INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 17 NO. 1 PAGE 199–206 (2024)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1384669
RESEARCH ARTICLE

Generalized Maximal Diameter Theorems

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

We prove Maximal Diameter Theorems for pointed Riemannian manifolds which are compared to surfaces of revolution with weaker radial attraction.

Keywords: *Maximal diameter theorem, maximal perimeter theorem, weaker radial attraction.*

AMS Subject Classification (2020): *Primary: 53C20 ; Secondary: 53C22.*

1. Introduction

The papers [1] and [5] provide generalized maximal diameter theorems for pointed n–dimensional Riemannian manifolds (M, o) whose radial curvature along geodesics emanating from o is bounded from below by that of a closed model surface, the model surface $\widetilde{M}$ being a closed, simply–connected two–dimensional Riemannian manifold which is rotationally symmetric about a vertex $\tilde{o}$ and whose metric takes the form:

$$ds^2 = dr^2 + y(r)^2 d\theta^2$$

in a geodesic polar coordinate system (r, θ) about $\tilde{o}$ with $0 \le r \le \ell$, $0 \le \theta \le 2\pi$. These theorems assert, under different additional technical assumptions on $\widetilde{M}$, that the diameter of M is less than or equal to ℓ, and if the diameter equals ℓ, then M is isometric to the n–model associated to $\widetilde{M}$, that is, to an n–sphere whose Riemannian metric in geodesic polar coordinates about o takes the form:

$$ds^2 = dr^2 + y(r)^2 d\theta_{n-1}^2$$

where $d\theta_{n-1}^2$ is the standard metric on S^{n-1} where $0 \le r \le \ell$.

The maximal diameter theorem for noncompact manifolds proved in [7] asserts that if (M, o) is a complete noncompact Riemannian manifold whose radial curvature is bounded from below by that of a complete, rotationally symmetric surface $\widetilde{M}$ with vertex $\tilde{o}$, whose metric takes the form

$$ds^2 = dr^2 + y(r)^2 d\theta^2$$

in a geodesic polar coordinate system (r, θ) about $\tilde{o}$ with $0 \le r$, $0 \le \theta \le 2\pi$, and if

$$\int_1^\infty y(r)^{-2}\, dr = \infty,$$

then M is isometric to the n–model associated to $\widetilde{M}$, that is, to $\mathbf{R}^n$ whose Riemannian metric in geodesic polar coordinates about o, takes the form:

$$ds^2 = dr^2 + y(r)^2 d\theta_{n-1}^2$$

where $d\theta_{n-1}^2$ is the standard metric on S^{n-1} and $0 \le r$.

The proofs of these theorems utilize different versions of the generalized Toponogov triangle theorem [8, 5, 6]. In a pair of papers [3, 4], the authors proved a version of the Toponogov Triangle Theorem in which

Received : *01-12-2023,* ***Accepted :*** *05-12-2023*
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the hypothesis of bounding radial curvature from below is replaced by a weaker condition. Here we apply this theorem [4, Theorem 1.3] to generalize the maximal diameter theorems found in [1, 5, 7].

We begin by reviewing the material from [3, 4] that we will need.

Given a complete pointed Riemannian manifold (M, o) and a complete simply–connected model surface $(\widetilde{M}, \tilde{o})$ which is rotationally symmetric about $\tilde{o}$, [4, Theorem 1.3] provides necessary and sufficient conditions so that for every geodesic triangle $\triangle opq$ in M there exists a corresponding Alexandrov triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$. The conditions are two–fold: (i) $\widetilde{M}$ has weaker radial attraction than M and (ii) no minimizing geodesic in M has a bad encounter with the cut locus in $\widetilde{M}$. In other words, the assumption of (i) and (ii) is equivalent to the existence of corresponding Alexandrov Triangles in $\widetilde{M}$ for every geodesic triangle $\triangle opq$ in M.

The notion of weaker radial attraction was introduced in [3, Definition 4.1] as a hinge comparison. It is equivalent to a comparison of the Hessians of the the distance functions from the base points [3, Theorem 5.3]. One should note that the assumption of radial curvature being bounded from below implies having weaker radial attraction, but not conversely [3].

