share the value a CM (counting multiplicities), if $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share the value a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. A meromorphic function $\alpha(\not\equiv 0, \infty)$ is called a small function with respect to f, if $T(r, \alpha) = S(r, f)$

Recently difference polynomials in the complex plane $\mathbb{C}$ become a subject of great interest among the researcher around the world. With the development of difference analogue of Nevanlinna theory[see [3, 4, 5, 6]], a large number of papers have focused

Date: March 11, 2017 and, in revised form, May 31, 2017.

2010 *Mathematics Subject Classification.* 30D35, 39A10.

Key words and phrases. Uniqueness, Entire functions, Difference-differential polynomial, Weighted Sharing.

The author is grateful to the referee for reading the manuscript carefully and making a number of valuable comments and suggestions for the improvement of the paper.

on value distribution and uniqueness of difference polynomials. In 2007, I. Laine and C.C. Yang [11] proved the following result for difference polynomials.

Theorem 1.1. *Let f be a transcendental entire function of finite order and c be a nonzero complex constant. Then for $n \geq 2$, $f^n(z)f(z+c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

In 2010, X.G. Qi, L.Z. Yang and K. Liu [13] proved the following uniqueness result which corresponded to Theorem 1.1.

Theorem 1.2. *Let f and g be two transcendental entire functions of finite order and c be a nonzero complex constant, and let $n \geq 6$ be an integer. If $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share the value 1 CM, then either $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.*

In 2012 L. Kai, L. Xin-ling, C. Ting-bin [8] considered the difference differential polynomials and proved the following results.

Theorem 1.3. *Let $f(z)$ be a transcendental entire function of finite order. If $n \geq k+2$, then the difference-differential polynomial $[f^n(z)f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.*

Theorem 1.4. *Let f and g be two transcendental entire functions of finite order $n \geq 2k+6$ and c is a nonzero complex constant. If $[f^n(z)f(z+c)]^{(k)}$ and $[g^n(z)g(z+c)]^{(k)}$ share the value 1 CM, then either $f(z) = c_1 e^{Cz}$, $g(z) = c_2 e^{-Cz}$, where c_1, c_2 and C are constant satisfying $(-1)^k (c_1 c_2)^{n+1}((n+1)C)^{2k} = 1$, or $f = tg$ for a constant such that $t^{n+1} = 1$.*

In the same direction J.L. Zhang [17] investigated the value distribution and uniqueness of difference polynomials of entire functions and obtained the following results.

Theorem 1.5. *Let f be a transcendental entire function of finite order $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to f and c be a nonzero complex constant. If $n \geq 2$ is an integer, then $f^n(z)(f(z)-1)f(z+c) - \alpha(z)$ has infinitely many zeros.*

In the same paper the author also proved the following uniqueness result which corresponds to Theorem 1.5.

Theorem 1.6. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to both f and g. Suppose that c is a nonzero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f = g$.*

In 2013, S.S. Bhoosnurmath and S.R. Kabbur [2] considered the zeros of difference polynomial of the form $f^n(z)(f^m(z)-1)f(z+c)$; where n, m are positive integers and c is a nonzero complex constant and obtained the following theorem.

Theorem 1.7. *Let f be an entire function of finite order and $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to f. Suppose that c is a nonzero complex constant and n, m are positive integers. If $n \geq 2$, then $f^n(z)(f^m(z)-1)f(z+c) - \alpha(z)$ has infinitely many zeros.*

The following two theorems are the uniqueness results corresponding to Theorem 1.7 proved by S.S. Bhoosnurmath and S.R. Kabbur [2].

Theorem 1.8. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to f and g. Suppose that c is a nonzero complex constant and n, m are positive integers such that $n \geq m+6$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f = tg$ where $t^m = 1$.*

Theorem 1.9. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to f and g. Suppose that c is a nonzero complex constant and n, m are positive integers satisfying $n \geq 4m+12$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share $\alpha(z)$ IM, then $f = tg$ where $t^m = 1$.*

An increment to uniqueness theory has been considering weighted sharing instead of sharing IM or CM, this implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing has been introduced by I. Lahiri around 2001, which measure how close a shared value is to being shared CM or to being shared IM. The definition are as follows.

