Konuralp Journal of Mathematics
Volume 5 No. 2 pp. 146–159 (2017) ©KJM

A NEW DEFINITION AND PROPERTIES OF QUANTUM INTEGRAL WHICH CALLS $\overline{q}$-INTEGRAL

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Abstract. In this paper, we present a new definition of q-integral by using trapezoid pieces and we name second sense q-integral which is showed $\overline{q}$-integral and we give some results and properties of $\overline{q}$-integral. Finaly, we establish some new $\overline{q}$-Hermite-Hadamard type inequalities for convex functions.

1. Introduction

Quantum calculus is the modern name for the investigation of calculus without limits.The quantum calculus or q-calculus began with FH Jackson in the early twentieth century, but this kind of calculus had already been worked out by Euler and Jacobi. Recently it arose interest due to high demand of mathematics that models quantum computing. q-calculus appeared as a connection between mathematics and physics. It has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and other sciences quantum theory, mechanics and the theory of relativity.

Many of the fundamental aspects of quantum calculus. It has been shown that quantum calculus is a subfield of the more general mathematical field of time scales calculus. Time scales provide a unified framework for studying dynamic equations on both discrete and continuous domains. In studying quantum calculus, we are concerned with a specific time scale, called the q-time scale, defined as follows: $T := q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0\}$ see [1]-[5], [7]-[13] and references cited therein.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard's inequality [6], due to its rich geometrical significance and applications, which is stated as follows:

Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$, be a convex mapping and $a, b \in J$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

Date: August 10, 2017 and, in revised form, October 14, 2017.
2010 *Mathematics Subject Classification.* 34A08, 26D10, 26D15.
Key words and phrases. Hermite-Hadamard's inequalities, convex functions, q-integrals.

Both the inequalities hold in reversed direction if f is concave. Since its discovery, Hermite-Hadamard's inequality has been considered the most useful inequality in mathematical analysis. This inequality has been extended in a number of ways and a number of papers have been written.

The main aim of this paper is to establish some new quantum integral inequalities for midpoint formula on convex functions. Many consequences of Hermite-Hadamard type inequalities are obtained as special cases when $q \to 1$.

2. Preliminaries

In this section, we give definition q-derivates. Let $J := [a, b] \subset \mathbb{R}$, $J^\circ := (a, b)$ be interval and $0 < q < 1$ be a constant. We define q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a, b]$ as follows.

Definition 2.1. Assume $f : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then the expression

$$\begin{aligned} {}_aD_q\ f(x) &= \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a, \\ {}_aD_q\ f(a) &= \lim_{x \to a} {}_aD_q\ f(x) \end{aligned} \tag{2.1}$$

is called the q-derivative on J of function f at x.

We say that f is q -differentiable on J provided ${}_aD_q\ f(x)$ exists for all $x \in J$. Note that if $a = 0$ in (2.1), then ${}_0D_q\ f = D_q f$, where D_q is the well-known q-derivative of the function $f(x)$ defined by

$$D_q\ f(x) = \frac{f(x) - f(qx)}{(1-q)x}$$

For more details, see [7].

Lemma 2.1. [12] *Let $\alpha \in \mathbb{R}$, then we have*

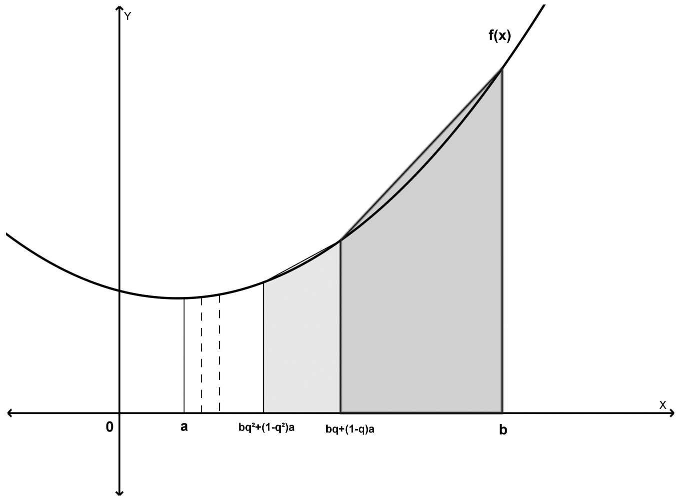
$${}_aD_q\ (x-a)^\alpha = \left(\frac{1-q^\alpha}{1-q}\right)(x-a)^{\alpha-1}. \tag{2.2}$$

