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Hyper Lattice Implication Algebras and Some of Their Characterizations

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Abstract: In this paper, we introduce the concept of hyper lattice implication algebras which is established by combining the concept of hyper lattices and hyper implication algebras. We give some important examples of hyper lattice implication algebras and obtain some of their properties. Then, we characterize hyper lattice implication algebras in which 1 is an implication scalar element. We prove that the implication reduction of hyper lattice implication algebras is a regular commutative fuzzy implication algebra, if 1 is an implication scalar element. Finally, we define the notion of isomorphisms between two hyper lattice implication algebras and obtain all of the hyper lattice implication algebras of order 2.

Keywords: Hyper lattice implication algebra, implication scalar element, fuzzy implication algebra, isomorphism.

1. Introduction

In order to research the logical system whose propositional value is given in a lattice from the semantic viewpoint, Xu proposed the concept of lattice implication algebras in [13] and discussed some of their properties. Since then, many researchers have studied this important logic algebra.

The theory of hyper structures, which is a generalization of the concept of algebraic structures was introduced by F. Marty in 1934 at the eighth congress of Scandinavian mathematicians [7]. The composition of two elements is an element in a classical algebraic structure, while the composition of two elements is a non-empty subset of elements in an algebraic hyper-structure. F. Marty introduced the concept of hyper-group. Since then, many researchers have worked on and developed hyper-structure theory. There are extensive applications in many branches of mathematics and applied sciences, such as Euclidian and non-Euclidian geometries, graphs and hyper-graphs, binary relations, lattices, fuzzy and rough sets, automata, cryptography, codes, probabilities, information sciences and so on. Some interesting applications of hyper structures can be found in the book [2].

R. A. Borzooei et al. introduced and studied hyper K-algebras [1] and S. Ghorbani et al. [3], applied the hyper structures to MV-algebras and introduced the concept of a hyper MV -algebra,

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which is a generalization of the MV-algebra. In [8], Mittas et al. applied the hyper structures to lattices and introduced the concepts of a hyper lattice. S. Rasouli and B. Davvaz proved that a hyper MV-algebra induced a hyper lattice in [10].

In this paper, we introduce the notion of hyper lattice implication algebras as an application of hyper structures to lattice implication lattices.

This paper is organized as follows. In Section 2, some basic definitions are mentioned. In section 3, we introduce the notion of hyper lattice implication algebras as a generalization of lattice implication algebras and give some discrete examples. We investigate some properties of hyper lattice implication algebras. In section 4, we characterize hyper lattice implication algebras in which 1 is an implication scalar. Also, we obtain conditions under which a hyper implication operation in a hyper implication algebra is an implication operation. In section 5, we characterize hyper implication algebras of order 2 and we obtain three non-isomorphic hyper lattice implication algebras of order 2 such that 1 is an implication scalar element and twenty eight non-isomorphic hyper lattice implication algebras of order 2 such that 1 is not an implication scalar element.

2. Preliminaries

Definition 1. ([13]) A lattice implication algebra is a structure $L = (L, \vee, \wedge, \rightarrow, ', 0, 1)$ of type $(2,2,2,1,0,0)$ such that:

(L1) $L = (L, \vee, \wedge, \rightarrow, ', 0, 1)$ is a bounded lattice with an order reversing involution $'$, 1 and 0 are the greatest and the smallest element of L respectively,

(L2) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(L3) $x \rightarrow x = 1$,

(L4) $x \rightarrow y = y' \rightarrow x'$,

(L5) $x \rightarrow y = y \rightarrow x = 1$ implies $x = y$,

(L6) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

(L7) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,

(L8) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,

for all $x, y, z \in L$.

Definition 2. ([9]) A fuzzy implication algebra, FI-algebra for short, is an algebra $(X, \rightarrow, 0)$ of type $(2,0)$ satisfying the following conditions for all $x, y, z \in X$:

(FI1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(FI2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,

(FI3) $x \rightarrow x = 1$,

(FI4) $x \rightarrow y = y \rightarrow x = 1$ implies $x = y$, (FI5) $0 \to x = 1$,

where $1 = 0 \to 0$.

An FI-algebra $(X, \to, 0)$ is called regular, if $CC(x) = x$ for all $x \in X$, where $C(x) = x \to 0$. It is called commutative, if $(x \to y) \to y = (y \to x) \to x$ holds for all $x, y \in X$ (see [11]).

Definition 3. ([7]) Let H be a non-empty set and "$\circ$" be a function from H^2 to $P(H) \backslash \{\emptyset\}$. Then "$\circ$" is called a hyper operation on H.
Note that if $\emptyset \neq A, B \subseteq H$, then by $A \circ B$ we mean the subset $\cup\{a \circ b : a \in A, b \in B\}$ of H, $a \circ B := \{a\} \circ B$ and $A \circ b := A \circ \{b\}$ for all $a, b \in H$.

Definition 4. ([8]) Let L be a nonempty set endowed with hyper operations $\wedge$ and $\vee$. Then $(L, \wedge, \vee)$ is called a hyper lattice if for any $x, y, z \in L$,the following conditions are satisfied:

(HL1) $x \in x \wedge x$, $x \in x \vee x$,
(HL2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,
(HL3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, $x \vee (y \vee z) = (x \vee y) \vee z$,
(HL4) $x \in x \wedge (x \vee y)$, $x \in x \vee (x \wedge y)$.

3. Hyper Lattice Implication Algebras

Definition 5. A hyper lattice implication algebra $(L, \wedge, \vee, \to, ', 0, 1)$ is a non-empty set L equipped with three hyper operations $\wedge$, $\vee$ and $\to$, a unary operation $'$ and two constants 0 and 1 which satisfy the following axioms:

(HLI1) $(L, \vee, \wedge)$ is a hyper lattice such that $0' = 1$ and $1' = 0$,
(HLI2) $1 \in x \to x$,
(HLI3) $1 \in x \to 1$,
(HLI4) $(x')' = x$,
(HLI5) $x \to y = y' \to x'$,
(HLI6) $(x \to y) \to y = (y \to x) \to x$,
(HLI7) $x \to (y \to z) = y \to (x \to z)$,
(HLI8) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,
(HLI9) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(HLI10) $1 \in x \to y$ and $1 \in y \to x$ implies $x = y$,

for all $x, y, z \in L$.

Now, we give some examples of hyper lattice implication algebras. From the following example, we know that the concept of hyper lattice implication algebra is a generalization of lattice implication algebras.

Example 3.1. Each lattice implication algebra is a hyper lattice implication algebra.