The condition that no minimizing geodesic in M has a bad encounter with the cut locus in $\widetilde{M}$ was introduced in [4, Definition 4.1]. Its purpose is to avoid obvious obstructions to the existence of corresponding Alexandrov triangles when the cut locus of a point in $\widetilde{M}$ is not contained in the meridian opposite. It is in the spirit of condition (2.1) of [5, Theorem 5], but is not equivalent to it. Note, in particular, that the assumption of no bad encounters is automatically satisfied whenever the cut loci of points in $\widetilde{M}$ are contained in the opposite meridians, e.g., when $\widetilde{M}$ is a von Mangoldt surface.

The gist of [4, Theorem 1.3] is that the triangle $\triangle opq$ and its corresponding Alexandrov triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ satisfy the conditions:

1. Equality of Corresponding sides:
$$d(o,p) = d(\tilde{o},\tilde{p}), d(o,q) = d(\tilde{o},\tilde{q}), d(p,q) = d(\tilde{p},\tilde{q}).$$
2. Alexandrov convexity:
   (a) from o: $d(\tilde{o},\tilde{\sigma}(t)) \leq d(o,\sigma(t)) \ \ \forall t \in [0, d(p,q)]$.
   (b) from p: $d(\tilde{p},\tilde{\gamma}(s)) \leq d(p,\gamma(s)) \ \ \forall s \in [0, d(o,q)]$.
   (c) from q: $d(\tilde{q},\tilde{\tau}(s)) \leq d(q,\tau(s)) \ \ \forall s \in [0, d(o,p)]$.
3. Angle Comparison:
   (a) The base angle: $\measuredangle\tilde{o} \leq \measuredangle o$.
   (b) The top left and right angles: $\measuredangle\tilde{p} \leq \measuredangle p$, and $\measuredangle\tilde{q} \leq \measuredangle q$.

(Here σ denotes the side $\widehat{pq}$ joining p to q, γ the side $\widehat{oq}$ joining o to q and τ the side $\widehat{op}$ joining o to p, and the corresponding sides in $\triangle \tilde{o}\tilde{p}\tilde{q}$ are denoted $\tilde{\sigma}$, $\tilde{\gamma}$ and $\tilde{\tau}$ respectively.)

We will also have need of the following theorem proved in [4, Theorem 1.5]:

Theorem 1.1 (Maximal Radius Theorem)**.** *Suppose $(\widetilde{M}, \tilde{o})$ is a compact model surface with radius $\ell < \infty$, whose metric takes the form*
$$ds^2 = dr^2 + y(r)^2 d\theta^2$$
in polar coordinates (r, θ) about $\tilde{o}$. Suppose that every geodesic triangle $\triangle opq$ in M has a corresponding Alexandrov triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$. If there is a point q in M with $d(o,q) = \ell$, then M is diffeomorphic to S^n and its metric takes the form
$$ds^2 = dr^2 + y(r)^2 d\theta^2_{n-1}$$
in geodesic coordinates about o where $d\theta^2_{n-1}$ is the standard metric on the unit $(n-1)$ sphere.

In Section 2 we prove facts about a closed model surface. Sections 3, 4, and 5 are devoted to proving the maximal diameter and maximal perimeter theorems.

2. Diameter in a Model Surface

Throughout this section let $\widetilde{M}$ be a closed simply–connected surface of revolution with Riemannian metric

$$ds^2 = dr^2 + y(r)^2 d\theta^2$$

in a geodesic polar coordinate system (r, θ) about the vertex $\tilde{o}$ with $0 \le r \le \ell$, $0 \le \theta \le 2\pi$. The distance between two points $\tilde{p}, \tilde{q} \in \widetilde{M}$ will be denoted $d(\tilde{p}, \tilde{q})$.

Proposition 2.1. *For any two points $\tilde{p}$ and $\tilde{q}$ in $\widetilde{M}$,*

$$d(\tilde{p}, \tilde{q}) \le \ell$$

and the perimeter of the geodesic triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ satisfies

$$d(\tilde{o}, \tilde{p}) + d(\tilde{p}, \tilde{q}) + d(\tilde{o}, \tilde{q}) \le 2\ell$$

where $\tilde{o}$ is the vertex $r = 0$ of $\widetilde{M}$. Moreover, if $d(\tilde{p}, \tilde{q}) = \ell$ holds, then

$$\ell = d(\tilde{p}, \tilde{q}) = d(\tilde{o}, \tilde{p}) + d(\tilde{o}, \tilde{q}).$$

Furthermore the perimeter of $\triangle\tilde{o}\tilde{p}\tilde{q}$ equals 2ℓ if and only if

$$d(\tilde{o}', \tilde{p}) + d(\tilde{o}', \tilde{q}) = d(\tilde{p}, \tilde{q})$$

where $\tilde{o}'$ is the antipode $r = \ell$ of $\tilde{o}$.

