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AN INTEGRAL CONNECTIVITY CONDITION FOR MULTI-EQUILIBRIA CONSENSUS IN NETWORKS EVOLVING OVER UNDIRECTED GRAPHS

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ABSTRACT

In this paper, we study the multi-equilibria consensus problem for a time-varying network of *n* agents where the agents are modeled as integrators. Instead of the joint connectivity condition which is widely used in the literature, we propose an integral *K* connectivity condition that allows us to examine the network through a constant matrix. Based on this new concept, we present necessary and sufficient conditions on networks modeled with undirected graphs so that multi-equilibrium consensus states are achieved. Theoretical results are verified by numerical simulations.

Keywords: Integral connectivity, Multi-equilibria consensus, Time-varying topology.

1. INTRODUCTION

Many of today's studies on multi-agent networks focus on distributed coordination problems. Particularly, there has been much recent interest in consensus problems wherein a group of autonomous agents try to cooperatively agree on a common decision. This interest is primarily motivated by the wide variety of practical applications, including social networks, unmanned vehicles, intelligent highway systems and wireless sensor networks [1-3]. The analysis and design of classical consensus protocols for such complex and large-scale systems has become a significant research topic due to the rapid developments in the network technologies [4-9].

Most of the existing work in the consensus literature concentrates on the consensus protocols that reach a single equilibrium value. Such results are quite restrictive in the sense that they ignore the possibility of the network converging to multiple distinct consensus values. Lately, considerable interest has been directed toward group consensus or cluster synchronization [10-15].

Though the convergence properties of networks under fixed topology has been extensively studied, determining the convergence properties of a time-varying network is not an easy task since the communication topology changes dynamically due to the creation of new links and/or breakage of existing links. In order to analyze convergence properties of the consensus protocol over time-varying networks, the concept of *joint connectivity condition* is introduced [3-5]. This condition requires the dynamic graph $G(t)$ not necessarily to be connected at each time t, but the union of graphs[1] to be connected. One of the shortcomings of this condition is to divide the time axis into intervals such that the weighting coefficients among the agents are continuous on each interval. Moreover, it is also assumed that the time intervals are upper bounded. However, these assumptions on the weighting coefficients or topology may be difficult to verify for a given network which motivated the integral connectivity based criteria studied in this paper.

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[1] The definition of union of graphs is given in Section 2.1.

The objective of this paper is to address the multi-equilibria consensus problem for multi-agent networks in which the agents' dynamics are modeled by single integrators. The main contribution of this work is to derive an *integral connectivity* condition for networks having time-varying topologies. The result relies on an integral graph being connected. Such a convergence criterion is easier to verify than the existing results in the literature. Moreover, since we do not require the network to be artificially divided into subnetworks a priori as in [10-14], the proposed approach helps determining the number of distinct groups for the problem of multi-equilibria consensus.

The organization of this paper is as follows. In Section 2, we introduce the distributed consensus protocol and formulate the problem of multi-equilibria consensus. The integral connectivity condition is presented in Section 3. Section 4 illustrates the theoretical results with examples, while Section 5 concludes the paper.

2. THE MULTI-EQUILIBRIA CONSENSUS PROBLEM

In this section, we focus on mathematical preliminaries and definitions that are used to study distributed consensus. Additionally, the multi-equilibria consensus problem is formulated for networks with fixed and time-varying topologies.

2.1. Graph Theoretic Concepts

We use a graph G=(V, E) to represent a network consisting of n agents. The elements of $V=\{v_1,v_2,\dots,v_n\}$ are defined as the nodes, and elements of $E\subseteq V\times V$ are the edges of the given graph. The node indices take values in a finite index set $I=\{1,2,...,n\}$. Let $A=[a_{ij}]\in R^{n\times n}$ and $L=[l_{ij}]\in R^{n\times n}$ denote the adjacency and Laplacian matrices, respectively. An edge which shows information exchange among agents in G is denoted by $e_{ij}=(v_i,v_j)$ and represented as an arrow from node i to j. Two nodes i, j of G are *neighbors*, if e_{ij} is an edge of G. The set of neighbors of node v_i is denoted by $N_i=\{v_j\in V:(v_j,v_i)\in E\}$, where we adopt the convention that $v_i\notin N_i$. The considered graph is said to be undirected, if for all $i,j\in V:(v_i,v_j)\in E$ implies $(v_j,v_i)\in E$. Otherwise, the graph is directed. A *(directed) path* in a (di)graph is defined as a finite sequence of nodes $v_1,...,v_m$ such that $v_i,v_{i+1}\in E$, $i=1,...,m-1$. An undirected graph is said to be *connected* if there is a path from every node to every other node.