3. Definition and Properties of $\overline{q}$-integral

In this section, we present a new definition of q-integral by using trapezoid pieces and we name second sense q-integral which is showed $\overline{q}$-integral and we present some results and properties of $\overline{q}$-integral. Let $J := [a, b] \subset \mathbb{R}$, $J^\circ := (a, b)$ be interval and $0 < q < 1$ be a constant.

Remember that the definition of an integral may be phrased in terms of Riemann sums, each part of some width, h. Let us replace the strips of length h with strips between $x = q^{n+1}b + \left(1 - q^{n+1}\right)a$ and $x = q^n b + (1 - q^n)a$, which means that the strips become thinner and thinner $x \to a$, as the figure below. Let's show that

$$A_n = (1-q)q^n(b-a)\frac{f\left(q^{n+1}b + \left(1-q^{n+1}\right)a\right) + f\left(q^n b + (1-q^n)a\right)}{2}.$$



The integral is simply the sum of each of the trapezoid pieces gives that

$$
\begin{aligned}
\sum_{n=0}^{\infty} A_n &= \frac{(1-q)(b-a)}{2}\left[\sum_{n=0}^{\infty} q^n f\left(q^{n+1}b+\left(1-q^{n+1}\right)a\right)+\sum_{n=0}^{\infty} q^n f\left(q^n b+\left(1-q^n\right)a\right)\right] \\
&= \frac{(1-q)(b-a)}{2}\left[\frac{1}{q}\sum_{n=0}^{\infty} q^{n+1} f\left(q^{n+1}b+\left(1-q^{n+1}\right)a\right)+\sum_{n=0}^{\infty} q^n f\left(q^n b+\left(1-q^n\right)a\right)\right] \\
&= \frac{(1-q)(b-a)}{2}\left[\frac{1}{q}\sum_{n=1}^{\infty} q^n f\left(q^n b+\left(1-q^n\right)a\right)+\sum_{n=0}^{\infty} q^n f\left(q^n b+\left(1-q^n\right)a\right)\right] \\
&= \frac{(1-q)(b-a)}{2}\left[\frac{1}{q}\left\{f(b)-f(b)+\sum_{n=1}^{\infty} q^n f\left(q^n b+\left(1-q^n\right)a\right)\right\}+\sum_{n=0}^{\infty} q^n f\left(q^n b+\left(1-q^n\right)a\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1-q)(b-a)}{2}\left[\frac{1}{q}\left\{-f(b)+\sum_{n=0}^{\infty} q^n f\left(q^n b+\left(1-q^n\right)a\right)\right\}+\sum_{n=0}^{\infty} q^n f\left(q^n b+\left(1-q^n\right)a\right)\right] \\
&= \frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f\left(q^n b+\left(1-q^n\right)a\right)-f(b)\right] \\
&= \int_a^b f(s)\ {}_a d_{\overline{q}} s\ .
\end{aligned}
$$

Definition 3.1. Let $f : J \to \mathbb{R}$ is continuous function. For $0 < q < 1$

$$
\int_a^b f(s)\ {}_a d_{\overline{q}} s = \frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f\left(q^n b+\left(1-q^n\right)a\right)-f(b)\right] \tag{3.1}
$$

which second sense quantum integral definition that call $\overline{q}$-integral for $x \in J$.