Proof. For $\tilde{p}, \tilde{q} \in \widetilde{M}$, we have

$$d(\tilde{o}', \tilde{p}) = \ell - d(\tilde{o}, \tilde{p}), \quad d(\tilde{o}', \tilde{q}) = \ell - d(\tilde{o}, \tilde{q}).$$

Thus by the triangle inequality

$$d(\tilde{p}, \tilde{q}) \le d(\tilde{o}, \tilde{p}) + d(\tilde{o}, \tilde{q}) \tag{2.1}$$

and

$$d(\tilde{p}, \tilde{q}) \le d(\tilde{o}', \tilde{p}) + d(\tilde{o}', \tilde{q}) = 2\ell - (d(\tilde{o}, \tilde{p}) + d(\tilde{o}, \tilde{q})). \tag{2.2}$$

Adding the inequalities (2.1) and (2.2) gives

$$2d(\tilde{p}, \tilde{q}) \le 2\ell$$

from which the result follows. Clearly from the proof, if $d(\tilde{p}, \tilde{q}) = \ell$, then equality holds in (2.1) and (2.2). Thus $\ell = d(\tilde{p}, \tilde{q}) = d(\tilde{o}, \tilde{p}) + d(\tilde{o}, \tilde{q})$.

Furthermore, rearranging (2.2), gives

$$d(\tilde{o}, \tilde{p}) + d(\tilde{p}, \tilde{q}) + d(\tilde{o}, \tilde{q}) \le 2\ell$$

which shows that the perimeter equals 2ℓ if and only if equality holds in (2.2). □

Some of the results in Proposition 2.1 can also be found in [1, Lemmas 2.1 and 2.2].

Proposition 2.2. *Suppose $d(\tilde{p}, \tilde{q}) = \ell$ with $0 < d(\tilde{o}, \tilde{p}) < \ell$ and $0 < d(\tilde{o}, \tilde{q}) < \ell$, then $\tilde{p}$ and $\tilde{q}$ are on opposite meridians. If $\tilde{z}$ is on the meridian containing $\tilde{p}$ with $d(\tilde{o}, \tilde{p}) < d(\tilde{o}, \tilde{z}) < \ell$, then $d(\tilde{p}, \tilde{z}) + d(\tilde{z}, \tilde{q}) = \ell$. This shows that the minimizing geodesic from $\tilde{z}$ to $\tilde{q}$ must pass through $\tilde{o}'$.*

Proof. We have by hypothesis $d(\tilde{o}, \tilde{z}) = d(\tilde{o}, \tilde{p}) + d(\tilde{p}, \tilde{z})$. By Proposition 2.1,

$$d(\tilde{o}, \tilde{p}) + d(\tilde{p}, \tilde{z}) + d(\tilde{z}, \tilde{q}) + d(\tilde{o}, \tilde{q}) = d(\tilde{o}, \tilde{z}) + d(\tilde{z}, \tilde{q}) + d(\tilde{o}, \tilde{q}) \le 2\ell,$$

and

$$d(\tilde{o}, \tilde{p}) + d(\tilde{o}, \tilde{q}) = \ell.$$

Subtracting we obtain

$$d(\tilde{p}, \tilde{z}) + d(\tilde{z}, \tilde{q}) \le \ell.$$

By the triangle inequality

$$\ell = d(\tilde{p}, \tilde{q}) \le d(\tilde{p}, \tilde{z}) + d(\tilde{z}, \tilde{q}).$$

Therefore $d(\tilde{p}, \tilde{z}) + d(\tilde{z}, \tilde{q}) = \ell$.