Example 3.2. Suppose $L = \{0, a, b, 1\}$ and consider the following tables:

$\wedge$	0	a	b	1
0	$\{0\}$	$\{0,a\}$	$\{0\}$	$\{0\}$
a	$\{0,a\}$	$\{0,a\}$	$\{0\}$	$\{0,a\}$
b	$\{0\}$	$\{0\}$	$\{b\}$	$\{b\}$
1	$\{0\}$	$\{0,a\}$	$\{b\}$	$\{b,1\}$

$\vee$	0	a	b	1
0	$\{0,a\}$	$\{a\}$	$\{b,1\}$	$\{1\}$
a	$\{a\}$	$\{0,a\}$	$\{1\}$	$\{b,1\}$
b	$\{b,1\}$	$\{1\}$	$\{b,1\}$	$\{1\}$
1	$\{1\}$	$\{b,1\}$	$\{1\}$	$\{b,1\}$

$\rightarrow$	0	a	b	1
0	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$
a	$\{0,a\}$	$\{0,a,b,1\}$	$\{0,a,b\}$	$\{0,a,b,1\}$
b	$\{0,b\}$	$\{0,a,b\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$
1	$\{0\}$	$\{0,a\}$	$\{0,b\}$	$\{0,a,b,1\}$

$'$	0	a	b	1
	1	a	b	0

Then, $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ is a hyper lattice implication algebra.

Proposition 1. Let $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ be a hyper lattice implication algebra. Then for all $x, y, z \in L$ and $A, B, C \subseteq L$ the following hold:

(1) $1 \in 1 \rightarrow x$ implies $x = 1$,
(2) $x \in 1 \rightarrow x$ and $x' \in x \rightarrow 0$,
(3) $x \rightarrow (y \rightarrow z) = (x \rightarrow y')' \rightarrow z$,
(4) $x \in 1 \rightarrow 0$ implies $x \neq x'$,
(5) $1 \rightarrow 0 = \{0\}$,
(6) $1 \rightarrow (1 \rightarrow x) = 1 \rightarrow x$,
(7) $y \in 1 \rightarrow x$ implies $1 \in y \rightarrow x$,
(8) $1 \rightarrow x = 1 \rightarrow y$ implies $x = y$,
(9) $(A')' = A$,
(10) $A \rightarrow x = x' \rightarrow A'$,
(11) $A \subseteq A \wedge A$, $A \subseteq A \vee A$,
(12) $A \wedge B = B \wedge A$, $A \vee B = B \vee A$,
(13) $(A \wedge B) \wedge C = A \wedge (B \wedge C)$, $(A \vee B) \vee C = A \vee (B \vee C)$,
(14) $A \subseteq A \vee (A \wedge B)$, $A \subseteq A \wedge (A \vee B)$.

Proof. By (HLI3), we have $1 \in x \rightarrow 1$. So by assumption and (HLI10), we get that $x = 1$.

(2) By (HLI2) and (HLI7), we obtain that

$$1 \in 1 \to 1 \subseteq 1 \to (x \to x) = x \to (1 \to x).$$

Thus, there exists $y \in 1 \to x$ such that $1 \in x \to y$. On the other hand, we have

$$1 \in y \to y \subseteq y \to (1 \to x) = 1 \to (y \to x).$$

by (HLI2) and (HLI7). So there exists $z \in y \to x$ such that $1 \in 1 \to z$. By part (1), we get that $z = 1$. Hence, $1 \in y \to x$. Therefore, $y = x$ by (HLI10) and so $x \in 1 \to x$. By (HLI5), (HLI4) and (HLI1), we have $x' \in 1' \to x' = x \to 0$.

(3) Applying (HLI5), (HLI4) and (HLI7), we obtain

$$(x \to y')' \to z = z' \to (x \to y') = x \to (z' \to y') = x \to (y \to z).$$

(4) Suppose that there exists $x \in 1 \to 0$ such that $x = x'$. Then

$$1 \in x \to x = x' \to x \subseteq (1 \to 0)' \to x = 1 \to (1 \to x).$$

Therefore, there exists $y \in 1 \to x$ such that $1 \in 1 \to y$. By part (1), we have $y = 1$. Hence, $1 \in 1 \to x$. We get $x = 1$ which is a contradiction.

(5) Let $x \in 1 \to 0$. By (HLI2), (HLI7) and (HLI5), we have

$$1 \in x \to x \subseteq x \to (1 \to 0) = 1 \to (1 \to x').$$

Thus, there exists $y \in 1 \to x'$ such that $1 \in 1 \to y$. By part (1), we have $y = 1$. Thus $x' = 1$. By (HLI1), we have $x = 0$.

(6) By part (5) and part (3), we obtain

$$1 \to (1 \to x) = (1 \to 0)' \to x = 1 \to x.$$

(7) Since $y \in 1 \to x$, then $1 \to y \subseteq 1 \to (1 \to x) = 1 \to x$ by part (6). Applying (HLI2), (HLI7), we obtain

$$1 \in 1 \to 1 \subseteq 1 \to (y \to y) = y \to (1 \to y) \subseteq y \to (1 \to x).$$

Thus, there exists $z \in y \to x$ such that $1 \in 1 \to z$. By part (1), we have $z = 1$. Hence $1 \in y \to x$.

(8) By (HLI2) and (HLI7), we get that

$$1 \in 1 \to 1 \subseteq 1 \to (x \to x) = x \to (1 \to x) = x \to (1 \to y) = 1 \to (x \to y).$$

Thus, there exists $z \in x \to y$ such that $1 \in 1 \to z$. By part (1), we have $z = 1$. Hence $1 \in x \to y$. Similarly, we can show that $1 \in y \to x$. Hence $x = y$ by (HLI10).

The proof of the other parts is straightforward. ■

Proposition 2. Let $(L, \sqcap, \sqcup, \hookrightarrow, ', 0, 1)$ be a lattice implication algebra. Define three binary hyper operations $\wedge, \vee$ and $\rightarrow$ on L as follows: for any $x, y \in L$,

$$x \wedge y = [0, x \sqcap y],$$
$$x \vee y = [x \sqcup y, 1],$$
$$x \rightarrow y = [0, x \hookrightarrow y].$$

Then $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ is a hyper lattice implication algebra.

Proof. The proof is straightforward. ■

Proposition 3. Let $(L, \rightarrow, ', 0, 1)$ be a non-empty set L equipped with a hyper operation $\rightarrow$ and a unary operation $'$ such that $0' = 1$ and $1' = 0$. If $(L, \rightarrow, ', 0, 1)$ satisfies conditions (HLI2)-(HLI7) and (HLI10), then $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ is a hyper lattice implication algebra, where

$$x \vee y = \{x, y\},$$
$$x \wedge y = \{x, y\},$$

for all $x, y \in L$.