Moreover, if $c \in (a, x)$ then the definite $\overline{q}$-integral on J is defined by

$$\begin{aligned}
(3.2)\qquad & \int_c^x f(s)\ {}_a d_{\overline{q}} s \\
&= \int_a^x f(s)\ {}_a d_{\overline{q}} s - \int_a^c f(s)\ {}_a d_{\overline{q}} s \\
&= \frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - f(x)\right] \\
&\quad -\frac{(1-q)(c-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a) - f(c)\right].
\end{aligned}$$

Theorem 3.1. *Let $f : J \to \mathbb{R}$ be a continuous function. Then we have*

$$(3.3)\qquad {}_aD_q \int_a^x f(s)\ {}_a d_{\overline{q}} s = \frac{qf(x) + f(qx + (1-q)a)}{2q}.$$

Proof. From definition of $\overline{q}$-integral, we have

$$\int_a^x f(s)\ {}_a d_{\overline{q}} s = \frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - f(x)\right]$$

and take q-derivative of above equality write that

$$\begin{aligned}
& {}_aD_q \int_a^x f(s)\ {}_a d_{\overline{q}} s \\
=\ & {}_aD_q \left\{\frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - f(x)\right]\right\} \\
=\ & \frac{1}{(1-q)(x-a)}\left\{\frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - f(x)\right]\right. \\
& \left. -\frac{(1-q)(x-a)q}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f\left(q^{n+1}x + \left(1-q^{n+1}\right)a\right) - f(qx+(1-q)a)\right]\right\} \\
=\ & \frac{1}{2q}\left[(1+q)\left(\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - \sum_{n=0}^{\infty} q^n f\left(q^{n+1}x + \left(1-q^{n+1}\right)a\right)\right)\right. \\
& \left. + f(qx+(1-q)a) - f(x)\right] \\
=\ & \frac{qf(x) + f(qx + (1-q)a)}{2q}.
\end{aligned}$$

The proof is completed. □

Theorem 3.2. *(Change of Variables Property) Let $f : J \to \mathbb{R}$ be a function and $0 < q < 1$. Then we have*

$$\int_0^1 f(sb + (1-s)a) \ {}_0d_{\overline{q}}s \ = \frac{1}{b-a}\int_a^b f(t) \ {}_ad_{\overline{q}}t \ . \tag{3.4}$$

Proof. From definition of $\overline{q}$-integral, we have

$$\begin{aligned}
&\int_0^1 f(sb + (1-s)a) \ {}_0d_{\overline{q}}s \\
= \ & \frac{(1-q)(1-0)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f([q^n 1 + (1-q^n)0]b + (1 - [q^n 1 + (1-q^n)0])a)\right. \\
& \left. -f(1b + (1-1)a)\right] \\
= \ & \frac{(1-q)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)a) - f(b)\right]
\end{aligned}$$

Multiplying by $\frac{b-a}{b-a}$ then we have

$$\int_0^1 f(sb + (1-s)a) \ {}_0d_{\overline{q}}s \ = \frac{1}{b-a}\int_a^b f(t) \ {}_ad_{\overline{q}}t$$

The proof is completed. □

Theorem 3.3. *Let $f : J \to \mathbb{R}$ be a continuous function. Then we have*

$$\begin{aligned}
&\int_c^x {}_aD_q \ f(s) \ {}_ad_{\overline{q}}s \\
= \ & \frac{qf(x) + f(qx + (1-q)a) - qf(c) - f(qc + (1-q)a)}{2q} \quad \textit{for } c \in (a,x).
\end{aligned} \tag{3.5}$$

Proof. Applying definitions of $\overline{q}$-integral, q-derivative and by change of variables, we have