Thus $\tilde{z}$ lies on a minimizing geodesic joining $\tilde{p}$ to $\tilde{q}$. However, since the minimizing geodesic from $\tilde{p}$ to $\tilde{z}$ is unique and lies in the meridian containing $\tilde{p}$, its extension to the minimizing geodesic joining $\tilde{p}$ to $\tilde{q}$ passing through $\tilde{z}$ must therefore pass through $\tilde{o}'$. Thus $\tilde{q}$ lies in the opposite meridian from $\tilde{p}$. □

3. A Maximal Diameter Theorem

Theorem 3.1 (Maximal Diameter Theorem)**.** *Let (M, o) be a complete pointed Riemannian manifold, and let $(\widetilde{M}, \tilde{o})$ be a closed simply–connected surface of revolution. Assume that $\widetilde{M}$ has weaker radial attraction than M and no minimizing geodesic in M has a bad encounter with the cut locus in $\widetilde{M}$. Then the diameter of M is less than or equal to ℓ, the perimeter of any geodesic triangle $\triangle opq$ in M is $\leq 2\ell$. Moreover, if the diameter of M is equal to ℓ then, M is isometric to the n–dimensional model based on $\widetilde{M}$.*

Proof. The inequalities on the diameter and the perimeter follows from Proposition 2.1 and the existence of Alexandrov triangles $\triangle\tilde{o}\tilde{p}\tilde{q}$ to every geodesic triangle $\triangle opq$ in M.

Suppose that the diameter on M is equal to ℓ. By hypothesis there exist p and q in M such that $d(p, q) = \ell$. If $d(o, p)$ or $d(o, q)$ is equal to ℓ the result follows from the generalized Maximal Radius Theorem [4, Theorem 1.5] cited above as Theorem 1.1. If not we can suppose that $0 < d(o, p) < \ell$ and $0 < d(o, q) < \ell$. The idea of the proof is to show that there exists a point in M whose distance from o is ℓ so we may apply the generalized Maximal Radius Theorem.

Consider the Alexandrov triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ corresponding to the geodesic triangle $\triangle opq$ in M. Then $d(\tilde{p}, \tilde{q}) = d(p, q) = \ell$. Thus, by Proposition 2.1,

$$d(p, o) + d(o, q) = d(\tilde{p}, \tilde{o}) + d(\tilde{o}, \tilde{q}) = \ell = d(p, q). \tag{3.1}$$

Thus any minimizing geodesic from p to o followed by a minimizing geodesic from o to q, must be an unbroken minimizing geodesic from p to q. Thus the minimizing geodesic from o to p is unique, for otherwise we can construct a broken geodesic from p to q of length ℓ showing that $d(p, q) < \ell$. Moreover o and p are not conjugate, else by the Morse index theorem the minimizing geodesic from p to q passing through o could not minimize. Thus $p \notin C(o)$, the cut locus of o in M. Thus the geodesic from o to p minimizes beyond p.

Thus there is a point z lying beyond p on the minimizing geodesic starting from o which minimizes from o to z and hence $d(o, z) = d(o, p) + d(p, z)$ and $d(o, p) < d(o, z) < \ell$. By the triangle inequality $d(p, z) + d(z, q) \geq d(p, q) = \ell$. Since the perimeter of $\triangle ozq$ is $\leq 2\ell$,

$$d(o, p) + d(p, z) + d(z, q) + d(o, q) = d(o, z) + d(z, q) + d(o, q) \leq 2\ell.$$

Subtracting equation (3.1), gives $d(p, z) + d(z, q) \leq d(p, q) = \ell$. Therefore $d(p, z) + d(z, q) = d(p, q) = \ell$ which shows that the minimizing geodesic from p to z followed by the minimizing geodesic from z to q is a minimizing geodesic from p to q. Thus $d(z, q) = \ell - d(p, z)$.

Let $\triangle\tilde{o}\tilde{p}\tilde{q}$ be the Alexandrov triangle in $\widetilde{M}$ corresponding to the geodesic triangle $\triangle opq$ in M. Let $\tilde{z}$ be the point on the meridian through $\tilde{p}$ such that $d(\tilde{o}, \tilde{z}) = d(o, z)$. By Proposition 2.2, $d(\tilde{z}, \tilde{q}) = \ell - d(\tilde{p}, \tilde{z}) = \ell - d(p, z) = d(z, q)$. Thus $\triangle\tilde{o}\tilde{z}\tilde{q}$ is the Alexandrov triangle corresponding to $\triangle ozq$. By Proposition 2.2, the minimizing geodesic from $\tilde{z}$ to $\tilde{q}$ must pass through $\tilde{o}'$ which has distance ℓ from $\tilde{o}$. Thus by Alexandrov convexity 2(a), the minimizing geodesic from z to q must contain a point at a distance ℓ from o. The generalized Maximal Radius Theorem (Theorem 1.1) applies and M is isometric to the n–dimensional model based on $\widetilde{M}$. □

4. A Maximal Perimeter Theorem

Let $\widetilde{M}$ be a closed simply–connected surface of revolution with Riemannian metric

$$ds^2 = dr^2 + y(r)^2 d\theta^2$$

in a geodesic polar coordinate system (r, θ) with $0 \leq r \leq \ell$, $0 \leq \theta \leq 2\pi$. A geodesic triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ whose perimeter is 2ℓ is said to be of *maximal perimeter*. A geodesic triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ of maximal perimeter is *standard* if either the three sides lie in the union of two opposite meridians or one of the three sides has length ℓ.