Proof. It is easy to prove that $(L, \wedge, \vee)$ is a hyper lattice. Let $x, y, z \in L$ be arbitrary. We have

$$\begin{aligned}(x \rightarrow z) \wedge (y \rightarrow z) &= \cup\{t \wedge s : t \in x \rightarrow z, s \in y \rightarrow z\} \\ &= \cup\{\{t, s\} : t \in x \rightarrow z, s \in y \rightarrow z\} \\ &= (\cup\{\{t\} : t \in x \rightarrow z\}) \cup (\cup\{\{s\} : s \in y \rightarrow z\}) \\ &= (x \rightarrow z) \cup (x \rightarrow z) \\ &= \{x, y\} \rightarrow z \\ &= (x \vee y) \rightarrow z.\end{aligned}$$

Similarly, we can prove (HLI9). ■

Example 3.3. Let $L = \{0, b, 1\}$ and consider the following tables:

$\rightarrow$	0	b	1
0	$\{1\}$	$\{b, 1\}$	$\{1\}$
b	$\{0, b\}$	$\{b, 1\}$	$\{b, 1\}$
1	$\{0\}$	$\{0, b\}$	$\{1\}$

$'$	0	b	1
	1	b	0

Then $(L, \rightarrow, ', 0, 1)$ satisfies conditions (HLI2)-(HLI7) and (HLI10). By Proposition 3, $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ is a hyper lattice implication algebra, where $x \wedge y = \{x, y\}$ and $x \wedge y = \{x, y\}$ for all $x, y \in L$.

Theorem 1. Let $(L, \rightarrow, ', 0, 1)$ be a non-empty set L equipped with a hyper operation $\rightarrow$ and a unary operation $'$ satisfying conditions (HLI2)-(HLI8) and $0' = 1$ and $1' = 0$. Define two binary hyper operations $\wedge, \vee$ on L as follows: for any $x, y \in L$,

$$x \vee y = (x' \to y)' \to y,$$
$$(x \wedge y)' = x' \vee y'.$$

Then $(L, \wedge, \vee)$ is a hyper lattice. Moreover, if $(L, \wedge, \vee, \to, ', 0, 1)$ satisfies conditions (HLI8) - (HLI10), then it is a hyper lattice implication algebra.

Proof. We will show that $(L, \vee, \wedge, 0, 1)$ is a hyper lattice. Suppose that $x, y, z \in L$. Then,

(HL1) By Proposition 1 part (2) and (HLI2), we have

$$x \in 1 \to x \subseteq (x \to x) \to x = x \vee x.$$

Since $x' \in x' \vee x'$, then $x' \in (x \wedge x)'$. By (HLI4), we get $x \in x \wedge x$.

(HL2) It follows from (HLI6) that $\vee$ is commutative. By commutativity of $\vee$, definition $\wedge$ and (HLI4), we conclude $\wedge$ is commutative.

(HL3) Using (HIL4), (HLI7) and (HL2), we have

$$\begin{aligned}
x \vee (y \vee z) &= (x \to (y \vee z)) \to (y \vee z) \\
&= (x \to ((z \to y) \to y)) \to ((z \to y) \to y) \\
&= (z \to y) \to (((z \to y) \to (x \to y)) \to y) \\
&= y' \to (((z \to y) \to (x \to y)) \to (z \to y)') \\
&= y' \to (((x \to y)' \to (z \to y)') \to (z \to y)') \\
&= y' \to ((x \to y)' \vee (z \to y)') \\
&= y' \to (((z \to y)' \to (x \to y)') \to (x \to y)') \\
&= y' \to (((x \to y) \to (z \to y)) \to (x \to y)') \\
&= (x \to y) \to (((x \to y) \to (z \to y)) \to y) \\
&= (z \to ((x \to y) \to y) \to ((x \to y) \to y \\
&= (z \to (x \vee y)) \to (x \vee y) \\
&= z \vee (x \vee y) \\
&= (x \vee y) \vee z.
\end{aligned}$$

It follows from associative of $\vee$, definition of $\wedge$ and (HLI4), that $\wedge$ is associative.

(HL4) By (HLI4), Proposition 1 part (2) and (HLI6), we get that

$$\begin{aligned} x = (x')' &\in (1 \to x')' \\ &\subseteq (x \to ((x \to y) \to y)) \to x')' \\ &= ((x \to (x \vee y)) \to x')' \\ &= (((x \vee y)' \to x') \to x')' \\ &= (x' \vee (x \vee y)')' = x \wedge (x \vee y). \end{aligned}$$

■

Remark 3.1. (1) The conditions (HLI8) and (HLI9) are necessary in the above theorem. Consider $(L, \to, ', 0, 1)$ in Example 3.3. If we define two hyper operations $\wedge, \vee$ as in Proposition 3, then $(L, \wedge, \vee, \to, ', 0, 1)$ is not a hyper lattice implication algebra, because $(0 \wedge 0) \to a = 0 \to a = \{a, 1\}$ but $(0 \to a) \vee (0 \to a) = L$.

(2) We know that every hyper lattice implication algebra satisfies the conditions (HLI2)-(HLI8) but $x \vee y = (x' \to y)' \to y$ may not be true in general. Consider Example 3.2.

4. Implication Scalar Elements

Definition 6. An element a of a hyper lattice implication algebra $(L, \wedge, \vee, \to, ', 0, 1)$ is called

(1) left implication scalar if $|a \to x| = 1$ for all $x \in L$,
(2) right implication scalar if $|x \to a| = 1$ for all $x \in L$,
(3) implication scalar if it is both right and left implication scalar.

Proposition 4. Let 1 be a right implication scalar element of a hyper lattice implication algebra $(L, \wedge, \vee, \to, ', 0, 1)$. Then,

(1) $1 \wedge 1 = \{1\}$ and $1 \vee 1 = \{1\}$,
(2) $0 \wedge 0 = \{0\}$, $0 \vee 0 = \{0\}$,
(3) $0 \to x = \{1\}$,
(4) $x \to x = \{1\}$, for all $x \in L$,
(5) $|x \to y| = 1$ for all $x, y \in L$,
(6) $0 \in x \to 0$ implies $x = 1$.