Definition 1.1. ([9]) Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write *f, g share* (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

If α is a small function of f and g, then f, g share α with weight k means that $f-\alpha$, $g-\alpha$ share the value 0 with weight k.

In 2016, Pulak Sahoo and the present author [14] considered the differential polynomial of the form $(f^n(z)(f^m(z)-1)f(z+c))^{(k)}$ and proved the following results.

Theorem 1.10. *Let f be a transcendental entire function of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that c is a nonzero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers. If $n \geq k+2$, then*

$$(f^n(z)(f^m(z)-1)f(z+c))^{(k)} - \alpha(z)$$

has infinitely many zeros.

Theorem 1.11. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a nonzero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+6$. If $(f^n(z)(f^m(z)-1)f(z+c))^{(k)}$ and $(g^n(z)(g^m(z)-1)g(z+c))^{(k)}$ share $(\alpha, 2)$ then $f = tg$ where $t^m = 1$.*

Theorem 1.12. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a* *nonzero complex constant,* $n(\geq 1)$, $m(\geq 1)$ *and* $k(\geq 0)$ *are integers satisfying* $n \geq 5k + 4m + 12$. *If* $(f^n(z)(f^m(z) - 1)f(z + c))^{(k)}$ *and* $(g^n(z)(g^m(z) - 1)g(z + c))^{(k)}$ *share* $\alpha(z)$ *IM then* $f = tg$ *where* $t^m = 1$.

Regarding Theorems 1.10, 1.11 and 1.12 the following questions are inevitable which is the motive of the author.

Question. *What can be said if we consider the difference-differential polynomials of the form* $(f^n(z)P(f)f(z+c))^{(k)}$, *where* $P(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$, *where* $a_0(\neq 0)$, a_1, ... , a_{m-1}, $a_m(\neq 0)$ *are complex constants and* $k(\geq 0)$ *is an integer ?*

We will concentrate our attention to the above question and provide an affirmative answer in this direction. Indeed the following theorems which are the main results of the paper justify our claim.

Theorem 1.13. *Let* f *be a transcendental entire function of finite order and* $\alpha(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f. *Suppose that* c *is a nonzero complex constant,* $n(\geq 1)$, $m(\geq 1)$ *and* $k(\geq 0)$ *are integers and let* $P(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$, *where* $a_0(\neq 0)$, a_1, ... , a_{m-1}, $a_m(\neq 0)$ *are complex constants. If* $n \geq k + 2$, *then* $(f^n(z)P(f)f(z + c))^{(k)} - \alpha(z)$ *has infinitely many zeros.*

Theorem 1.14. *Let* f *and* g *be two transcendental entire functions of finite order and* $\alpha(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f *and* g *and let* $P(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$, *where* $a_0(\neq 0)$, a_1, ... , a_{m-1}, $a_m(\neq 0)$ *are complex constants. Suppose that* c *is a nonzero complex constant,* $n(\geq 1)$, $m(\geq 1)$ *and* $k(\geq 0)$ *are integers satisfying* $n \geq 2k + m + 6$. *If* $(f^n(z)P(f)f(z + c))^{(k)}$ *and* $(g^n(z)P(g)g(z + c))^{(k)}$ *share* $(\alpha, 2)$ *then* $f = tg$ *for a constant* t *such that* $t^d = 1$, *where* $d = \gcd(n + m + 1, ..., n + m + 1 - i, ..., n + 1)$ *and* $i = 0, 1, 2, ..., m$.

Remark 1.1. Theorem 1.14 improves and generalizes Theorem 1.11.

Theorem 1.15. *Let* f *and* g *be two transcendental entire functions of finite order and* $\alpha(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f *and* g *and let* $P(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$, *where* $a_0(\neq 0)$, a_1, ... , a_{m-1}, $a_m(\neq 0)$ *are complex constants. Suppose that* c *is a nonzero complex constant,* $n(\geq 1)$, $m(\geq 1)$ *and* $k(\geq 0)$ *are integers satisfying* $n \geq 5k + 4m + 12$. *If* $(f^n(z)P(f)f(z + c))^{(k)}$ *and* $(g^n(z)P(g)g(z + c))^{(k)}$ *share* $\alpha(z)$ *IM then* $f = tg$ *for a constant* t *such that* $t^d = 1$, *where* $d = \gcd(n + m + 1, ..., n + m + 1 - i, ..., n + 1)$ *and* $i = 0, 1, 2, ..., m$.