$$
\begin{aligned}
& \int_c^x {}_aD_q\ f(s)\ {}_ad_{\overline{q}}s \\
= & \int_c^x \frac{f(s)-f(qs+(1-q)a)}{(1-q)(s-a)}\ {}_ad_{\overline{q}}s \\
= & \int_a^x \frac{f(s)-f(qs+(1-q)a)}{(1-q)(s-a)}\ {}_ad_{\overline{q}}s\ -\int_a^c \frac{f(s)-f(qs+(1-q)a)}{(1-q)(s-a)}\ {}_ad_{\overline{q}}s \\
= & \int_a^x \frac{f(s)}{(1-q)(s-a)}\ {}_ad_{\overline{q}}s\ - \int_a^{qx+(1-q)a} \frac{f(s)}{(1-q)(s-a)}\ {}_ad_{\overline{q}}s \\
& -\int_a^c \frac{f(s)}{(1-q)(s-a)}\ {}_ad_{\overline{q}}s\ + \int_a^{qc+(1-q)a} \frac{f(s)}{(1-q)(s-a)}\ {}_ad_{\overline{q}}s \\
= & \frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty}\frac{q^n f(q^nx+(1-q^n)a)}{(1-q)q^n(x-a)}-\frac{f(x)}{(1-q)(x-a)}\right] \\
& -\frac{(1-q)q(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty}\frac{q^n f\left(q^{n+1}x+\left(1-q^{n+1}\right)a\right)}{(1-q)q^{n+1}(x-a)}-\frac{f(qx+(1-q)a)}{(1-q)q(x-a)}\right] \\
& -\frac{(1-q)(c-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty}\frac{q^n f(q^nc+(1-q^n)a)}{(1-q)q^n(c-a)}-\frac{f(c)}{(1-q)(c-a)}\right] \\
& +\frac{(1-q)q(c-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty}\frac{q^n f\left(q^{n+1}c+\left(1-q^{n+1}\right)a\right)}{(1-q)q^{n+1}(c-a)}-\frac{f(qc+(1-q)a)}{(1-q)q(c-a)}\right] \\
= & \frac{1+q}{2q}\sum_{n=0}^{\infty}\left[f(q^nx+(1-q^n)a)-f\left(q^{n+1}x+\left(1-q^{n+1}\right)a\right)\right. \\
& \left.-f(q^nc+(1-q^n)a)+f\left(q^{n+1}c+\left(1-q^{n+1}\right)a\right)\right] \\
& +\frac{1}{2q}\left[-f(x)+f(qx+(1-q)a)+f(c)-f(qc+(1-q)a)\right] \\
= & \frac{f(x)-f(c)}{2}+\frac{f(qx+(1-q)a)-f(qc+(1-q)a)}{2q}
\end{aligned}
$$

The proof is complated. □

Theorem 3.4. *Assume $f,g:J\to\mathbb{R}$ are continuous functions. Then, for $x\in J$,*

$$
\int_a^x [f(s)+g(s)]\ {}_ad_{\overline{q}}s\ = \int_a^x f(s)\ {}_ad_{\overline{q}}s\ + \int_a^x g(s)\ {}_ad_{\overline{q}}s\ . \tag{3.6}
$$

Proof. Using definition of $\overline{q}$-integral write that

$$\begin{aligned}
&\int_a^x [f(s)+g(s)] \ \ {}_ad_{\overline{q}}s \\
=\ & \frac{(1-q)(x-a)}{2q}\left\{(1+q)\sum_{n=0}^{\infty} q^n \left[f(q^n x+(1-q^n)a)+g(q^n x+(1-q^n)a)\right]\right. \\
&\left. -f(x)-g(x)\right\} \\
=\ & \frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n x+(1-q^n)a)-f(x)\right] \\
&+\frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n g(q^n x+(1-q^n)a)-g(x)\right] \\
=\ & \int_a^x f(s) \ \ {}_ad_{\overline{q}}s \ \ +\int_a^x g(s) \ \ {}_ad_{\overline{q}}s \ \ .
\end{aligned}$$

The proof is completed. □

Theorem 3.5. *Assume $f,g:J\to\mathbb{R}$ are continuous functions. $\alpha\in\mathbb{R}$. Then, for $x\in J$,*

$$\int_a^x (\alpha f)(s) \ \ {}_ad_{\overline{q}}s \ \ =\alpha\int_a^x f(s) \ \ {}_ad_{\overline{q}}s \ \ . \tag{3.7}$$

Proof. Using definition of $\overline{q}$-integral we have

$$\begin{aligned}
\int_a^x (\alpha f)(s) \ \ {}_ad_{\overline{q}}s &= \frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n (\alpha f)(q^n x+(1-q^n)a)-(\alpha f)(x)\right] \\
&= \alpha\frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n x+(1-q^n)a)-f(x)\right] \\
&= \alpha\int_a^x f(s) \ \ {}_ad_{\overline{q}}s \ \ .
\end{aligned}$$