Lemma 4.1. *Let $\triangle\tilde{o}\tilde{p}\tilde{q}$ be a standard geodesic triangle in $\widetilde{M}$ of maximal perimeter all of whose sides have length strictly less than ℓ. Then the side $\widehat{\tilde{p}\tilde{q}}$ must pass through the antipode $\tilde{o}'$ of $\tilde{o}$.*

Proof. Since $\triangle\tilde{o}\tilde{p}\tilde{q}$ is a standard geodesic triangle in $\widetilde{M}$ of maximal perimeter all of whose sides have length strictly less than ℓ, the three sides are contained in the union of two opposite meridians. The points $\tilde{p}$ and $\tilde{q}$

must lie in opposite meridians, for if the were both on the same meridian, we may assume, without loss of generality, that $d(\tilde{o}, \tilde{p}) < d(\tilde{o}, \tilde{q})$. Then we would have $d(\tilde{o}, \tilde{p}) + d(\tilde{p}, \tilde{q}) = d(\tilde{o}, \tilde{q})$, and thus

$$2\ell = d(\tilde{o}, \tilde{p}) + d(\tilde{p}, \tilde{q}) + d(\tilde{o}, \tilde{q}) = 2d(\tilde{o}, \tilde{q})$$

contradicting $d(\tilde{o}, \tilde{q}) < \ell$. By Proposition 2.1, $d(\tilde{o}', \tilde{p}) + d(\tilde{o}', \tilde{q}) = d(\tilde{p}, \tilde{q})$, which shows that a minimizing geodesic from $\tilde{p}$ to $\tilde{q}$ passes through $\tilde{o}'$. It remains to prove that the minimizing geodesic $\widehat{\tilde{p}\tilde{q}}$ does not pass through $\tilde{o}$. If it did, then $d(\tilde{o}, \tilde{p}) + d(\tilde{o}, \tilde{q}) = d(\tilde{p}, \tilde{q})$, and thus

$$2\ell = d(\tilde{o}, \tilde{p}) + d(\tilde{p}, \tilde{q}) + d(\tilde{o}, \tilde{q}) = 2d(\tilde{p}, \tilde{q})$$

contradicting $d(\tilde{p}, \tilde{q}) < \ell$. We conclude that the side $\widehat{\tilde{p}\tilde{q}}$ passes through $\tilde{o}'$.

□

Theorem 4.1 (Maximal Perimeter Theorem)**.** *Let (M, o) be a complete pointed Riemannian manifold, and let $(\widetilde{M}, \tilde{o})$ be a closed simply–connected surface of revolution which has no nonstandard triangles of maximal perimeter. Assume that $\widetilde{M}$ has weaker radial attraction than M and no minimizing geodesic in M has a bad encounter with the cut locus in $\widetilde{M}$. If M contains a geodesic triangle $\triangle opq$ with perimeter 2ℓ, then M is isometric to the n–dimensional model based on $\widetilde{M}$.*

Proof. If one of the sides of $\triangle opq$ has length ℓ, then M is isometric to the n–dimensional model based on $\widetilde{M}$ by the Maximal Diameter Theorem. Thus we may suppose that the three sides of $\triangle opq$ have length less than ℓ. Let $\triangle\tilde{o}\tilde{p}\tilde{q}$ be the Alexandrov Triangle corresponding to $\triangle opq$. Then $\triangle\tilde{o}\tilde{p}\tilde{q}$ has maximal perimeter and thus must be standard. Its three sides lie in the union of two opposite meridians. By Lemma 4.1, the side $\widehat{\tilde{p}\tilde{q}}$ must pass through $\tilde{o}'$. By Alexandrov convexity, then the side $\widehat{pq}$ must contain a point at a distance ℓ from o. Hence M is isometric to the n–dimensional model based on $\widetilde{M}$ by the generalized Maximal Radius Theorem (Theorem 1.1). □

Theorem 4.1 can be compared to [5, Corollary 33].