Proof. (1) Since 1 is a right implication scalar element and $1 \in x \to 1$, we have

$$\{1\} = (x \wedge y) \to 1 = (x \to 1) \vee (x \to 1) = 1 \vee 1$$

by (HLI8). Hence $1 \vee 1 = \{1\}$. Similarly, we can show that $1 \wedge 1 = \{1\}$.

(2) By part (1), Proposition 1 part (5) and (HLI9), we have

$$\{0\} = 1 \to 0 = (1 \vee 1) \to 0 = (1 \to 0) \wedge (1 \to 0) = 0 \wedge 0.$$

Therefore, $0 \wedge 0 = \{0\}$. Similarly, we can prove $0 \vee 0 = \{0\}$.

(3) It follows from Definition 6 part (2) and (HLI5).

(4) By (HLI3), (HLI6) and Proposition 1 part (2), we get that

$$1 \in x \to x \subseteq (1 \to x) \to x = (x \to 1) \to 1 = 1 \to 1 = \{1\}.$$

Hence, $x \to x = \{1\}$ for all $x \in L$.

(5) Suppose that $a, b \in x \to y$ are arbitrary. By (HLI7) and (HLI6), we have

$$\begin{aligned}(x \to y) \to (x \to y) &= x \to ((x \to y) \to y) = x \to ((y \to x) \to x) \\ &= (y \to x) \to (x \to x) = (y \to x) \to 1 = \{1\}.\end{aligned}$$

Since $a \to b, b \to a \subseteq (x \to y) \to (x \to y) = \{1\}$, we obtain that $a \to b = b \to a = \{1\}$. Then $a = b$ by (HLI10). Therefore, $|x \to y| = 1$ for all $x, y \in L$.

(6) Let $0 \in x \to 0$. By part (4), (HLI6) and part (3), we have

$$\{1\} = 0 \to 0 = (x \to 0) \to 0 = (0 \to x) \to x = 1 \to x.$$

Hence, $x = 1$ by Proposition 1 part (1). ∎

Theorem 2. Let 1 be a right implication scalar element of a hyper lattice implication algebra $(L, \wedge, \vee, \to, ', 0, 1)$. Then, 1 is an implication scalar element of L.

Proof. We will show that 1 is a left implication scalar element of L. Suppose that $y \in 1 \to x$. By Proposition 4 part (5), we have $1 \to x = \{y\}$. Using (HLI5), (HLI6) and Proposition 4 part (3), we have

$$1 \to y' = y \to 0 \subseteq (x \to 0) \to 0 = (0 \to x) \to x = 1 \to x.$$

Thus, $x = y'$ by Proposition 1 part (8), that is $1 \to x = \{x\}$. Hence, 1 is a left implication scalar element of L. ∎

Proposition 5. Let 1 be a left implication scalar element of a hyper lattice implication algebra $(L, \wedge, \vee, \to, ', 0, 1)$. Then,

(1) $1 \to x = \{x\}$,
(2) $x \to 0 = \{x'\}$,
(3) $(0 \to x) \to x = \{x\}$,
(4) $0 \to 0 = \{1\}$,

(5) $0, x \notin 0 \to x$, for all $x \in L - \{1\}$.

Proof. (1) By Proposition 1 part (2), $x \in 1 \to x$. Since 1 is a left implication scalar element, we have $|1 \to x| = 1$ for all $x \in L$. Hence $1 \to x = \{x\}$.

(2) We have $x \to 0 = 1 \to x'$ by (HLI5). Thus (2), follows from part (1).

(3) By (HLI6), part (2) and (HLI4), we obtain that

$$(0 \to x) \to x = (x \to 0) \to 0 = x' \to 0 = \{(x')'\} = \{x\}.$$

(4) It follows from part (2).

(5) if $x = 0$, then $0 \notin 0 \to 0$ by part (3). Now, suppose that $x \neq 0$.

If $0 \in 0 \to x$, then

$$0 \in 0 \to x \subseteq (0 \to x) \to x = \{x\}$$

by part (3). Hence $x = 0$, which is a contradiction.

If $x \in 0 \to x$, then

$$1 \in x \to x \subseteq (0 \to x) \to x = \{x\}$$

by (HLI2) and part (3). Hence, $x = 1$, which is a contradiction. ■

Proposition 6. Let 1 be a left implication scalar element of a hyper lattice implication algebra $(L, \wedge, \vee, \to, ', 0, 1)$. Then,

(1) $(x \vee y)' = x' \wedge y'$,
(2) $(x \wedge y)' = x' \vee y'$,

for all $x, y \in L$.

Proof. (1) By Proposition 5 part (2) and (HLI8), we obtain

$$x' \wedge y' = (x \to 0) \wedge (y \to 0) = (x \vee y) \to 0 = (x \vee y)'.$$

Similarly, part (2) can be proved. ■

Proposition 7. Let 1 be a left implication scalar element of a hyper lattice implication algebra $(L, \wedge, \vee, \to, ', 0, 1)$. Then, $0 \to 1 = \{1\}$.

Proof. We have $1 \in 0 \to 1$ by (HLI3). Suppose that $x \in 0 \to 1$, where $x \in L - \{1\}$. By Proposition 5 part (3), we have $x \to 1 \subseteq (0 \to 1) \to 1 = \{1\}$. Hence, $x \to 1 = \{1\}$. Applying Proposition 1 part (3) and (HLI6), we get that

$$\begin{aligned} 0 \to 1 &= ((x \to 1) \to 1)' \to 1 = (x \to 1) \to (0 \to 1) = 0 \to ((x \to 1) \to 1) \\ &= 0 \to ((1 \to x) \to x) = 0 \to (x \to x) = (0 \to x')' \to x \\ &= (x \to 1)' \to x = 0 \to x. \end{aligned}$$

Since $x \in 0 \to 1 = 0 \to x$, then $x \in 0 \to x$, which is a contradiction by Proposition 5 part (5). ■

Theorem 3. Let 1 be a left implication scalar element of a hyper lattice implication algebra $(L, \wedge, \vee, \to, ', 0, 1)$ such that $x' = x$, for all $x \in L - \{0,1\}$. Then, 1 is an implication scalar element.