We obtained (3.7) as required. □

Theorem 3.6. *Assume $f,g:J\to\mathbb{R}$ are continuous functions. Then, for $x\in J$*

$$\begin{aligned}
&\int_c^x f(s) \ \ {}_aD_q \ g(s) \ \ {}_ad_{\overline{q}}s \\
=\ & \left.\frac{qf(s)g(s)+f(qs+(1-q)a)g(qs+(1-q)a)}{2q}\right|_c^x \\
&-\int_c^x g(qs+(1-q)a) \ \ {}_aD_q \ f(s) \ \ {}_ad_{\overline{q}}s \ \ .
\end{aligned} \tag{3.8}$$

Proof. Using definition of q-derivative

$$\begin{aligned}
& {}_aD_q \; f(s)\, g(s) \\
= & \frac{f(s)\, g(s) - f(qs + (1-q)a)\, g(qs + (1-q)a)}{(1-q)(s-a)} \\
= & f(s) \frac{g(s) - g(qs + (1-q)a)}{(1-q)(s-a)} + g(qs + (1-q)a) \frac{f(s) - f(qs + (1-q)a)}{(1-q)(s-a)} \\
= & f(s) \;\; {}_aD_q \; g(s) + g(qs + (1-q)a) \;\; {}_aD_q \; f(s).
\end{aligned}$$

From here take $\overline{q}$-integral

$$\begin{aligned}
& \int_c^x {}_aD_q \; f(s)\, g(s) \;\; {}_ad_{\overline{q}}s \\
= & \int_c^x f(s) \;\; {}_aD_q \; g(s) \;\; {}_ad_{\overline{q}}s \; + \int_c^x g(qs + (1-q)a) \;\; {}_aD_q \; f(s) \;\; {}_ad_{\overline{q}}s
\end{aligned}$$

Applying (3.5) we have

$$\begin{aligned}
& \left. \frac{qf(s)\, g(s) + f(qs + (1-q)a)\, g(qs + (1-q)a)}{2q} \right|_c^x \\
= & \int_c^x f(s) \;\; {}_aD_q \; g(s) \;\; {}_ad_{\overline{q}}s \; + \int_c^x g(qs + (1-q)a) \;\; {}_aD_q \; f(s) \;\; {}_ad_{\overline{q}}s
\end{aligned}$$

and following equality holds

$$\begin{aligned}
\int_c^x f(s) \;\; {}_aD_q \; g(s) \;\; {}_ad_{\overline{q}}s \quad = & \quad \left. \frac{qf(s)\, g(s) + f(qs + (1-q)a)\, g(qs + (1-q)a)}{2q} \right|_c^x \\
& - \int_c^x g(qs + (1-q)a) \;\; {}_aD_q \; f(s) \;\; {}_ad_{\overline{q}}s \; .
\end{aligned}$$

The proof is completed. □

Theorem 3.7. *For $\alpha \in \mathbb{R} \backslash \{-1\}$, the following formula holds:*

$$\int_a^x (s-a)^\alpha \;\; {}_ad_{\overline{q}}s \; = \left(\frac{1-q}{1-q^{\alpha+1}} \right) \left(\frac{1+q^\alpha}{2} \right) (x-a)^{\alpha+1}. \tag{3.9}$$

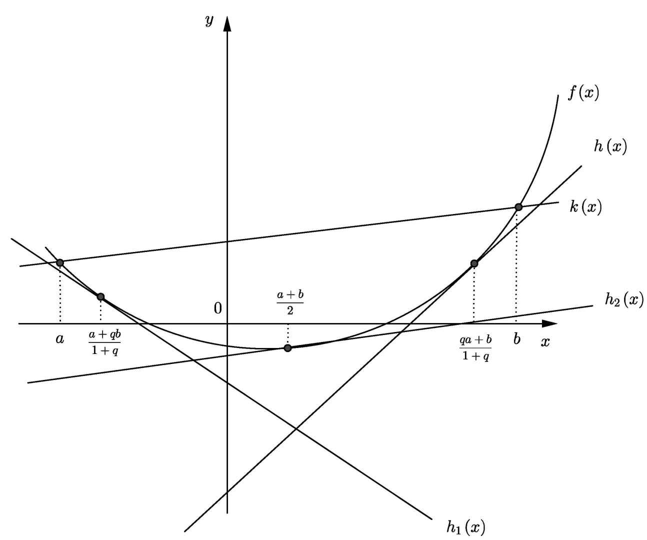
Proof. Using definition of $\overline{q}$-integral, we have