Remark 1.2. Theorem 1.15 improves and generalizes Theorem 1.12.

2. Lemmas

Let F and G be two nonconstant meromorphic functions defined in the open, complex plane $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 2.1. [3] *Let* $f(z)$ *be a transcendental meromorphic function of finite order then*

$$T(r, f(z + c)) = T(r, f) + S(r, f).$$

Lemma 2.2. [15] *Let f be a nonconstant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be small functions of f. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.3. [3, 4] *Let $f(z)$ be a meromorphic function of finite order and c is a nonzero complex constant. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Lemma 2.4. [7, 16] *Let $f(z)$ be a nonconstant meromorphic function and $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f)$, $i = 1, 2$. Then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f).$$

Lemma 2.5. [12] *Let f be a meromorphic function of finite order and let $c(\not\equiv 0)$ be a fixed nonzero complex constant. Then*

$$\overline{N}(r, \infty, f(z+c)) \leq \overline{N}(r, \infty, f) + S(r, f),$$

outside a possible exceptional set of finite logarithmic measure.

Lemma 2.6. [18] *Let f be a nonconstant meromorphic function and p, k be two positive integers. Then*

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.1}$$

and

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.2}$$

Lemma 2.7. [9] *Let f and g be two nonconstant meromorphic functions sharing $(1, 2)$. Then one of the following three cases hold:*
(i) $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$,
(ii) $f = g$,
(iii) $fg = 1$,
Where $T(r) = \max\{T(r, f), T(r, g)\}$ *and* $S(r) = o\{T(r)\}$.

Lemma 2.8. [1] *Let F and G be two nonconstant meromorphic functions sharing the value 1 IM and $H \not\equiv 0$. Then*
$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, F) + S(r, G)$,
and the same inequality holds for $T(r, G)$.

Lemma 2.9. *Let $f(z)$ be a transcendental entire function of finite order and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$, where $a_0(\neq 0)$, a_1, ... , a_{m-1}, $a_m(\neq 0)$ are complex constants. Let $F = f(z)^n P((f)) f(z+c)$. Then*

$$T(r, F) = (n + m + 1)T(r, f) + S(r, f). \tag{2.3}$$

Proof. Since f is entire function of finite order. We deduce from Lemma 2.3 and the standard Valiron Mohon'ko theorem that,

$$\begin{aligned} (n + m + 1)T(r, f(z)) &= T(r, f(z)^{n+1} P(f)) + S(r, f) \\ &= m(r, f(z)^{n+1} P(f)) + S(r, f) \end{aligned}$$

or,

$$\begin{aligned}(n+m+1)T(r,f(z)) &= m\left(r,\frac{f(z)^{n+1}P(f)}{f(z)^nP((f))f(z+c)}\right)+m(r,F)+S(r,f)\\ &= m\left(r,\frac{f(z)}{f(z+c)}\right)+m(r,F)+S(r,f)\\ &= T(r,F)+S(r,f)\end{aligned}$$

Therefore we have

$$(n+m+1)T(r,f(z)) \leq T(r,F)+S(r,f) \tag{2.4}$$

On the other hand, by Lemma 2.1 and the fact that f is a transcendental entire function of finite order we get

$$\begin{aligned}T(r,F) &\leq T(r,f(z)^nP(f))+T(r,f(z+c))+S(r,f)\\ &= (n+m)T(r,f(z))+T(r,f(z+c))+S(r,f)\\ &\leq (n+m+1)T(r,f(z))+S(r,f)\end{aligned}$$

$$i.e., \qquad T(r,F) \leq (n+m+1)T(r,f(z))+S(r,f). \tag{2.5}$$

Thus (2.3) follows from (2.4) and (2.5).

□

Lemma 2.10. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, let n, k be two positive integers with $n > k+2$ and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to f and g and let $P(z) = a_mz^m + a_{m-1}z^{m-1} + \dots + a_1z + a_0$, where a_0, a_1, ... , a_{m-1}, a_m are complex constants. If*

$$(f^n(z)P(f)f(z+c))^{(k)}(g^n(z)P(g)g(z+c))^{(k)} \equiv \alpha^2,$$

then $P(z)$ is reduced to a nonzero monomial, that is $P(z) = a_iz^i \neq 0$ for some $i = 0,1,2,...,m$.