Proof. We know that $0 \to 1 = \{1\}$, by Proposition 7. Now, suppose that $y \in x \to 1$, where $x \in L - \{0\}$ and $y \in L - \{1\}$. Then, $y \to x \subseteq (0 \to x) \to x = x$ by Proposition 5 part (3). Hence, $y \to x = x$ and we have

$$\begin{aligned} x' \to 1 &= ((y \to 1) \to x)' \to 1 = (y \to 1) \to (x' \to 1) = x' \to ((y \to 1) \to 1) \\ &= x' \to ((1 \to y) \to y) = x' \to (y \to y) = (x' \to y')' \to y \\ &= (y \to x)' \to y = x' \to y \end{aligned}$$

by Proposition 1 part (3) and (HLI6). Since $1 \in x' \to 1 = x' \to y = y' \to x$, $x' = x$ and $y' = y$, we get that $1 \in x \to y = y \to x$, that is $x = y$ by (HLI10). Hence, $x' = x \in x \to 1 = 0 \to x'$, which is a contradiction by Proposition 5 part (5). Thus, $x \to 1 = \{1\}$ for all $x \in L$. Therefore, 1 is a right implication scalar element. ■

Theorem 4. Let 1 be a left implication scalar of a finite hyper lattice implication algebra $(L, \wedge, \vee, \to, ', 0, 1)$. Then, 1 is an implication scalar of L.

Proof. We will show that $x \to 1 = \{1\}$. Suppose that there exists $x \in L$ such that $x \to 1 \neq \{1\}$. Since L is finite and $x \to 1 \subseteq L$, there exist $y_1, \ldots, y_n \in L - \{1\}$ such that $x \to 1 = \{y_1, \ldots, y_n, 1\}$. By Theorem 7 and Proposition 1 part (3),

$$\{y_1, \ldots, y_n, 1\} = x \to 1 = x \to (0 \to 1) = x \to 1)' \to 1 = (0 \to 1) \cup (y_1' \to 1) \cup \ldots \cup (y_n' \to 1).$$

By Proposition 5 part (5), we have $y_i \notin 0 \to y_i$ for all $1 \leq i \leq n$. Hence, there exists $2 \leq i \leq n$ such that $y_1 \in 0 \to y_i$. Without loss of generality, suppose that $y_1 \in 0 \to y_2$. We claim that $y_2 \notin 0 \to y_1$. If $y_2 \in 0 \to y_1$, then

$$y_2 \to y_1 \subseteq (0 \to y_1) \to y_1 = y_1.$$

Hence $y_2 \to y_1 = y_1$. Also, we have

$$\begin{aligned} y_1' \to 1 &= ((y_2 \to 1) \to y_1)' \to 1 \\ &= (y_2 \to 1) \to (y_1' \to 1) \\ &= y_1' \to ((y_2 \to 1) \to 1) \\ &= y_1' \to ((1 \to y_2) \to y_2) \\ &= y_1' \to (y_2 \to y_2) \\ &= (y_1' \to y_2')' \to y_2 \\ &= (y_2 \to y_1)' \to y_2 \\ &= y_1' \to y_2. \end{aligned}$$

Similarly, we can prove that $y_2' \to 1 = y_2' \to y_1$. By (HLI5), we get that

$$y_2 \in 0 \to y_1 = y_1' \to 1 = y_1' \to y_2 = y_2' \to y_1 = y_2' \to 1 = 0 \to y_2$$

which is a contradiction. Hence, $y_2 \notin 0 \to y_1$. Thus, there exists $3 \leq i \leq n$ such that $y_2 \in 0 \to y_i$. Without loss of generality, suppose that $y_3 \in 0 \to y_2$ such that $y_2 \notin 0 \to y_3$. Similarly, we can show that $y_{n-1} \in 0 \to y_n$ such that $y_n \notin 0 \to y_{n-1}$. Hence, there exists $1 \leq k \leq n-2$ such that $y_n \in 0 \to y_k$. We have

$$0 \to y_k \subseteq 0 \to (0 \to y_{k+1}) = 0 \to y_{k+1} \subseteq \ldots \subseteq 0 \to y_{n-1} \subseteq 0 \to y_n.$$

On the other hand, $0 \to y_n \subseteq 0 \to (0 \to y_k) = 0 \to y_k$. We get that $0 \to y_k = 0 \to y_n$, that is $y_n \in 0 \to y_n$, which is a contradiction. Hence, $x \to 1 \neq \{1\}$ for all $x \in L$. Therefore, 1 is a right implication scalar of L. ■

Theorem 5. Let $(L, \wedge, \vee, \to, ', 0, 1)$ be a hyper lattice implication algebra such that $x \to x = \{1\}$. Then, 1 is an implication scalar element of L.

Proof. We will prove that 1 is a left implication element of L. By Proposition 1 part (2), we have $x \in 1 \to x$. Suppose that $y \in 1 \to x$ is arbitrary. Then,

$$x \to y = x \to (1 \to x) = 1 \to (x \to x) = 1 \to 1 = \{1\}.$$

Since $y \in 1 \to x$, then $1 \in y \to x$ by Proposition 1 part (6). Hence, $y = x$, that is $1 \to x = \{x\}$ for all $x \in L$. Hence, 1 is a left implication element of L.

Now, we will prove that 1 is a right implication element of L. Suppose that $x \in L$ is arbitrary. We have $1 \in x \to 1$ by (HLI3). Applying (HLI5) and Proposition 1 part (3), we obtain

$$\begin{aligned} x \to 1 \subseteq x \to (x \to 1) = (x \to 1)' \to x' \\ = (0 \to x')' \to x' = 0 \to (x \to x) = 0 \to 1 = \{1\}. \end{aligned}$$

Hence, $x \to 1 = \{1\}$ for all $x \in L$, that is 1 is a right implication element of L. ■

Corollary 1. Let $(L, \wedge, \vee, \to, ', 0, 1)$ be a hyper lattice implication algebra. Then, 1 is an implication scalar element of L if and only if $x \to x = \{1\}$ for all $x \in L$.

Proof. It follows from Proposition 4 part (4), Theorem 2 and Theorem 5. ■

Theorem 6. Let $(L, \wedge, \vee, \to, ', 0, 1)$ be a hyper lattice implication algebra such that 1 is an implication scalar element of L. Then, $(L, \to, 0)$ is a regular commutative FI-algebra.

Proof. We have that (FI1), (FI3), (FI4) and (FI5) follow by (HLI7), Proposition 4 part (4), (HLI10) and Proposition 4 part (3) respectively. We will prove (FI2). Let $x, y, z \in L$ be arbitrary. Since 1 is an implication scalar element of L, then $|x \to y| = 1$ by Proposition 4 part (5). Therefore, there exists $a \in L$ such that $x \to y = a$. We obtain $(x \to y) \to (x \to y) = 1$. Applying (HLI7) and (HLI6), we have

$$\begin{aligned} (x \to y) \to ((y \to z) \to (x \to z)) &= (x \to y) \to (x \to ((y \to z) \to z)) \\ &= (x \to y) \to (x \to ((z \to y) \to y)) \\ &= (x \to y) \to ((z \to y) \to (x \to y)) \\ &= (z \to y) \to ((x \to y) \to (x \to y)) \\ &= (z \to y) \to 1 = 1. \end{aligned}$$

Then, $(L, \to, 0)$ is a FI-algebra and it is regular by Proposition 5 part (2) and (HLI4). Finally, $(L, \to, 0)$ is commutative by (HLI6). ■

5. Hyper Lattice Implication Algebras of Order 2

In this section, we will obtain all hyper lattice implication algebras of order 2. For this, we define the concept of homomorphism of hyper lattice implication algebras.