$$
\begin{aligned}
\int_a^x (s-a)^\alpha \ {}_a d_{\overline{q}} s &= \frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \left(q^n x + (1-q^n) a - a\right)^\alpha - (x-a)^\alpha\right] \\
&= \frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \left(q^n (x-a)\right)^\alpha - (x-a)^\alpha\right] \\
&= \frac{(1-q)(x-a)}{2q}\left[(1+q)(x-a)^\alpha \sum_{n=0}^{\infty} \left(q^{\alpha+1}\right)^n - (x-a)^\alpha\right] \\
&= \frac{(1-q)(x-a)^{\alpha+1}}{2q}\left[\frac{(1+q)}{1-q^{\alpha+1}} - 1\right] \\
&= \left(\frac{1-q}{1-q^{\alpha+1}}\right)\left(\frac{1+q^\alpha}{2}\right)(x-a)^{\alpha+1}.
\end{aligned}
$$

The proof is completed. □

4. $\overline{q}$-Hermite-Hadamard Inequalities

In this section, we present $\overline{q}$-Hermite-Hadamard type inequalities for convex functions. Let $0 < q \le 1$, we consider the following graphics:



Theorem 4.1 ($\overline{q}$-Hermite-Hadamard)**.** *Let $f : J \to \mathbb{R}$ be a convex continuous function on J and $0 < q < 1$. Then we have*

$$
f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x) \ {}_a d_{\overline{q}} x \ \le \frac{f(a)+f(b)}{2} \tag{4.1}
$$

Proof. Since f is differentiable function on $[a,b]$, there is a tangent line for the function f at the point $\frac{a+b}{2} \in (a,b)$. This tangent line can be expressed as a function $h_2(x) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right)$. Since f is a convex function on $[a,b]$, then we have the following inequality

$$h_2(x) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) \leq f(x) \tag{4.2}$$

for all $x \in [a,b]$ (see Figure 2). $\overline{q}$-integrating the inequality (4.2) on $[a,b]$, we have

$$\begin{aligned}
& \int_a^b h_2(x)\ {}_a d_{\overline{q}} x \\
= & \int_a^b \left[f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right)\right]\ {}_a d_{\overline{q}} x \\
= & (b-a) f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right) \int_a^b \left(x - a + \frac{a-b}{2}\right)\ {}_a d_{\overline{q}} x \\
= & (b-a) f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right) \\
& \times \left[\left(\frac{1-q}{1-q^2}\right)\left(\frac{1+q}{2}\right)(x-a)^2\Big|_a^b + \frac{a-b}{2}(b-a)\right] \\
= & (b-a) f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left[\frac{(b-a)^2}{2} - \frac{(b-a)^2}{2}\right] \\
= & (b-a) f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)\ {}_a d_{\overline{q}} x \ .
\end{aligned} \tag{4.3}$$