If M is a complete Riemannian manifold with sectional curvature bounded from below by $K > 0$ and $\widetilde{M}$ is the two–sphere of constant curvature K, then the hypothesis of Theorem 4.1 is obviously satisfied for any choice of base points. We immediately obtain the following:

Corollary 4.1. *If M is a complete Riemannian manifold with sectional curvature bounded from below by $K > 0$, then M is isometric to a sphere of constant curvature K if and only if there exists a geodesic triangle of perimeter $2\pi/\sqrt{K}$ in M.*

Many model surfaces of revolution have only standard triangles of maximal perimeter. These include round spheres, ellipsoids of revolution, λ–spheres, etc. On the other hand one can construct a surface of revolution that contains nonstandard triangles of maximal perimeter.

On the standard two–sphere S^2, with south pole $\tilde{o}$ and north pole $\tilde{o}'$, fix a point $\tilde{p}$ (the west pole) on the equator, and let $\tilde{p}'$ (the east pole) be the antipodal point to $\tilde{p}$. Pick a point $\tilde{q}$ near $\tilde{p}'$ in the northern hemisphere on the great circle passing through $\tilde{p}$, $\tilde{o}'$, and $\tilde{p}'$. Using ideas from Gluck and Singer [2], we can deform the metric on S^2 in a neighborhood of the north pole to obtain a metric that is still rotationally symmetric about the poles $\tilde{o}$ and $\tilde{o}'$, and such that the arc $\widehat{\tilde{q}\tilde{p}'}$ is the cut locus of $\tilde{p}$ and for which there is a 1–parameter family of minimizing geodesics joining $\tilde{p}$ to $\tilde{q}$. See Figure 1. By construction each of the geodesic triangles $\triangle\tilde{o}\tilde{p}\tilde{q}$ have maximal perimeter no matter which of the minimizing geodesics joining $\tilde{p}$ to $\tilde{q}$ is chosen for the side $\widehat{\tilde{p}\tilde{q}}$, but only one, the one passing through the north pole $\tilde{o}'$, is standard.Therefore the constructed surface has nonstandard geodesic triangles of maximal perimeter.

Remark 4.1. The given proof of Theorem 4.1 requires the assumption that $\widetilde{M}$ have no nonstandard geodesic triangles of maximal perimeter. However it is an open question whether that assumption is actually necessary, as we have been unable to construct a counterexample.

5. Noncompact case

One ingredient used in proving a maximal diameter theorem for noncompact manifolds is Lemma 3.1 in [7]. This lemma states that if K and $\bar{K}$ are continuous functions on $[0, \infty)$ satisfying $K \geq \bar{K}$ on $[0, \infty)$, and

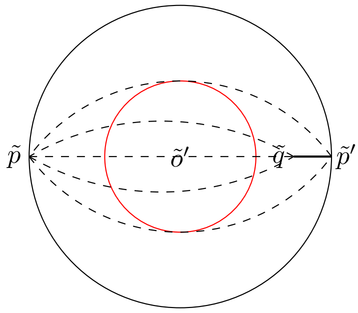

Figure 1. The northern hemisphere of a surface of revolution with nonstandard triangles of maximal perimeter viewed looking down from above the north pole. The arc $\widehat{\tilde{q}\tilde{p}'}$ is the cut locus of $\tilde{p}$. Outside of the small red circle the metric agrees with the standard metric on S^2. The dashed curves represent minimizing geodesics.

if y and $\bar{y}$ are the respective solutions of the Jacobi equations $y'' + Ky = 0$ and $\bar{y}'' + \bar{K}\bar{y} = 0$ satisfying the initial conditions $y(0) = \bar{y}(0) = 0$ and $y'(0) = \bar{y}'(0) = 1$, and if $y(r) > 0$ for $r \in (0, \infty)$ and $\int_1^\infty (\bar{y}(r))^{-2} dr = \infty$, then $K = \bar{K}$ on $(0, \infty)$. Consequently, by uniqueness of the solutions of the Jacobi equation, $y(r) = \bar{y}(r)$ for $r \in [0, \infty)$. This result does not hold if the inequality on curvature is replaced by the condition of weaker radial attraction.