Definition 7. Let L_1 and L_2 be two lattice implication algebras. A mapping $f : L_1 \to L_2$ is said to be a homomorphism, if

$$f(0) = 0,$$
$$f(x') = f(x)',$$

$$f(x\wedge y) = f(x)\wedge f(y),$$
$$f(x\vee y) = f(x)\vee f(y),$$
$$f(x\to y) = f(x)\to f(y),$$

for all $x,y\in L_1$.

Clearly, if f is a homomorphism, then $f(1)=1$.

If f is one-to-one (or onto), then we say that f is a monomorphism (or epimorphism), and if f is both one-to-one and onto, we say that f is an isomorphism.

Theorem 7. There are three non-isomorphic hyper lattice implication algebras of order 2 such that 1 is an implication scalar element.

Proof. Consider $L=\{0,1\}$. Since L is a hyper lattice implication algebra, then $1\to 0=\{0\}$. By assumption, 1 must be an implication scalar element. Hence, we have $1\to 1=\{1\}=0\to 0$ and $0\to 1=\{1\}$. Therefore, we have the following hyper operation $\to$ on L such that 1 is an implication scalar element:

$\to$	0	1
0	$\{1\}$	$\{1\}$
1	$\{0\}$	$\{1\}$

Also, we have $1\wedge 1=1=1\vee 1$ and $0\wedge 0=0=0\vee 0$ by Proposition 4. Since $0\wedge 1=1\wedge 0\subseteq\{0,1\}$ and $0\vee 1=(0\wedge 1)'$ by Proposition 6, we have the following hyper operations $\wedge,\vee$ on L:

(1) if $0\wedge 1=\{0\}$, then $0\vee 1=\{1\}$. Hence, we obtain

$\wedge_1$	0	1
0	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$

$\vee_1$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{1\}$

(2) if $0\wedge 1=\{1\}$, then $0\vee 1=\{0\}$. Therefore, we obtain

$\wedge_2$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{1\}$

$\vee_2$	0	1
0	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$

(3) if $0\wedge 1=\{0,1\}$, then $0\vee 1=\{0,1\}$, then we get

$\wedge_3$	0	1
0	$\{0\}$	$\{0, 1\}$
1	$\{0, 1\}$	$\{1\}$

$\vee_3$	0	1
0	$\{0\}$	$\{0, 1\}$
1	$\{0, 1\}$	$\{1\}$

We can check that $(L,\wedge_1,\vee_1,\rightarrow,0,1)$, $(L,\wedge_2,\vee_2,\rightarrow,0,1)$ and $(L,\wedge_3,\vee_3,\rightarrow,0,1)$ are three non-isomorphic hyper lattice implication algebras such that 1 is an implication scalar.

■

Lemma 1. Let L be a hyper lattice implication algebra of order 2. If $1 \rightarrow 1 = \{1\}$, then 1 is an implication scalar element.

Proof. By assumption and (HLI5), we have $0 \rightarrow 0 = 1 \rightarrow 1 = \{1\}$. By Proposition 1 part (5), we have $1 \rightarrow 0 = \{0\}$. Hence, 1 is a left implication scalar element. By Theorem 4, 1 is an implication scalar element. ■

Theorem 8. There are twenty eight non-isomorphic hyper lattice implication algebras of order 2 such that 1 is not an implication scalar element.

Proof. Suppose that $L = \{0,1\}$. By Proposition 1 part (5), we have $1 \rightarrow 0 = \{0\}$. Since 1 is not an implication scalar and $1 \in 1 \rightarrow 1 \subseteq \{0,1\}$, we have $0 \rightarrow 0 = 1 \rightarrow 1 = \{0,1\}$ by Lemma 1. Also, $1 \in 0 \rightarrow 1$. Hence, $0 \rightarrow 1 = \{1\}$ or $0 \rightarrow 1 = \{0,1\}$. Therefore, we have two hyper operations $\rightarrow_1$ and $\rightarrow_2$ on L as follows:

$\rightarrow_1$	0	1
0	{0, 1}	{ 1}
1	{0}	{0, 1}

$\rightarrow_2$	0	1
0	{0, 1}	{0, 1}
1	{0}	{0, 1}

By (HL1), we have $1 \in 1 \wedge 1 \subseteq \{0,1\}$ and $1 \in 1 \vee 1 \subseteq \{0,1\}$. Hence, $1 \wedge 1 = \{1\}$ or $1 \wedge 1 = \{0,1\}$. Also, $1 \vee 1 = \{1\}$ or $1 \vee 1 = \{0,1\}$.

(1) Suppose that $1 \wedge 1 = \{1\}$ and $1 \vee 1 = \{1\}$. By (HLI8) and (HLI9), we obtain

$$\{0\} = 1 \rightarrow 0 = (1 \wedge 1) \rightarrow 0 = (1 \rightarrow 0) \vee (1 \rightarrow 0) = 0 \vee 0,$$

$$\{0\} = 1 \rightarrow 0 = (1 \vee 1) \rightarrow 0 = (1 \rightarrow 0) \wedge (1 \rightarrow 0) = 0 \wedge 0.$$

We have $0 \vee 1 = \{1\}$ or $0 \vee 1 = \{0\}$ or $0 \vee 1 = \{0,1\}$. Consider the following cases:

(i) Suppose that $0 \vee 1 = \{1\}$. Since $0 = 0 \wedge 0$. Then,

$$\{0\} = 1 \rightarrow 0 = (0 \vee 1) \rightarrow 0 = (0 \rightarrow 0) \wedge (1 \rightarrow 0) = (0 \wedge 0) \cup (1 \wedge 0).$$

Hence, $0 \wedge 1 = \{0\}$. Therefore, $(L,\wedge_1,\vee_1,\rightarrow_1,0,1)$ and $(L,\wedge_1,\vee_1,\rightarrow_2,0,1)$ are two non-isomorphic hyper lattice implication algebras where