Proof. If $P(z)$ is not reduced to a nonzero monomial, then without loss of generality, we assume that $P(z) = a_mz^m + a_{m-1}z^{m-1} + \dots + a_1z + a_0$, where $a_0(\neq 0)$, a_1, ... , a_{m-1}, $a_m(\neq 0)$ are complex constants. Since

$$(f^n(z)P(f)f(z+c))^{(k)}(g^n(z)P(g)g(z+c))^{(k)} \equiv \alpha^2. \tag{2.6}$$

From $n > k+2$ and the assumption that $f(z)$ and $g(z)$ are two transcendental entire functions we deduce by (2.6) that

$$f(z) \neq 0, g(z) \neq 0.$$

Let $f(z) = e^{\beta(z)}$, where $\beta(z)$ is an entire function. Thus, by induction we have

(2.7)

$$[a_if^{i+n}f(z+c)]^{(k)} = P_i\big(\beta^{'},....\beta^{(k)},\beta^{'}(z+c),....\beta^{(k)}(z+c)\big)e^{(i+n)\beta}e^{\beta(z+c)}.$$

where P_i $(i = 1,2,...,m)$ are difference-differential polynomials. Obviously $P_0 \neq 0,...,P_m \neq 0$, where if $a_i \neq 0$ for some $i \in \{0,1,...,m-1\}$, then $P_i \not\equiv 0$. Since $g(z)$ is an entire function, we get from (2.6) that $(f^n(z)P(f)f(z+c))^{(k)} \neq 0$. Thus by (2.7) we have

$$P_me^{m\beta} + \dots + P_0 \not\equiv 0. \tag{2.8}$$

Since $\beta(z)$ and $\beta(z+c)$ are entire function, we obtain

$$T(r,P_i)=S(r,f) \tag{2.9}$$

and

$$N(r,0;P_me^{m\beta}+...+P_0)=S(r,f). \tag{2.10}$$

Thus by (2.8), (2.9), (2.10) and Lemma 2.2 and Lemma 2.4, we get

$$\begin{aligned} mT(r,f) &= T(r,P_me^{m\beta}+...+P_1e^{\beta})+S(r,f)\\ &\le \overline{N}(r,0;P_me^{m\beta}+...+P_1e^{\beta})+\overline{N}(r,0;P_me^{m\beta}+...+P_1e^{\beta}+P_0)\\ &\quad +S(r,f)\\ &\le \overline{N}(r,0;P_me^{(m-1)\beta}+...+P_2e^{\beta}+P_1)+S(r,f)\\ &\le (m-1)T(r,f)+S(r,f) \end{aligned}$$

which is a contradiction. This shows that $P(z)$ is reduced to a nonzero monomial, that is $P(z)=a_iz^i\neq 0$ for some $i=0,1,2,...,m$. This completes the proof of the lemma. □

Lemma 2.11. *Let f and g be two entire functions and let $P(z)=a_mz^m+a_{m-1}z^{m-1}+...+a_1z+a_0$, where $a_0(\neq 0)$, a_1, ... , a_{m-1}, $a_m(\neq 0)$ are complex constants and $k(\geq 0)$, be integer. let $F=(f^n(z)P(f)f(z+c))^{(k)}$, $G=(g^n(z)P(g)g(z+c))^{(k)}$. If there exists nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F)=\overline{N}(r,0;G)$ and $\overline{N}(r,c_2;G)=\overline{N}(r,0;F)$, then $n\leq 2k+m+3$.*

Proof. We put $F_1=f^n(z)P(f)f(z+c)$, $G_1=g^n(z)P(g)g(z+c)$. By the second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r,F) &\le \overline{N}(r,0;F)+\overline{N}(r,c_1;F)+S(r,F)\\ &\le \overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F). \end{aligned} \tag{2.11}$$

Using (2.11), Lemmas 2.6 and 2.9, we obtain

$$\begin{aligned} (n+m+1)T(r,f) &\le T(r,F)-\overline{N}(r,0;F)+N_{k+1}(r,0;F_1)\\ &\quad +S(r,f)\\ &\le \overline{N}(r,0;G)+N_{k+1}(r,0;F_1)+S(r,f)\\ &\le N_{k+1}(r,0;F_1)+N_{k+1}(r,0;G_1)\\ &\quad +S(r,f)+S(r,g)\\ &\le (k+m+2)(T(r,f)+T(r,g))\\ &\quad +S(r,f)+S(r,g). \end{aligned} \tag{2.12}$$