The weaker radial attraction version would assume that the two functions y and $\bar{y}$ with $y(0) = \bar{y}(0) = 0$ and $y'(0) = \bar{y}'(0) = 1$ satisfy

$$\frac{y'(r)}{y(r)} \leq \frac{\bar{y}'(r)}{\bar{y}(r)}.$$

Following [3], set $y(r) = m(r)\bar{y}(r)$, then

$$\frac{y'(r)}{y(r)} = \frac{\bar{y}'(r)}{\bar{y}(r)} + \frac{m'(r)}{m(r)}$$

with $m(0) = 1$. Thus if $\bar{y}(r) > 0$ for $r \in (0, \infty)$ with $\int_1^\infty (\bar{y}(r))^{-2} dr = \infty$, and $m(r)$ is a positive decreasing function on $[0, \infty)$ with $m(0) = 1$, for example $m(r) = \frac{1}{1+r^2}$, then $y(r) = m(r)\bar{y}(r)$ will satisfy $y(r) > 0$ and

$$\frac{y'(r)}{y(r)} \leq \frac{\bar{y}'(r)}{\bar{y}(r)}$$

on $(0, \infty)$ and yet $y \neq \bar{y}$. Thus the result of [7, Lemma 3.1] does not hold. This is interesting because here is a situation where a lower bound on curvature is needed; weaker radial attraction is not enough. (Another example of this situation is rigidity in Toponogov's Theorem [3]. It would be interesting to find other results that require curvature bounded from below and not just weaker radial attraction.)

In this case we can obtain a weaker conclusion using the generalized Toponogov Theorem in [4] in place of [7, Theorem 3.1], but because of the above example we cannot conclude the space is isometric to the n–dimensional model.

Theorem 5.1. *Let $\widetilde{M}$ be a noncompact model surface with vertex $\tilde{o}$ that satisfies the condition that through every $\tilde{p} \in \widetilde{M} \backslash \{\tilde{o}\}$ there passes a unique geodesic ray. Let (M, o) be a complete noncompact pointed Riemannian manifold such that $\widetilde{M}$ has weaker radial attraction and none of the geodesics in M have bad encounters with cut loci in $\widetilde{M}$. Then M is diffeomorphic to $\mathbb{R}^n$ under $\exp_o$.*

By [7, Corollary to Theorem 1.1] the hypothesis that through every $\tilde{p} \in \widetilde{M} \backslash \{\tilde{o}\}$ there passes a unique geodesic ray is a consequence of assuming that one of the following equations holds:

$$\int_1^\infty (\bar{y}(r))^{-2} dr = \infty \tag{5.1}$$

$$\liminf_{r \to \infty} y(r) = 0. \tag{5.2}$$

This leads to a question: Is there a model $\widetilde{M}$ that satisfies the hypothesis of Theorem 5.1 but neither (5.1) nor (5.2)? The answer is no. In [10, Theorem 1], M. Tanaka proves that if $\int_1^\infty \tilde{y}^{-2}(t) dt < \infty$ and $\liminf_{t \to \infty} \tilde{y}(t) > 0$, then there is a ball about $\tilde{o}$ consisting of poles of $\widetilde{M}$. (For an alternative proof, see [7, Theorem 1.2].) Since through a pole every geodesic emanating from it is a geodesic ray, there are many points $\tilde{p} \in \widetilde{M}\backslash\{\tilde{o}\}$ through which pass many geodesic rays . Thus we see that the assumption that through every $\tilde{p} \in \widetilde{M}\backslash\{\tilde{o}\}$ there passes a unique geodesic ray is equivalent to assuming that either (5.1) or (5.2) holds.

Proof. Let $X \in T_oM$ be any unit tangent vector. We need to prove that $t \mapsto \exp_o(tX)$ is a geodesic ray. Since X is arbitrary, it follows that the cut locus $C(o) = \emptyset$ and hence that $\exp_o : T_oM \to M$ is a diffeomorphism.