$\wedge_1$	0	1
0	{0}	{0}
1	{0}	{1}

$\vee_1$	0	1
0	{0}	{1}
1	{1}	{1}

(ii) Let $0 \vee 1 = \{0\}$. Since $1 \in 1 \wedge (0 \vee 1) = 0 \wedge 1$ and $0 \in 1 \wedge (0 \vee 1) = 0 \wedge 1$ by (HL4), then $0 \wedge 1 = \{0,1\}$. Therefore, $(L, \wedge_2, \vee_2, \rightarrow_1, 0, 1)$ and $(L, \wedge_2, \vee_2, \rightarrow_2, 0, 1)$ are two non-isomorphic hyper lattice implication algebras where

$\wedge_2$	0	1
0	$\{0\}$	$\{0, 1\}$
1	$\{0, 1\}$	$\{1\}$

$\vee_2$	0	1
0	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$

(iii) Let $0 \vee 1 = \{0,1\}$. Then, $0 \wedge 1 \neq \{0\}$. If $0 \wedge 1 = \{0\}$, then

$$\{0,1\} = (1 \vee 0) \rightarrow 0 = (1 \rightarrow 0) \wedge (0 \rightarrow 0) = 0 \wedge \{0,1\} = (0 \wedge 0) \cup (0 \wedge 1),$$

which is a contradiction. Also, $0 \wedge 1 \neq \{1\}$. If $0 \wedge 1 = \{1\}$, then

$$\{0\} = 1 \rightarrow 0 = (1 \wedge 0) \rightarrow 0 = (1 \rightarrow 0) \vee (0 \rightarrow 0) = 0 \vee \{0,1\} = (0 \vee 0) \cup (0 \vee 0) = \{0,1\}$$

which is a contradiction. Hence, we have $0 \wedge 1 = \{0,1\}$. Therefore, $(L, \wedge_3, \vee_3, \rightarrow_1, 0, 1)$ and $(L, \wedge_3, \vee_3, \rightarrow_2, 0, 1)$ are two non-isomorphic hyper lattice implication algebras where

$\wedge_3$	0	1
0	$\{0\}$	$\{0, 1\}$
1	$\{0, 1\}$	$\{1\}$

$\vee_3$	0	1
0	$\{0\}$	$\{0, 1\}$
1	$\{0, 1\}$	$\{1\}$

(2) Suppose that $1 \wedge 1 = \{1\}$ and $1 \vee 1 = \{0,1\}$. Then, we have

$$\{0\} = 1 \rightarrow 0 = (1 \wedge 1) \rightarrow 0 = (1 \rightarrow 0) \vee (1 \rightarrow 0) = 0 \vee 0,$$

$$\{0,1\} = \{0,1\} \rightarrow 0 = (1 \vee 1) \rightarrow 0 = (1 \rightarrow 0) \wedge (1 \rightarrow 0) = 0 \wedge 0.$$

We have $0 \vee 1 \neq \{1\}$. Suppose that $0 \vee 1 = \{1\}$. Since $0 \wedge 0 = \{0,1\}$, then

$$\{0\} = 1 \rightarrow 0 = (0 \vee 1) \rightarrow 0 = (0 \rightarrow 0) \wedge (1 \rightarrow 0) = (0 \wedge 0) \cup (1 \wedge 0) = \{0,1\},$$

which is a contradiction. Hence, we have or $0 \vee 1 = \{0\}$ or $0 \vee 1 = \{0,1\}$. Consider the following cases:

(i) Let $0 \vee 1 = \{0\}$. Since $1 \in 1 \wedge (0 \vee 1) = 0 \wedge 1$ and $0 \in 1 \wedge (0 \vee 1) = 0 \wedge 1$ by (HL3), then $0 \wedge 1 = \{0,1\}$. Thus, $(L, \wedge_4, \vee_4, \rightarrow_1, 0, 1)$ and $(L, \wedge_4, \vee_4, \rightarrow_2, 0, 1)$ are two non-isomorphic hyper lattice implication algebras where

$\wedge_4$	0	1
0	$\{0, 1\}$	$\{0, 1\}$
1	$\{0, 1\}$	$\{1\}$

$\vee_4$	0	1
0	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{0,1\}$

(ii) Let $0 \vee 1 = \{0,1\}$. If $0 \wedge 1 = \{0\}$, then there are two non-isomorphic hyper lattice implication algebras $(L, \wedge_5, \vee_5, \rightarrow_1, 0, 1)$ and $(L, \wedge_5, \vee_5, \rightarrow_2, 0, 1)$ where

$\wedge_5$	0	1
0	{0, 1}	{0}
1	{0}	{1}

$\vee_5$	0	1
0	{0}	{0, 1}
1	{0, 1}	{0,1}

If $0 \wedge 1 = \{1\}$, then there are two non-isomorphic hyper lattice implication algebras $(L, \wedge_6, \vee_6, \rightarrow_1, 0, 1)$ and $(L, \wedge_6, \vee_6, \rightarrow_2, 0, 1)$ where

$\wedge_6$	0	1
0	{0, 1}	{1}
1	{1}	{1}

$\vee_6$	0	1
0	{0}	{0, 1}
1	{0, 1}	{0,1}

If $0 \wedge 1 = \{0, 1\}$, then there are two non-isomorphic hyper lattice implication algebras $(L, \wedge_7, \vee_7, \rightarrow_1, 0, 1)$ and $(L, \wedge_7, \vee_7, \rightarrow_2, 0, 1)$ where

$\wedge_7$	0	1
0	{0, 1}	{0, 1}
1	{0, 1}	{1}

$\vee_7$	0	1
0	{0}	{0, 1}
1	{0, 1}	{0,1}

(3) Suppose that $1 \wedge 1 = \{0, 1\}$ and $1 \vee 1 = \{1\}$. Then,

$$\{0,1\} = \{0,1\} \rightarrow 0 = (1 \wedge 1) \rightarrow 0 = (1 \rightarrow 0) \vee (1 \rightarrow 0) = 0 \vee 0.$$

$$0 = 1 \rightarrow 0 = (1 \vee 1) \rightarrow 0 = (1 \rightarrow 0) \wedge (1 \rightarrow 0) = 0 \wedge 0.$$

We have $0 \wedge 1 \neq \{1\}$. Assume that $0 \wedge 1 = \{1\}$. Since $0 \vee 0 = \{0, 1\}$, then

$$\{0\} = 1 \rightarrow 0 = (0 \wedge 1) \rightarrow 0 = (0 \rightarrow 0) \vee (1 \rightarrow 0) = (0 \vee 0) \cup (1 \vee 0) = \{0, 1\},$$

which is a contradiction. Hence, we have $0 \wedge 1 = \{0\}$ or $0 \wedge 1 = \{0, 1\}$.