For the case that the topology is time-varying, the network can be described by a dynamic graph $\mathcal{G}=(V,E(t))$. The network topology will switch among a set of topologies given by $\mathcal{G}=\{\mathcal{G}_1,...,\mathcal{G}_N\}$. The union of graphs $\{\mathcal{G}_1,...,\mathcal{G}_m\}\subset\mathcal{G}$ is referred to as an undirected graph with nodes given by $v_i, i\in I$ and edge set defined by the union of edge sets $\mathcal{G}_j$, $j=1,...,m$. The notions, agent and node, will be used interchangeably throughout the paper.

2.2. Distributed Consensus Protocol

We consider a network of n agents with dynamics

$$x_i(t)=u_i(t),\qquad i\in I, \tag{1}$$

where $x_i(t) \in R$ is the state value and $u_i(t) \in R$ is the control input associated with the *i*-th agent in the network. The state value of each agent is updated according to the following continuous-time distributed consensus protocol

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}(t)(x_j(t) - x_i(t)) \tag{2}$$

where $a_{ij}(t)$ denotes the (i, j)-th entry of the corresponding weighted adjacency matrix $A(t) \in R^{n \times n}$ at time *t*. Note that, the weighting coefficients $a_{ij}(t)$ satisfy the condition: If $e_{ji} \notin E$, then $a_{ij}(t) = 0$. If $e_{ji} \in E$, there exists a positive parameter δ such that $a_{ij}(t) > 0$, $\forall t \geq 0$, $\forall j \neq i$.

Let $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$ be the state vector of the agents in the network. By applying protocol (2), (1) can be represented in matrix form as

$$x(t) = -L(t)x(t), \qquad x(0) = x_0, \tag{3}$$

where $L(t)$ is the graph Laplacian of G at time t. The elements of the graph Laplacian $L(t) = [l_{ij}(t)] \in R^{n \times n}$ are defined as follows:

$$l_{ij}(t) = \begin{cases} \sum_{j=1}^{n} a_{ij}(t), & \text{if } i = j \\ -a_{ij}(t), & \text{if } i \neq j. \end{cases} \tag{4}$$

For the time-varying network $G(t)$, we assume that the following condition holds.

Assumption 1: $L(t)$ is symmetric, i.e., $l_{ij}(t) = l_{ji}(t)$.

In most of the studies, the network topology is assumed to be fixed under ideal communication channels where the graph Laplacian becomes time-invariant, i.e., $L(t)=L$. By definition, each row of the Laplacian matrix adds up to zero. Therefore, L has a zero eigenvalue associated with the right eigenvector $\mathbf{1} = [1, ..., 1]^T \in R^n$, so $span\{\mathbf{1}\} \subseteq \ker L$, where $\ker L$ is the null space $\{x \in R^n \mid Lx = 0\}$.

The goal of the classical consensus problem is to ensure that the agents agree on a common (single) value asymptotically, i.e.,

$$\lim_{t \to \infty} x(t) = c\mathbf{1}$$

for all initial conditions $x(0) = x_0$ where c denotes the agreement value of the agents.

This is an important problem which finds many practical applications. However, this setting does not consider the occurence of multiple disctinct consensus states which may arise due to the working space, tasks, time etc. Therefore, our objective is to investigate the conditions on the time-varying network topology such that the network achieves multi-equilibria consensus.

2.3. Problem Statement

We say that the network represented by system (2) converges to K equilibrium consensus states if there exist K distinct constants c_l, and K non-empty sets $S_l, l = 1, ..., K$ such that

$$\bigcup_{l=1}^{K} S_l = V,\ \ S_l \cap S_m = \varnothing,\ \text{for } l \neq m,\ \text{and } l,m = 1,\ldots,K$$

and for the set S_l we have

$$\lim_{t\to\infty} x_i(k) = c_l,\ \forall v_i \in S_l,\ i = 1,\ldots,n$$

for any initial condition $[x_1(0), x_2(0), \ldots, x_n(0)]^T \in R^n$.