Similarly,

$$\begin{aligned} (n+m+1)T(r,g) &\le (k+m+2)(T(r,f)+T(r,g))\\ &\quad +S(r,f)+S(r,g). \end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13) we obtain

$$(n-2k-m-3)(T(r,f)+T(r,g))\le S(r,f)+S(r,g),$$

which gives $n\leq 2k+m+3$. This proves the lemma. □

Lemma 2.12. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order. let n, m be two positive integers with $n \geq m+5$ and $P(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$, where a_0, a_1, ... , a_{m-1}, a_m are complex constants. If*

$$(f^n(z)P(f)f(z+c)) = (g^n(z)P(g)g(z+c)) \tag{2.14}$$

then $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = \gcd(n+m+1, ..., n+m+1-i, ..., n+1)$ and $i = 0, 1, 2, ..., m$.

Proof. Let $h(z) = \frac{f(z)}{g(z)}$. If $h(z)^{n+m}h(z+c) \neq 1$ then from (2.14) we have

$$\begin{aligned}(gh)^n[a_m(gh)^m + a_{m-1}(gh)^{m-1} + ... + a_0]g(z+c)h(z+c)\\ = g^n(a_m g^m + ... + a_0)g(z+c)\end{aligned}$$

i.e.,

$$g^{n+m}[a_m(h^{m+n}(z)h(z+c) - 1)] + ... + g^n[a_0(h^n(z)h(z+c) - 1)] = 0$$

i.e.,

(2.15)

$$g^m = \frac{-[g^{m-1}[a_{m-1}(h^{m+n-1}(z)h(z+c) - 1)] + ... + [a_0(h^n(z)h(z+c) - 1]]}{a_m(h^{m+n}(z)h(z+c) - 1)}.$$

Denote,

$$h(z)^{n+m}h(z+c) = H. \tag{2.16}$$

We have

$$T(r, H) = (n+m+1)T(r, h) + S(r, h).$$

If 1 is a Picard exceptional value of H, applying the Nevanlinna second main theorem with Lemma 2.1, we get

$$\begin{aligned} T(r, H) &\leq \overline{N}(r, \infty; H) + \overline{N}(r, 0; H) + \overline{N}(r, 1; H) + S(r, H)\\ &\leq 2T(r, h) + 2T(r, h) + S(r, h)\end{aligned}$$

i.e.,

$$(n+m+1)T(r, h) \leq 4T(r, h) + S(r, h),$$

which is contradiction to $n \geq m+5$.

Therefore 1 is not a Picard exceptional value of H. Thus there exists z_0 such that $h(z_0)^{n+m}h(z_0+c) = 1$ then by (2.15), we have $h^d(z_0) = 1$, where $d = \gcd(n+m+1, ..., n+m+1-i, ..., n+1)$ and $i = 0, 1, 2, ..., m$. Then

$$\overline{N}(r, 1; H) \leq \overline{N}(r, 1; h^d(z)) \leq dT(r, h) + O(1) \leq dT(r, h) + O(1) \tag{2.17}$$

Applying the second fundamental theorem to H and using lemma 2.1 and (2.16) and (2.17), we get

$$\begin{aligned} T(r, H) &\leq \overline{N}(r, H) + \overline{N}(r, 0; H) + \overline{N}(r, 1; H) + S(r, h)\\ &\leq \overline{N}(r, H) + \overline{N}(r, 0; H) + mT(r, H) + S(r, h)\\ &\leq (4+m)T(r, h) + S(r, h)\end{aligned}$$

Regarding this we have

$$\begin{aligned}(n+m)T(r,h) &= T(r,h^{n+m}(z)) = T\left(r,\frac{H}{h(z+c)}\right)\\ &\le T(r,H)+T(r,h(z+c))+\ O(1)\\ &\le (4+m)T(r,h)+T(r,h)+S(r,h)\\ &= (5+m)T(r,h)+S(r,h)\end{aligned}$$

which contradicts our hypothesis, $n \ge m+5$. Therefore, $h(z)^{n+m}h(z+c) \equiv 1$, then from (2.15), we get $h(z)^n h(z+c) \equiv 1 \Rightarrow h^d(z) \equiv 1$. Hence, we get $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = \gcd(n+m+1, ..., n+m+1-i, ..., n+1)$ and $i = 0, 1, 2, ..., m$. This completes the proof of the Lemma 2.12.