On the other hand, line connecting the points $(a, f(a))$ and $(b, f(b))$ can be expressed as a function $k(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. Since f is a convex function on $[a,b]$, then we have the following inequality

$$f(x) \leq k(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \tag{4.4}$$

for all $x \in [a,b]$ (see Figure 2). $\overline{q}$-integrating the inequality (4.4) on $[a,b]$, we have

$$\begin{aligned}
&\int_a^b f(x) \quad {}_a d_{\overline{q}} x \\
\leq &\int_a^b k(x) \quad {}_a d_{\overline{q}} x \\
= &\int_a^b \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right) \quad {}_a d_{\overline{q}} x \\
= &\; (b-a) f(a) + \frac{f(b)-f(a)}{b-a} \int_a^b (x-a) \quad {}_a d_{\overline{q}} x \\
= &\; (b-a) f(a) + \frac{f(b)-f(a)}{b-a} \left(\frac{1-q}{1-q^2} \right) \left(\frac{1+q}{2} \right) (x-a)^2 \Big|_a^b \\
= &\; (b-a) f(a) + \frac{f(b)-f(a)}{b-a} \frac{(b-a)^2}{2}
\end{aligned}$$

and then

$$(b-a)\frac{f(a)+f(b)}{2} \geq \int_a^b f(x) \quad {}_a d_{\overline{q}} x \quad . \tag{4.5}$$

A combination of (4.3) and (4.5) gives (4.1) and the proof is completed. □

Remark 4.1. In Theorem 4.1, if we take $q \to 1^-$, we recapture the well known Hermite-Hadamard inequality for convex function.

Theorem 4.2. *Let $f : [a,b] \to \mathbb{R}$ be a convex differentiable function on $[a,b]$ and $0 < q < 1$. Then we have*

(4.6)
$$f\left(\frac{qa+b}{1+q}\right) + \frac{q-1}{1+q}\frac{(b-a)}{2} f'\left(\frac{qa+b}{1+q}\right) \leq \frac{1}{b-a}\int_a^b f(x) \quad {}_a d_{\overline{q}} x \quad \leq \frac{f(a)+f(b)}{2}.$$

Proof. Since f is differentiable function on $[a,b]$, there is a tangent line for the function f at the point $\frac{qa+b}{1+q} \in (a,b)$. This tangent line can be expressed as a function $h(x) = f\left(\frac{qa+b}{1+q}\right) + f'\left(\frac{qa+b}{1+q}\right)\left(x - \frac{qa+b}{1+q}\right)$. Since f is a convex function on $[a,b]$, then we have the following inequality

$$h(x) = f\left(\frac{qa+b}{1+q}\right) + f'\left(\frac{qa+b}{1+q}\right)\left(x - \frac{qa+b}{1+q}\right) \leq f(x) \tag{4.7}$$

for all $x \in [a,b]$ (see Figure 2). $\overline{q}$-integrating the inequality (4.7) on $[a,b]$, we have

$$\begin{aligned}
&\int_a^b f(x) \ {}_a d_{\overline{q}} x \\
\geq &\int_a^b h(x) \ {}_a d_{\overline{q}} x \\
= &\int_a^b \left[f\left(\frac{qa+b}{1+q}\right) + f'\left(\frac{qa+b}{1+q}\right)\left(x - \frac{qa+b}{1+q}\right)\right] \ {}_a d_{\overline{q}} x \\
= &(b-a) f\left(\frac{qa+b}{1+q}\right) + f'\left(\frac{qa+b}{1+q}\right)\left[\int_a^b \left(x - a + \frac{a-b}{1+q}\right) \ {}_a d_{\overline{q}} x \ \right] \\
= &(b-a) f\left(\frac{qa+b}{1+q}\right) + f'\left(\frac{qa+b}{1+q}\right)\left[\left(\frac{1-q}{1-q^2}\right)\left(\frac{1+q}{2}\right)(x-a)^2\Big|_a^b + \frac{a-b}{1+q}(b-a)\right] \\
= &(b-a) f\left(\frac{qa+b}{1+q}\right) + f'\left(\frac{qa+b}{1+q}\right)\left[\frac{(b-a)^2}{2} - \frac{(b-a)^2}{1+q}\right]
\end{aligned}$$

and then

$$(b-a) f\left(\frac{qa+b}{1+q}\right) + \frac{q-1}{1+q}\frac{(b-a)^2}{2} f'\left(\frac{qa+b}{1+q}\right) \leq \int_a^b f(x) \ {}_a d_{\overline{q}} x \ . \tag{4.8}$$