3. THE INTEGRAL CONNECTIVITY CONDITION

In this section, we analyze the multi-equilibria consensus problem for the system in (2) by extending the results of [16] under the assumption that $L(t)$ is symmetric. One of the objectives is to relax some of the restrictions on the weighing coefficients. Note that Assumption 1 in [16] requires the existence of some positive parameter δ such that $a_{ij}(t) \geq \delta$ if $e_{ij} \in E$.

However, there may be systems with time-varying coefficients where such an assumption is not satisfied. For instance take $a_{ij}(t) = e^{-t}$. Note that $a_{ij}(t)$ is always positive. Nonetheless, there does not exist a positive parameter δ such that $a_{ij}(t) \geq \delta$ holds for all $t \geq 0$.

In order to study multi-equilibria consensus in time-varying networks, we require the following integral graph definition.

Definition 1: (*Integral graph*) Let $G(t) = (V, E(t), A(t))$ be a time-varying graph. The integral graph of $G(t)$ on the interval $[0,\infty)$ is defined as a constant graph $G_{[0,\infty)} := (V, E, A)$ with the same vertex set V, and the elements of adjancency matrix $A = (a_{ij})$ are given by

$$a_{ij} = \begin{cases} 1, & \int_0^\infty a_{ij}(t)dt = \infty, \\ 0, & \int_0^\infty a_{ij}(t)dt < \infty. \end{cases} \tag{5}$$

The above integral graph notion has been used in [17] to state the following condition on consensus.

Lemma: ([17], Th 4.1) For the dynamics given in (2), the undirected network topology $G(t)$ achieves consensus if and only if $G_{[0,\infty)}$ is connected.

We will now define integral K connectedness of a time-varying graph which can be regarded as an extension of integral graph definition.

Definition 2: (*Integral K connected graph*) A time-varying graph $G(t)$ is said to be integral K connected over $[0,\infty)$, if the corresponding integral graph $G_{[0,\infty)}$ is K connected.

Using Definition 2 and the Lemma, we can state the following result.

Theorem: Consider a network of n agents represented by a time-varying graph $G(t) = (V, E(t), A(t))$. Under Assumption 1, consensus protocol (2) achieves K equilibria consensus if and only if the corresponding integral graph $G_{[0,\infty)}$ is K connected.

Proof. (*Sufficiency*) Suppose that the corresponding integral graph $G_{[0,\infty)}$ is K connected. Then L can be represented in the form

$$L = blockdiag\{L_1, L_2, ..., L_K\}$$

where each $L_l, l = 1,...,K$ has exactly one eigenvalue at 0 and the rest of its eigenvalues (if any) are positive. Note that, if L_l is not in this form, it can be transformed into this form with a proper permutation matrix. By the Lemma, it can be concluded that each subgraph with the corresponding $L_l, l = 1,...,K$ reaches consensus. Therefore, K equilibria consensus is achieved in the sense of Definition 2.

(*Necessity*): Suppose that the corresponding integral graph $G_{[0,\infty)}$ is not K connected. Then it has to be either at least *K+1* connected or at most *K-1* connected. Thus, by using the arguments in the sufficiency part, we conclude that the system achieves either *K+1*or *K-1* equilibria consensus which leads to a contradiction.

Remark: One obvious question is whether it is possible extend the Theorem to the networks modeled with digraphs. Although connectivity notion can be defined similarly, the agents may not achieve K equilibria consensus in the case that the corresponding integral graph is K connected. Therefore, it is not possible to extend the result to the case of directed graphs.

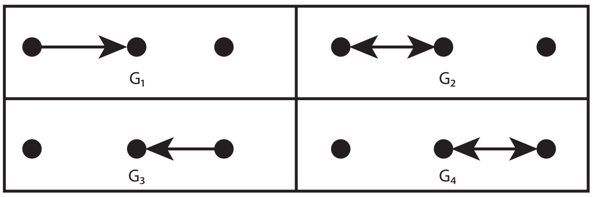

Figure 1. Four different communication topologies with 3 nodes.

Consider a network of three agents with the graphs given in Figure 1. Given $m \in N$, suppose that the sequence of the graph G_m corresponds to the concatenation of the finite sequences of the form

$$G_m = \underbrace{G_1 \quad G_1}_{2m}, G_2, \underbrace{G_3 \quad G_3}_{2m+1}, G_4 .$$

Although the union of edge sets over the interval $[t_0, \infty)$ forms an undirected connected graph, the time-varying graph G_m may not necessarily achieve K equilibria consensus (c.f. [4]).