□

3. Proof of the Theorems

Proof of Theorem 1.13. Let $F_1 = f^n(z)P(f)f(z+c))$. Then F_1 is a transcendental entire function. If possible, we assume that $F_1^{(k)} - \alpha(z)$ has only finitely many zeros. Then we have

$$N(r,\alpha;F_1^{(k)}) = O\{logr\} = S(r,f). \tag{3.1}$$

Using (2.1), (3.1) and Nevanlinna's three small function theorem we obtain

$$\begin{aligned}T(r,F_1^{(k)}) &\le \overline{N}(r,0;F_1^{(k)}) + \overline{N}(r,\alpha;F_1^{(k)}) + S(r,f)\\ &\le T(r,F_1^{(k)}) - T(r,F_1) + N_{k+1}(r,0;F_1) + S(r,f).\end{aligned} \tag{3.2}$$

Applying Lemma 2.6 we obtain from (3.2)

$$\begin{aligned}(n+m+1)T(r,f) &\le N_{k+1}(r,0;F_1) + S(r,f)\\ &\le (k+m+2)T(r,f) + S(r,f).\end{aligned}$$

This gives

$$(n-k-1)T(r,f) \le S(r,f),$$

a contradiction with the assumption that $n \ge k+2$. This proves the theorem 1.13. □

Proof of Theorem 1.14. Let $F_1 = f^n(z)P(f)f(z+c)$, $G_1 = g^n(z)P(g)g(z+c)$, $F = \frac{F_1^{(k)}}{\alpha(z)}$ and $G = \frac{G_1^{(k)}}{\alpha(z)}$. Then F and G are transcendental meromorphic functions that share $(1,2)$ except the zeros and poles of $\alpha(z)$. Using (2.1) and Lemma 2.9 we get

$$\begin{aligned}N_2(r,0;F) &\le N_2(r,0;(F_1)^{(k)}) + S(r,f)\\ &\le T(r,(F_1)^{(k)}) - (n+m+1)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f)\\ &\le T(r,F) - (n+m+1)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f).\end{aligned}$$

From this we get

$$\begin{aligned}(n+m+1)T(r,f) \le\ & T(r,F) + N_{k+2}(r,0;F_1) - N_2(r,0;F)\\ & + S(r,f).\end{aligned} \tag{3.3}$$

Again by (2.2) we have

$$N_2(r,0;F) \le N_2(r,0;F_1^{(k)}) + S(r,f) \le N_{k+2}(r,0;F_1) + S(r,f). \tag{3.4}$$

Suppose, if possible, that (i) of Lemma 2.7 holds. Then using (3.4) we obtain from (3.3)

(3.5)

$$\begin{aligned}(n+m+1)T(r,f) &\le N_2(r,0;G) + N_2(r,1;F) + N_2(r,1;G) + N_{k+2}(r,0;F_1)\\ &\quad +S(r,f) + S(r,g)\\ &\le N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + S(r,f) + S(r,g)\\ &\le (k+m+3)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g).\end{aligned}$$

In a similar manner we obtain

$$\begin{aligned}(n+m+1)T(r,g) &\le (k+m+3)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g).\end{aligned} \tag{3.6}$$

(3.5) and (3.6) together gives $(n-2k-m-5)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)$, contradicting with the fact that $n \ge 2k+m+6$. Therefore, by Lemma 2.7 we have either $FG = 1$ or $F = G$. Let $FG = 1$. By assumption that $a_m \ne 0$, $a_n \ne 0$, we can arrive at a contradiction by Lemma 2.10.