(i) Let $0 \wedge 1 = \{0\}$. Since $1 \in 1 \vee (0 \wedge 1) = 0 \vee 1$ and $0 \in 1 \vee (0 \wedge 1) = 0 \vee 1$ by (HL3), then $0 \vee 1 = \{0, 1\}$. Therefore, $(L, \wedge_8, \vee_8, \rightarrow_1, 0, 1)$ and $(L, \wedge_8, \vee_8, \rightarrow_2, 0, 1)$ are two non-isomorphic hyper lattice implication algebras where

$\wedge_8$	0	1
0	{0}	{0}
1	{0}	{0,1}

$\vee_8$	0	1
0	{0, 1}	{0, 1}
1	{0, 1}	{1}

(ii) Let $0 \wedge 1 = \{0, 1\}$. If $0 \vee 1 = \{0\}$, then there are two non-isomorphic hyper lattice implication algebras $(L, \wedge_9, \vee_9, \rightarrow_1, 0, 1)$ and $(L, \wedge_9, \vee_9, \rightarrow_2, 0, 1)$ where

$\wedge_9$	0	1
0	{0}	{0, 1}
1	{0,1}	{0,1}

$\vee_9$	0	1
0	{0, 1}	{0}
1	{0}	{1}

If $0 \vee 1 = \{1\}$, then there are two non-isomorphic hyper lattice implication algebras $(L, \wedge_{10}, \vee_{10}, \rightarrow_1, 0, 1)$ and $(L, \wedge_{10}, \vee_{10}, \rightarrow_2, 0, 1)$ where

$\wedge_{10}$	0	1
0	{0}	{0, 1}
1	{0,1}	{0,1}

$\vee_{10}$	0	1
0	{0, 1}	{1}
1	{1}	{1}

If $0 \vee 1 = \{0, 1\}$, then there are two non-isomorphic hyper lattice implication algebras $(L, \wedge_{11}, \vee_{11}, \rightarrow_1, 0, 1)$ and $(L, \wedge_{11}, \vee_{11}, \rightarrow_2, 0, 1)$ where

$\wedge_{11}$	0	1
0	{0}	{0, 1}
1	{0,1}	{0,1}

$\vee_{11}$	0	1
0	{0, 1}	{0, 1}
1	{0, 1}	{1}

(4) Suppose that $1 \wedge 1 = \{0, 1\}$ and $1 \vee 1 = \{0, 1\}$. Then,

$$\{0,1\} = \{0,1\} \rightarrow 0 = (1 \wedge 1) \rightarrow 0 = (1 \rightarrow 0) \vee (1 \rightarrow 0) = 0 \vee 0.$$

$$\{0,1\} = \{0,1\} \rightarrow 0 = (1 \vee 1) \rightarrow 0 = (1 \rightarrow 0) \wedge (1 \rightarrow 0) = 0 \wedge 0.$$

We have $0 \wedge 1 \neq \{1\}$. Suppose that $0 \wedge 1 = \{1\}$. Since $0 \vee 0 = \{0, 1\}$, then

$$\{0\} = 1 \rightarrow 0 = (0 \wedge 1) \rightarrow 0 = (0 \rightarrow 0) \vee (1 \rightarrow 0) = (0 \vee 0) \cup (1 \vee 0) = \{0, 1\},$$

which is a contradiction. Hence we have $0 \wedge 1 = \{0\}$ or $0 \wedge 1 = \{0, 1\}$.

Also, we have $0 \vee 1 \neq \{1\}$. Assume that $0 \vee 1 = \{1\}$. Since $0 \wedge 0 = \{0, 1\}$, then

$$\{0\} = 1 \rightarrow 0 = (0 \vee 1) \rightarrow 0 = (0 \rightarrow 0) \wedge (1 \rightarrow 0) = (0 \wedge 0) \cup (1 \wedge 0) = \{0, 1\},$$

which is a contradiction. Hence we have $0 \vee 1 = \{0\}$ or $0 \vee 1 = \{0, 1\}$.

(i) Let $0 \vee 1 = \{0\}$. Since $1 \in 1 \wedge (0 \vee 1) = 0 \wedge 1$ and $0 \in 1 \wedge (0 \vee 1) = 0 \wedge 1$ by (HL3), then $0 \wedge 1 = \{0, 1\}$. So $(L, \wedge_{12}, \vee_{12}, \rightarrow_1, 0, 1)$ and $(L, \wedge_{12}, \vee_{12}, \rightarrow_2, 0, 1)$ are two non-isomorphic hyper lattice implication algebras where

$\wedge_{12}$	0	1
0	{0, 1}	{0, 1}
1	{0, 1}	{0, 1}

$\vee_{12}$	0	1
0	{0, 1}	{0}
1	{0}	{0,1}

(ii) Let $0 \vee 1 = \{0, 1\}$. If $0 \wedge 1 = \{0\}$, then there are two non-isomorphic hyper lattice implication algebras $(L, \wedge_{13}, \vee_{13}, \rightarrow_1, 0, 1)$ and $(L, \wedge_{13}, \vee_{13}, \rightarrow_2, 0, 1)$ where

$\wedge_{13}$	0	1
0	{0, 1}	{0}
1	{0}	{0, 1}

$\vee_{13}$	0	1
0	{0, 1}	{0, 1}
1	{0, 1}	{0,1}

If $0 \wedge 1 = \{0,1\}$, then there are two non-isomorphic hyper lattice implication algebras $(L, \wedge_{14}, \vee_{14}, \rightarrow_1, 0, 1)$ and $(L, \wedge_{14}, \vee_{14}, \rightarrow_2, 0, 1)$ where

$\wedge_{14}$	0	1
0	$\{0, 1\}$	$\{0, 1\}$
1	$\{0, 1\}$	$\{0, 1\}$

$\vee_{14}$	0	1
0	$\{0, 1\}$	$\{0, 1\}$
1	$\{0, 1\}$	$\{0,1\}$

■

Corollary 2. There are thirty one non-isomorphic hyper lattice implication algebras of order 2.

6. Conclusions

In this paper, we introduce the notion of hyper lattice implication algebras and study their basic properties. We obtain some conditions under which a hyper implication operation in a hyper implication algebra is an implication operation. Finally, hyper lattice implication algebras of order 2 are considered. In future work, we will study the relations between hyper lattice implication algebra and hyper MV-algebra, hyper K-algebra and (weak) hyper residuated lattices.

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