Fix any small $t_0 > 0$ so that if $p = \exp_o(t_0X)$ then $d(o,p) = t_0$. Since M is noncompact, there exists a geodesic ray γ emanating from o. For each $s > 0$ consider the geodesic triangle $\triangle op\gamma(s)$, and let $\triangle\tilde{o}\tilde{p}\tilde{\gamma}(s)$ be the corresponding Alexandrov triangle in $\widetilde{M}$. For each s, let $\alpha_s = \measuredangle op\gamma(s)$ and $\tilde{\alpha}_s = \measuredangle\tilde{o}\tilde{p}\tilde{\gamma}(s)$. Thus $\alpha_s \geq \tilde{\alpha}_s$. Let $\sigma_s : [0, d(p,\gamma(s))] \to M$ and $\tilde{\sigma}_s : [0, d(p,\gamma(s))] \to \widetilde{M}$ denote the respective sides of $\triangle op\gamma(s)$ and $\triangle\tilde{o}\tilde{p}\tilde{\gamma}(s)$ joining p to $\gamma(s)$ and $\tilde{p}$ to $\tilde{\gamma}(s)$ respectively. Set $Y_s = \sigma_s'(0) \in T_pM$ and $\tilde{Y}_s = \tilde{\sigma}_s'(0) \in T_{\tilde{p}}\widetilde{M}$. By the triangle inequality

$$L(\sigma_s) = L(\tilde{\sigma}_s) = d(p,\gamma(s)) \geq d(o,\gamma(s)) - d(p,o) = s - t_0.$$

Let $\tilde{\alpha} = \limsup_{s\to\infty} \tilde{\alpha}_s$. We can find an increasing sequence $s_i > 0$ such that $s_i \to \infty$ as $i \to \infty$ and for which $\lim_{i\to\infty} \tilde{\alpha}_{s_i} = \tilde{\alpha}$. Moreover, by compactness of the unit tangent spheres we can also assume there exist $Y \in T_pM$ and $\tilde{Y} \in T_{\tilde{p}}\widetilde{M}$ such that $\lim_{i\to\infty} Y_{s_i} = Y$ and $\lim_{i\to\infty} \tilde{Y}_{s_i} = \tilde{Y}$. It follows that the minimizing geodesics $\tilde{\sigma}_{s_i}$ converge to a geodesic ray $t \mapsto \exp_{\tilde{p}}(t\tilde{Y})$ and the minimizing geodesics σ_{s_i} converge to a geodesic ray $t \mapsto \exp_p(tY)$ for $t \in [0,\infty)$. (The limit is a geodesic ray since the lengths of the minimizing geodesics become arbitrarily large. Here is the calculation: if $t > 0$, then $d(p, \exp_p(tY)) = \lim_{i\to\infty} d(p, \exp_p(tY_{s_i})) = \lim_{i\to\infty} t = t$, and similarly for $t \mapsto \exp_{\tilde{p}}(t\tilde{Y})$.) The angle that $t \mapsto \exp_{\tilde{p}}(t\tilde{Y})$ makes with the geodesic segment from $\tilde{p}$ to $\tilde{o}$ is $\tilde{\alpha}$ since $\lim_{i\to\infty} \tilde{\alpha}_{s_i} = \tilde{\alpha}$. But by hypothesis, there is a unique geodesic ray through $\tilde{p}$ which is the portion of the meridian through $\tilde{p}$ directed away from $\tilde{o}$. Hence $\exp_{\tilde{p}}(t\tilde{Y})$ is that portion of the meridian, and therefore $\lim_{i\to\infty} \tilde{\alpha}_{s_i} = \tilde{\alpha} = \pi$. Since $\pi \geq \alpha_{s_i} \geq \tilde{\alpha}_{s_i}$ it follows that $\lim_{i\to\infty} \alpha_{s_i} = \pi$. Hence the angle Y makes with the geodesic segment from p to o is π. This implies that the geodesic ray $t \mapsto \exp_p(tY)$ for $t > 0$ is the portion of the geodesic $t \mapsto \exp_o(tX)$ for $t \geq t_0$. Since $t_0 > 0$ is an arbitrary small number, we conclude that $t \mapsto \exp_p(tX)$ for $t > 0$ is a geodesic ray. This follows from the calculation: for any $t > 0$, $d(o, \exp_o(tX)) = \lim_{t_0\to 0^+} d(\exp_o(t_0X), \exp_o(tX)) = \lim_{t_0\to 0^+}(t - t_0) = t$. □

The above theorem shows that if $\liminf_{r\to\infty} y(r) = 0$, then M is diffeomorphic to $\mathbf{R}^n$. Weakening the hypothesis to $\liminf_{r\to\infty} \frac{y(r)}{r} < \frac{2}{\pi}$ implies that M has at most one end by [4, Proposition 7.3].

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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