Therefore, we must have $F = G$, and then

$$(f^n(z)P(f)f(z+c))^{(k)} = (g^n(z)P(g)g(z+c))^{(k)},$$

Integrating above we obtain

$$(f^n(z)P(f)f(z+c))^{(k-1)} = (g^n(z)P(g)g(z+c))^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \ne 0$, using Lemma 2.11 it follows that $n \le 2k+m+1$, a contradiction. Hence $c_{k-1} = 0$. Repeating the process k-times, we deduce that

$$f^n(z)P(f)f(z+c) = g^n(z)P(g)g(z+c),$$

which by Lemma 2.12 gives $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = \gcd(n+m+1, ..., n+m+1-i, ..., n+1)$ and $i = 0,1,2,...,m$. This proves Theorem 1.14.

□

Proof of Theorem 1.15. Let F, G, F_1 and G_1 be defined as in the proof of Theorem 1.14. Then F and G are transcendental meromorphic functions that share the value 1 IM except the zeros and poles of $\alpha(z)$. We assume, if possible, that $H \not\equiv 0$. Using Lemma 2.8 and (3.4) we obtain from (3.3)

(3.7)

$$\begin{aligned}(n+m+1)T(r,f) &\leq N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)\\&\quad+\overline{N}(r,0;G)+N_{k+2}(r,0;F_1)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\&\quad+S(r,f)+S(r,g)\\&\leq N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+2N_{k+1}(r,0;F_1)\\&\quad+N_{k+1}(r,0;G_1)+S(r,f)+S(r,g)\\&\leq (3k+3m+7)T(r,f)+(2k+2m+5)T(r,g)\\&\quad+S(r,f)+S(r,g)\\&\leq (5k+5m+12)T(r)+S(r).\end{aligned}$$

Similarly,

$$(n+m+1)T(r,g)\leq(5k+5m+12)T(r)+S(r). \tag{3.8}$$

(3.7) and (3.8) together yields

$$(n-5k-4m-11)T(r)\leq S(r),$$

which is a contradiction with the assumption that $n\geq 5k+4m+12$. We now assume that $H\equiv 0$. Then

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)=0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{3.9}$$

where $A(\neq 0)$ and B are constants. From (3.9) it is obvious that F, G share the value 1 CM and hence they share $(1,2)$. Therefore $n\geq 2k+m+6$. We now discuss the following three cases separately.

Case 1. Suppose that $B\neq 0$ and $A=B$. Then from (3.9) we obtain

$$\frac{1}{F-1}=\frac{BG}{G-1}. \tag{3.10}$$

If $B=-1$, then from (3.10) we obtain

$$FG=1,$$

which is a contradiction by Lemma 2.10.

If $B\neq -1$, from (3.10), we have $\frac{1}{F}=\frac{BG}{(1+B)G-1}$ and so $\overline{N}(r,\frac{1}{1+B};G)=\overline{N}(r,0;F)$. Using (2.1), (2.2) and the second fundamental theorem of Nevanlinna, we deduce that

$$\begin{aligned}T(r,G) &\leq \overline{N}(r,0;G)+\overline{N}\left(r,\frac{1}{B+1};G\right)+\overline{N}(r,\infty;G)+S(r,G)\\&\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,G)\\&\leq N_{k+1}(r,0;F_1)+T(r,G)+N_{k+1}(r,0;G_1)\\&\quad-(n+m+1)T(r,g)+S(r,g).\end{aligned}$$

This gives

$$(m+n+1)T(r,g)\leq(k+m+2)\{T(r,f)+T(r,g)\}+S(r,g).$$

In a similar manner we can get

$$(m+n)T(r,f)\leq(k+m+2)T(r,f)+(2k+m+1)T(r,g)+S(r,g).$$

Thus we obtain

$$(n-2k-m-3)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

which is a contradiction as $n \geq 2k+m+6$.

Case 2. Let $B \neq 0$ and $A \neq B$. Then from (3.9) we get $F = \frac{(B+1)G-(B-A+1)}{BG(A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r, 0; F)$. Proceeding in a manner similar to case 1 we can arrive at a contradiction.

Case 3. Let $B = 0$ and $A \neq 0$. Then from (3.9) we get $F = \frac{G+A-1}{A}$ and $G = AF-(A-1)$. If $A \neq 1$, it follows that $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, 1-A; G) = \overline{N}(r, 0; F)$. Now applying Lemma 2.11 it can be shown that $n \leq 2k+m+3$, which is a contradiction. Thus $A = 1$ and then $F = G$. Now the result follows from the proof of Theorem 1.14. This completes the proof of Theorem 1.15.

□

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