4. ILLUSTRATIVE EXAMPLE

In this section we consider a numerical example in order to illustrate the theoretical result on multi-equilibria consensus with integral connectivity.

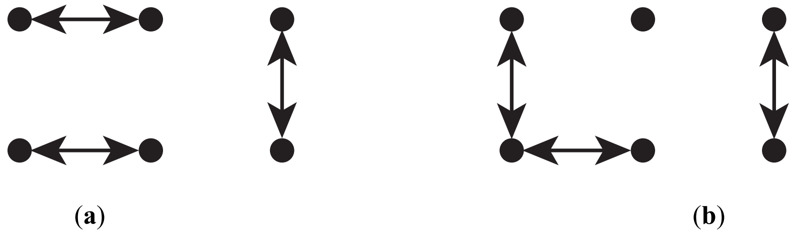

Figure 2. (a), (b) Two different communication topologies with 6 nodes.

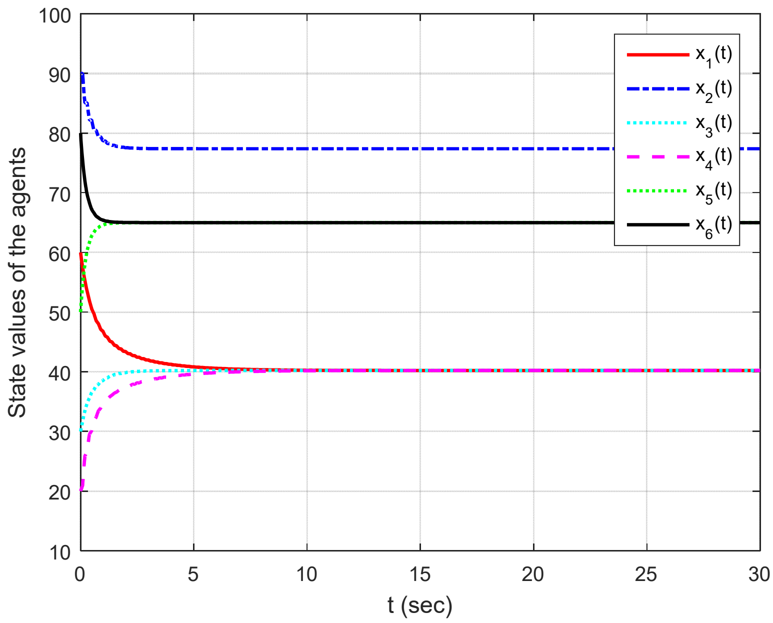

Figure 3. Convergence of the time-varying network given in Figure 2.

Consider the finite set of Laplacian matrices $\mathcal{L} = \{L_a(t), L_b\}$, associated with the topologies given in Figure 2. The Laplacian matrices are given by

$$L_a(t) = e^{-t}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-t} & -e^{-t} & 0 & 0 \\ 0 & 0 & -e^{-t} & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-t/2} & -e^{-t/2} \\ 0 & 0 & 0 & 0 & e^{-t/2} & e^{-t/2} \end{bmatrix}, \quad L_b = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix}.$$

Suppose that the network topology varies between the topologies Figure 2 (a) and (b) which switches every 10 s in the sequence of $\mathcal{L} = \{L_a(t), L_b\}$. Note that the network matrix $L_a(t)$ has time-varying weighting coefficients. Note also that, there does not exist δ, $a_{ij}(t) \geq \delta$ (e.g. we cannot choose δ such that $a_{12}(t) \geq \delta$). In line with the Theorem, the time-varying network is expected to converge to three equilibria consensus. The state trajectories of the agents with arbitrary initial states are illustrated in Figure 3 from which it is seen that the time-varying network reaches three equilibria in the sense of Definition 2.

5. CONCLUSION

In this paper, we have presented a complimentary condition for checking multi-equilibria consensus in undirected networks with underlying distributed communication infrastructure. The condition requires the integral graph being K-connected. Since the network is not artificially divided into subnetworks a priori as opposed to most of the literature, the proposed technique offers a computationally easy means to check multi-equilibria consensus in undirected time-varying networks. Future related work might include the study of time-delays in this model and its extension to directed networks.

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