A combination of (4.8) and (4.5) gives (4.6) and the proof is completed. □

Theorem 4.3. *Let $f : [a,b] \to \mathbb{R}$ be a convex differentiable function on $[a,b]$ and $0 < q < 1$. Then we have*

$$f\left(\frac{a+qb}{1+q}\right) + \frac{1-q}{1+q}\frac{b-a}{2} f'\left(\frac{a+qb}{1+q}\right) \leq \frac{1}{b-a}\int_a^b f(x) \ {}_a d_{\overline{q}} x \ \leq \frac{f(a)+f(b)}{2}. \tag{4.9}$$

Proof. Since f is differentiable function on $[a,b]$, there is a tangent line for the function f at the point $\frac{a+qb}{1+q} \in (a,b)$. This tangent line can be expressed as a function $h_1(x) = f\left(\frac{a+qb}{1+q}\right) + f'\left(\frac{a+qb}{1+q}\right)\left(x - \frac{a+qb}{1+q}\right)$. Since f is a convex function on $[a,b]$, then we have the following inequality

$$h_1(x) = f\left(\frac{a+qb}{1+q}\right) + f'\left(\frac{a+qb}{1+q}\right)\left(x - \frac{a+qb}{1+q}\right) \leq f(x) \tag{4.10}$$

for all $x \in [a,b]$ (see Figure 2). $\overline{q}$-integrating the inequality (4.10) on $[a,b]$, we have

$$\begin{aligned}
&\int_a^b f(x) \; {}_a d_{\overline{q}} x \\
\geq &\int_a^b h_1(x) \; {}_a d_{\overline{q}} x \\
= &\int_a^b \left[f\left(\frac{a+qb}{1+q}\right) + f'\left(\frac{a+qb}{1+q}\right)\left(x - \frac{a+qb}{1+q}\right)\right] \; {}_a d_{\overline{q}} x \\
= &(b-a) f\left(\frac{a+qb}{1+q}\right) + f'\left(\frac{a+qb}{1+q}\right) \int_a^b \left(x - a + q\frac{a-b}{1+q}\right) \; {}_a d_{\overline{q}} x \\
= &(b-a) f\left(\frac{a+qb}{1+q}\right) + f'\left(\frac{a+qb}{1+q}\right)\left[\left(\frac{1-q}{1-q^2}\right)\left(\frac{1+q}{2}\right)(x-a)^2\Big|_a^b + (b-a)\, q\frac{a-b}{1+q}\right] \\
= &(b-a) f\left(\frac{a+qb}{1+q}\right) + f'\left(\frac{a+qb}{1+q}\right)\left[\frac{(b-a)^2}{2} - q\frac{(b-a)^2}{1+q}\right]
\end{aligned}$$

and we can write

$$(b-a) f\left(\frac{a+qb}{1+q}\right) + \frac{1-q}{1+q}\frac{(b-a)^2}{2} f'\left(\frac{a+qb}{1+q}\right) \leq \int_a^b f(x) \; {}_a d_{\overline{q}} x \tag{4.11}$$

A combination of (4.5) and (4.11) gives (4.9) and the proof is completed. □

Theorem 4.4 (Generalized $\overline{q}$-Hermite-Hadamard inequality)**.** *Let $f : [a,b] \to \mathbb{R}$ be a convex differentiable function on $[a,b]$ and $0 < q < 1$. Then we have*

$$\max\{I_1, I_2, I_3\} \leq \frac{1}{b-a}\int_a^b f(x) \; {}_a d_{\overline{q}} x \; \leq \frac{f(a)+f(b)}{2}. \tag{4.12}$$

where

$$I_1 = f\left(\frac{a+b}{2}\right),$$
$$I_2 = f\left(\frac{qa+b}{1+q}\right) + \frac{q-1}{1+q}\frac{(b-a)}{2} f'\left(\frac{qa+b}{1+q}\right),$$
$$I_3 = f\left(\frac{a+qb}{1+q}\right) + \frac{1-q}{1+q}\frac{b-a}{2} f'\left(\frac{a+qb}{1+q}\right).$$

Proof. A combination of (4.1), (4.6), and (4.9) gives (4.12) and the proof is completed. □

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